Friday Lecture

We'll start with RATIO TEST EXAMPLES

Recall the test: assume \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = R \)

if \( R > 1 \) \( \sum a_k \) diverges
if \( R < 1 \) \( \sum a_k \) converges

Easy ratios: \( \frac{k+1}{R} = 1 + \frac{1}{k} \) \( \frac{k}{k+1} = \frac{1}{1 + \frac{1}{k}} \)

\[ \frac{2^{k+1}}{2^k} = \frac{2 \cdot 2^1}{2} = 2^1 \]

\[ \frac{(k+1)!}{k!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots k \cdot (k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots k} = k+1 \]

Example: converge or diverge? \( \sum \frac{k^2 \cdot 3^k}{k!} \)

There are factorials so we must use ratio
There are no \( (-1)^k \) so we don't need \( 1 \) 1

\( \left( \frac{a_{k+1}}{a_k} \right) = \frac{(k+1)^2 \cdot 3^{k+1}}{(k+1)!} \frac{k!}{k^2 \cdot 3^k} \)

\[ \frac{(k+1)^2}{k^2} \cdot \frac{3^{k+1}}{3^k} \cdot \frac{k!}{(k+1)!} \]

Put similar terms together instead of running up and down of numer and denom of oh
\[ = (1 + \frac{1}{k})^2 \cdot 3^1 \cdot \frac{1}{k+1} \leq \text{use the three easy ratios} \]

so, I did my algebra before I took limits, but now I need to connect back with limit

\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{(1 + \frac{1}{k})^2 \cdot 3^1 \cdot \frac{1}{k+1}}{1^2 \cdot 3 \cdot \frac{1}{k}} = 0
\]

\[ R = 0 < 1 \text{ so } \sum a_k \text{ converges by ratio test} \]

Example: this one compares \( k^k \) with \( k! \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( k^k )</th>
<th>( k! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2^2 = 4</td>
<td>2! = 2</td>
</tr>
<tr>
<td>3</td>
<td>3^3 = 27</td>
<td>3! = 6</td>
</tr>
<tr>
<td>4</td>
<td>4^4 = 256</td>
<td>4! = 24</td>
</tr>
<tr>
<td>5</td>
<td>5^5 = 3125</td>
<td>5! = 120</td>
</tr>
</tbody>
</table>

It looks like \( k^k \) is a lot longer than \( k! \), so \( \frac{k^k}{k!} \to \infty \)

so \( \sum \frac{k^k}{k!} \) ought to diverge.

But \( \sum \frac{k^k}{k!} \) has a factorial, so we must use ratio test
\[
\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}
\]

now flip denominator

\[
= \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \frac{(k+1)^{k+1}}{k^k} \cdot \frac{k!}{(k+1)!}
\]

The first factor has neither common factor nor common exponent, so I can't combine them. But, watch this

\[
= \frac{(k+1)^k}{k^k} \cdot \frac{k!}{(k+1)!} = \left( \frac{k+1}{k} \right)^k \cdot \frac{k!}{(k+1)!}
\]

\[
= (1 + \frac{1}{k})^k
\]

So \( \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} (1 + \frac{1}{k})^k = e \) Freesle free test

\( e > 1 \) so \( \sum a_k \) diverges by ratio test.

Remark \( k! = \int_0^\infty x^k e^{-x} dx \) so \( k! \) isn't

observable. But if you mess with the integral, you can get
Stirling's approximation for $k!$

$$k! = \sqrt{2\pi k} \frac{k^k}{e^k} \left(1 + \frac{1}{12k} + \frac{1}{288k^2} + \ldots\right)$$

So $$\frac{k^k}{k!}$$ will be like $$\frac{k^k}{\sqrt{2\pi k} \frac{k^k}{e^k}} = \frac{e^k}{\sqrt{2\pi k}}$$

So that's where the $e$ comes from in the ratio test.

Note: What is $\pi$ doing there?!!

If you work hard at \((\frac{1}{2})! = \int x^{\frac{1}{2}} e^{-x} \, dx\) you can show

\[(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}\]

This is just for "fun" not spring break level fun, but math fun.

You won't need to know stuff on this page.
Alternating Series

\[ \sum (-1)^k a_k \quad \text{or} \quad \sum (-1)^{k+1} a_k \quad \text{for} \quad a_k \geq 0 \]

Example 1

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \left( -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \right) \]

\[ = \left( -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \right) \]

Whereas

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \left( -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \right) \]

\[ = \left( -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \right) \]

hence

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = a_1 - a_2 + a_3 - a_4 + \ldots \]

\[ \sum_{k=1}^{\infty} (-1)^{k+1} a_k = -a_1 + a_2 - a_3 + \ldots \]

\[ = -\left[ a_1 - a_2 + a_3 - \ldots \right] \]

\[ = -\left[ \sum_{k=1}^{\infty} (-1)^k a_k \right] \]

So \[ \sum (-1)^k a_k \] and \[ \sum (-1)^{k+1} a_k \] are negatives of each other, so they both "do the same thing" - both converge or both diverge. So sometimes I'll just write about one of them, because the other sometimes plan ahead.