One For You: Root Test

Do the following converge or diverge? Why?

1) \[ \sum \frac{(-1)^{k+1}}{k^2 2^k} \]

2) \[ \sum \frac{1}{k^2 + k^2 2^k} \quad \text{This is good but too tricky for a quiz} \]

3) \[ \sum \frac{2^k + 3^k}{(\sqrt{k})^k} \quad \text{This is good but too tricky for a quiz} \]

4) \[ \sum \frac{(\sqrt{k})^k}{2^k + 3^k} \quad \text{This is insane. I must have been in a weird space when I wrote this problem} \]
Root Test: \[ 14 u \]

\[ \Sigma \frac{(-1)^{k+1}}{k^2 2^k} \]

The presence of the \( 2^k \) makes me think of root.

-- also I know root will make the \( k^2 \) go away.

So, first the absolute value:

\[ \left| \frac{(-1)^{k+1}}{k^2 2^k} \right| = \left| \frac{(-1)^k}{k^2 2^k} \right| = \frac{1}{k^2 2^k} \]

Because \(|(-1)^k| = 1\) and because \( k^2 \geq 0, 2^k \geq 0 \)

Now root:

\[ \left( \frac{1}{k^2 2} \right)^{1/2} = \frac{1}{k^{1/2} 2^{1/2}} = \frac{1}{2^{1/2} k^{1/2}} \]

New limit:

\[ \lim_{k \to \infty} \left| a_k \right| = \lim_{k \to \infty} \frac{1}{(k^2)^{1/2} 2^{1/2}} = \frac{1}{2} = \frac{1}{2} \]

\[ R = \frac{1}{2} < 1 \]

So \( \Sigma a_k \) converges by root.

Note: never write \( k^2 = 1 \), \( 2^k \lim k^{1/2} = 1 \)

\[ \Sigma \frac{1}{k^{1/2} + k^2} \]

This is a simple sum in the denom. Sense comp is easier.
I need to prove part of base case: \( a_k \leq b_k \)

\[
\frac{1}{k^2+k^2} \leq \frac{1}{k^2} \quad ? \quad k^2 \leq b^2 + k^2 \quad ?
\]

\[
0 \leq k^2 \quad \text{yes, } \quad k \geq 0
\]

But \( \frac{1}{k^2} \) isn't finished! I need to figure out that it converges! I'll use a root on it

\[
\lim_{k \to \infty} \frac{1}{k^2} = \lim_{k \to \infty} \left( \frac{1}{k} \right)^2 \quad \text{simplify using right}
\]

\[
= \lim_{k \to \infty} \frac{1}{k^2} = \frac{1}{2} = 2
\]

\( R < 1 \) so \( \sum \frac{1}{k^2} \) converges by root so

\[
\sum \frac{1}{k^2+k^2}
\]

**Note:** I could make this easier by collapsing the comparison with \( \sum \frac{1}{k^2} \) converges \( p > 1 \)

then \( \frac{1}{k^2+k^2} \leq \frac{1}{k^2} \quad ? \quad k^2 \leq k^2 + k^2 \quad ? \quad 0 \leq k^2 \quad \text{yes, } \quad k = 1, 2, \ldots \) and \( 2^k > 0 \)

so \( \sum \frac{1}{k^2+k^2} \) converges by

Solve comparing \( \sum \frac{1}{k^2} \)

But this doesn't work with \( \sum \frac{1}{k^2+k^2} \)

\( 2 \) doesn't converge so you'd have to do the first method.
\[ \sum \frac{2^k + 3^k}{(\sqrt{k})^k} \] 

Says "root" but root doesn't like sums - so I'll use algebra of \( \sum \) to break this into two \( \sum \)

\[ = \sum \frac{2^k}{(\sqrt{k})^k} + \sum \frac{3^k}{(\sqrt{k})^k} \] 

Now, root each

\[ \lim_{k \to \infty} \left( \frac{2^k}{(\sqrt{k})^k} \right)^{\frac{1}{k}} = \lim_{k \to \infty} \frac{2^{k/k}}{((\sqrt{k})^k)^{1/k}} = \lim_{k \to \infty} \frac{2}{\sqrt{k}} = 0 \]

So \( \sum \frac{2^k}{(\sqrt{k})^k} \) converges by root.

Same trick works for \( \sum \frac{3^k}{(\sqrt{k})^k} \).

Remark: This wouldn't work for \( \sum \frac{2^k \sqrt{k}}{2^k + 3^k} \)!!

Because remember \( \frac{1}{2} + \frac{1}{3} \neq \frac{1}{2 + 3} \)!!

So how would you do it? Base case: simple

Sum in denom \( \sqrt{k} \) \( \leq \sqrt{k} \) ?

\[ \frac{2^k + 3^k}{2^k + 3^k} \leq \frac{3^k}{3^k} \]

\[ 2^k \leq 3^k ? \quad \text{and} \quad 2^k \leq 2^k + 3^k \quad \text{and} \quad 2^k \quad \text{yes} \]

\( 0 \leq 2^k \) yes

\( \text{only} \geq 0 \).

Now what does \( \sum \frac{\sqrt{k}}{3^k} \) do?
\[ \lim_{k \to 0} \frac{\sqrt{k}}{\frac{1}{3^k}} = \lim_{k \to 0} \frac{\sqrt{k}}{3^{-k}} \]  
(keep some edges)

\[ \lim_{k \to 0} \frac{\sqrt{k}}{3} = 0 - 0 = 0 \]

\[ R > 1 \] so \( \sum \frac{\sqrt{k}}{3^k} \) diverges so ---

wait for it --- I can't use the cond part of base comp!!!! I got it wrong. Like the Australian Outback, you have to know when you are doing the wrong thing, so you can go back.

Let's do \( \lim \) comp?

\[ \lim_{k \to 0} \frac{\sqrt{k}}{\frac{2^k}{3^k}} = \lim_{k \to 0} \frac{\sqrt{k}}{2^k} \cdot \lim_{k \to 0} \frac{3^k}{\sqrt{k}} \]

\[ = \lim_{k \to 0} \frac{1}{(\frac{2}{3})^k + 1} \text{ or log, uk class result} \]

\[ \lim_{k \to 0} \left(\frac{2}{3}\right)^k = 0 \text{ since } r = \frac{2}{3} \text{ and } |r| < 1 \]

so \( L = \frac{1}{0+1} = 1 \). \( 0 < L < \infty \) so

\[ \sum \frac{\sqrt{k}}{2^k + 3^k} \text{ dv by } \lim \text{ comp with } \sum \frac{\sqrt{k}}{3^k} \]