Lecture 2

FOURIER TRANSFORMS

Change of Basis

Let's start with 'why'. The cheap answer is 'to get a different perspective; to see things that couldn't be seen otherwise'.

Or, we could just do an example. Eigenvalues and eigenvectors were made for this. Let's look at the matrix $T = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$. Since we'll be changing bases, we ought to specify that the matrix is taken with respect to the standard basis of $\mathbf{R}^2$, $\mathbf{s_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\mathbf{s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.Then, as always, the first column of T is $T\mathbf{s_1}$, and the second is $T\mathbf{s_2}$.

<u>Problem</u> Show that T has eigenvalues $\lambda_1 = 6;\ \lambda_2 = 4$, and that the corresponding eigenvectors may be taken to be

$$\mathbf{b_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix};\ \mathbf{b_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Note that the two are orthonormal. Of course the change-of-basis map from the $\mathbf{s}$ to the $\mathbf{b}$ basis is

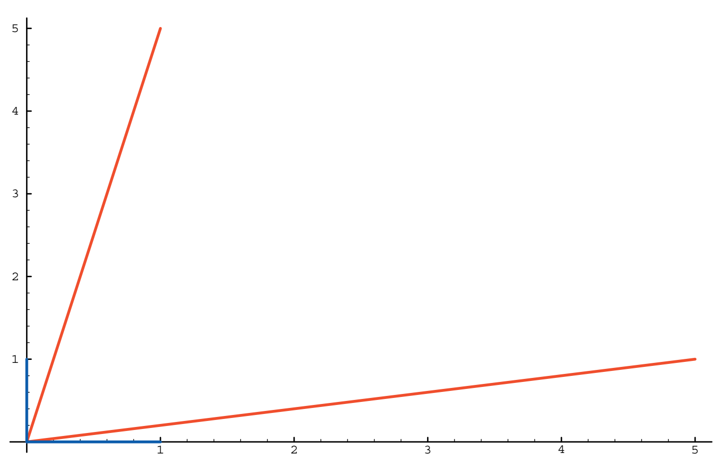
$$C_{\mathbf{sb}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Prove that the inverse of $C_{\mathbf{sb}}$, that is, $C_{\mathbf{bs}}$, is given by

$$C_{\mathbf{bs}} = C_{\mathbf{sb}}^{-1} = C_{\mathbf{sb}}^{*}$$

where we use the * to denote complex-conjugate and transpose. Now, show that the matrix T, in the $\mathbf{b}$ basis, is $C_{\mathbf{bs}} T C_{\mathbf{sb}} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$, which is what we'd expect.

In this 'perspective', the matrix T just stretches the basis differently in the two orthogonal directions. Compare the picture below, showing the action of T on the standard basis.



Homework Problem As we'll be changing bases rather a lot in this course, let's prove some basic results. Let V be a finite-dimensional inner product space. Given any pair of orthonormal bases **b,** **s** for V, show that $C_{\mathbf{bs}} = C_{\mathbf{sb}}^{-1} = C_{\mathbf{sb}}^{*}$. Moreover, show that the change of bases maps are *unitary*, that is, that for any pair of vectors $x\ y \in V$,

$$(C_{\mathbf{bs}}x, C_{\mathbf{bs}}y) = (x,\ y)$$

That is, unitary transformations preserve inner products. As angles and lengths are defined in terms of inner products . . .

If $V = \mathbf{R}^n$ with the usual inner product, show that if $x = (x_1,\ x_2,\ \dots\ ,\ x_n)$ in the **s** basis, but $x = (w_1,\ w_2,\ \dots\ ,\ w_n)$ in the **b** basis, then

$$\sum |x_j|^2 = \sum |w_j|^2$$