The Continuous Wavelet Transform

Mathematically, the process of Fourier analysis is represented by the Fourier transform:

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \]

which is the sum over all time of the signal \( f(t) \) multiplied by a complex exponential. (Recall that a complex exponential can be broken down into real and imaginary sinusoidal components.)

The results of the transform are the Fourier coefficients \( F(\omega) \), which when multiplied by a sinusoid of frequency \( \omega \) yield the constituent sinusoidal components of the original signal. Graphically, the process looks like

Similarly, the continuous wavelet transform (CWT) is defined as the sum over all time of the signal multiplied by scaled, shifted versions of the wavelet function \( \psi \):

\[ C(\text{scale, position}) = \int_{-\infty}^{\infty} f(t) \psi(\text{scale, position, } t) dt \]

The results of the CWT are many wavelet coefficients \( C \), which are a function of scale and position.
Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelets of the original signal:

![Wavelet Transform](image)

**Scaling**

We’ve already alluded to the fact that wavelet analysis produces a time-scale view of a signal, and now we’re talking about scaling and shifting wavelets. What exactly do we mean by *scale* in this context?

Scaling a wavelet simply means stretching (or compressing) it.

To go beyond colloquial descriptions such as “stretching,” we introduce the *scale factor*, often denoted by the letter $a$. If we’re talking about sinusoids, for example, the effect of the scale factor is very easy to see:

$$f(t) = \sin(t) ; \quad a = 1$$

$$f(t) = \sin(2t) ; \quad a = \frac{1}{2}$$

$$f(t) = \sin(4t) ; \quad a = \frac{1}{4}$$
The scale factor works exactly the same with wavelets. The smaller the scale factor, the more “compressed” the wavelet.

$$f(t) = \psi(t) ; \quad a = 1$$

$$f(t) = \psi(2t) ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) ; \quad a = \frac{1}{4}$$

It is clear from the diagrams that, for a sinusoid $\sin(\omega t)$, the scale factor $a$ is related (inversely) to the radian frequency $\omega$. Similarly, with wavelet analysis, the scale is related to the frequency of the signal. We’ll return to this topic later.

**Shifting**

Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function $f(t)$ by $k$ is represented by $f(t - k)$:

**Five Easy Steps to a Continuous Wavelet Transform**

The continuous wavelet transform is the sum over all time of the signal multiplied by scaled, shifted versions of the wavelet. This process produces wavelet coefficients that are a function of scale and position.

It’s really a very simple process. In fact, here are the five steps of an easy recipe for creating a CWT:
1. Take a wavelet and compare it to a section at the start of the original signal.

2. Calculate a number, \( C \), that represents how closely correlated the wavelet is with this section of the signal. The higher \( C \) is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one, \( C \) may be interpreted as a correlation coefficient.

Note that the results will depend on the shape of the wavelet you choose.

3. Shift the wavelet to the right and repeat steps 1 and 2 until you’ve covered the whole signal.

4. Scale (stretch) the wavelet and repeat steps 1 through 3.

5. Repeat steps 1 through 4 for all scales.
When you're done, you'll have the coefficients produced at different scales by different sections of the signal. The coefficients constitute the results of a regression of the original signal performed on the wavelets.

How to make sense of all these coefficients? You could make a plot on which the x-axis represents position along the signal (time), the y-axis represents scale, and the color at each x-y point represents the magnitude of the wavelet coefficient $C$. These are the coefficient plots generated by the graphical tools.

These coefficient plots resemble a bumpy surface viewed from above. If you could look at the same surface from the side, you might see something like this:

The continuous wavelet transform coefficient plots are precisely the time-scale view of the signal we referred to earlier. It is a different view of signal data from the time-frequency Fourier view, but it is not unrelated.
Scale and Frequency
Notice that the scales in the coefficients plot (shown as y-axis labels) run from 1 to 31. Recall that the higher scales correspond to the most “stretched” wavelets. The more stretched the wavelet, the longer the portion of the signal with which it is being compared, and thus the coarser the signal features being measured by the wavelet coefficients.

Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:

- Low scale $a \Rightarrow$ Compressed wavelet $\Rightarrow$ Rapidly changing details $\Rightarrow$ High frequency $\omega$.
- High scale $a \Rightarrow$ Stretched wavelet $\Rightarrow$ Slowly changing, coarse features $\Rightarrow$ Low frequency $\omega$.

The Scale of Nature
It’s important to understand that the fact that wavelet analysis does not produce a time-frequency view of a signal is not a weakness, but a strength of the technique.

Not only is time-scale a different way to view data, it is a very natural way to view data deriving from a great number of natural phenomena.
Consider a lunar landscape, whose ragged surface (simulated below) is a result of centuries of bombardment by meteorites whose sizes range from gigantic boulders to dust specks.

If we think of this surface in cross section as a one-dimensional signal, then it is reasonable to think of the signal as having components of different scales — large features carved by the impacts of large meteorites, and finer features abraded by small meteorites.

Here is a case where thinking in terms of scale makes much more sense than thinking in terms of frequency. Inspection of the CWT coefficients plot for this signal reveals patterns among scales and shows the signal’s possibly fractal nature.

Even though this signal is artificial, many natural phenomena — from the intricate branching of blood vessels and trees, to the jagged surfaces of mountains and fractured metals — lend themselves to an analysis of scale.
What’s Continuous About the Continuous Wavelet Transform?

Any signal processing performed on a computer using real-world data must be performed on a discrete signal — that is, on a signal that has been measured at discrete time. So what exactly is “continuous” about it?

What’s “continuous” about the CWT, and what distinguishes it from the discrete wavelet transform (to be discussed in the following section), is the set of scales and positions at which it operates.

Unlike the discrete wavelet transform, the CWT can operate at every scale, from that of the original signal up to some maximum scale that you determine by trading off your need for detailed analysis with available computational horsepower.

The CWT is also continuous in terms of shifting: during computation, the analyzing wavelet is shifted smoothly over the full domain of the analyzed function.