

THE BERNSTEIN CENTER OF THE CATEGORY OF SMOOTH $W(k)[\mathrm{GL}_n(F)]$ -MODULES

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ABSTRACT. We consider the category of smooth $W(k)[\mathrm{GL}_n(F)]$ -modules, where F is a p -adic field and k is an algebraically closed field of characteristic ℓ different from p . We describe a factorization of this category into blocks, and show that the center of each such block is a reduced, ℓ -torsion free, finite type $W(k)$ -algebra. Moreover, the k -points of the center of a such a block are in bijection with the possible “supercuspidal supports” of the smooth $k[\mathrm{GL}_n(F)]$ -modules that lie in the block. Finally, we describe a large explicit subalgebra of the center of each block and a description of the action of this algebra on the simple objects of the block, in terms of the description of the classical “characteristic zero” Bernstein center of [BD].

1. INTRODUCTION

The *center* of an abelian category is \mathcal{A} the endomorphism ring of the identity functor of that category. It is a commutative ring that acts naturally on every object of \mathcal{A} , a fact which often allows one to approach questions about \mathcal{A} from a module-theoretic point of view.

One spectacular success of this approach is due to Bernstein and Deligne [BD], who computed the centers of categories of smooth complex representations of p -adic algebraic groups. The center of such a category is called the Bernstein center. Bernstein and Deligne give a factorization of this category into blocks, known as Bernstein components, as well as a simple and explicit description of the center of each block, which is a finite type \mathbb{C} -algebra. In particular they showed that the \mathbb{C} -points of the Bernstein center were in bijection with the supercuspidal supports of irreducible smooth complex representations.

The results of [BD] made it possible to give purely algebraic proofs of theorems about smooth representations that previously could only be proven via deep results from Fourier theory; Bushnell-Henniart’s results about Whittaker models in [BH] are an example of this approach.

In recent years there has been considerable interest in studying smooth representations over fields other than the complex numbers, or even over more general rings. To apply similar techniques in such a setting one needs to understand the centers of categories of smooth representations over \mathbb{F}_ℓ or \mathbb{Z}_ℓ (or even over \mathbb{Z}). Some progress along these lines was made by Dat [D1]; in particular he was able to give an explicit description of the center of the category of smooth representations of a p -adic algebraic group G over \mathbb{Z}_ℓ , for ℓ a *banal* prime; that is, for ℓ prime to the order of $G(\mathbb{F}_p)$.

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More recently Paskunas [Pa] has studied the center of a category of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over \mathbb{F}_p ; his results allow him to characterize the image of the Colmez functor.

We fix our attention on the category $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))$ of smooth representations of $\mathrm{GL}_n(F)$, where F is a p -adic field, over a ring of Witt vectors $W(k)$, for k an algebraically closed field of characteristic ℓ different from p . We obtain a factorization of this category into blocks that parallels the Bernstein decomposition over \mathbb{C} . This description is closely related to the decomposition due to Vigneras [V2] of the category of smooth representations of $\mathrm{GL}_n(F)$ over k ; in both decompositions the blocks are parameterized by inertial equivalence classes of pairs (L, π) , where L is a Levi subgroup of $\mathrm{GL}_n(F)$ and π is an irreducible supercuspidal representation of L over k .

Let $A_{[L, \pi]}$ be the center of the block of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))$ corresponding to (L, π) . We obtain a description of $A_{[L, \pi]}$ as a concrete subalgebra of the endomorphism algebra of a certain projective object. When $\ell > n$ this yields a completely explicit description of $A_{[L, \pi]}$, but the description falls short of being totally explicit for small ℓ . In spite of this, for all $\ell \neq p$ we are able to construct a subalgebra $C_{[L, \pi]}$ of $A_{[L, \pi]}$ such that $A_{[L, \pi]}$ is a finitely generated $C_{[L, \pi]}$ -module, and give a simple, concrete description of $C_{[L, \pi]}$. Indeed, after making certain choices we obtain an isomorphism:

$$C_{[L, \pi]} \cong W(k)[Z]^{W_L(\pi)},$$

where Z is a certain finitely generated free abelian subgroup of the center of L , $W_L(\pi)$ is the subgroup of the Weyl group of $\mathrm{GL}_n(F)$ consisting of elements w such that $wLw^{-1} = L$, and π^w is inertially equivalent to π . In particular it follows that $A_{[L, \pi]}$ and $C_{[L, \pi]}$ are finite type $W(k)$ -algebras.

The action of $C_{[L, \pi]}$ on irreducible representations of $\mathrm{GL}_n(F)$, both in characteristic zero and in characteristic ℓ , can be made completely explicit in terms of a choice of certain “compatible systems of cuspidals”. (We refer the reader to Theorem 7.3 and the discussion preceding it for a description of these systems.) This allows us to show that the k -points of $A_{[L, \pi]}$ are in bijection with the supercuspidal supports of irreducible smooth representations of $\mathrm{GL}_n(F)$ over k that lie in the block corresponding to (L, π) . This gives a “mod ℓ ” analogue of the corresponding result of Bernstein-Deligne for complex points of the classical Bernstein center. (See section 12 for precise statements of these results.)

This is the first part of a three-part paper. The second part of this paper will be devoted to exploring the relationship between $A_{L, \pi}$ and the deformation theory of Galois representations, via the local Langlands correspondence. In particular we will give a conjectural description of the completion of $A_{[L, \pi]}$ at a point x as a subalgebra of the universal framed deformation ring of the semisimple representation of G_F attached to x via local Langlands. Although our techniques fall short of proving this conjecture, we also give a conjectural description of $C_{[L, \pi]}$ in this manner, as the subalgebra generated by the coefficients of the characteristic polynomials of certain “Frobenius elements” of the universal framed deformation ring. This conjecture turns out to be more tractable, and we prove it in many cases. When ℓ is a “banal” prime (that is, when $1, \dots, q^n$ are distinct mod ℓ , where q is the order of the residue field of F .) then $C_{[L, \pi]} = A_{[L, \pi]}$ and we can prove both conjectures.

The third paper in this series will consider the implications of these conjectures for the “local Langlands correspondence in families” of [EH]. In particular we will apply the structure theory of $A_{[L,\pi]}$ to questions that arise from the theory of Whittaker models, that were first studied over the complex numbers in [BH]. We establish versions of several of these results that hold for smooth representations over $W(k)$, and use these results, together with our conjectural description of $A_{L,\pi}$, to prove a conjecture of [EH] attaching families of admissible representations of $\mathrm{GL}_n(F)$ to families of Galois representations.

Our approach to the Bernstein center is necessarily different from that of Bernstein and Deligne, who rely on properties of cuspidal representations that hold only over fields of characteristic zero. Instead, we proceed by constructing faithfully projective objects in certain direct factors of the category of smooth $W(k)[\mathrm{GL}_n(F)]$ -modules. Central endomorphisms of such objects then yield elements of the Bernstein center via standard arguments that we recall in section 2.

Our construction of these projective $W(k)[\mathrm{GL}_n(F)]$ -modules relies heavily on the theory of types. (This is why we must restrict our attention to the group GL_n , where the theory of types is well-developed.) The construction occurs in several steps, but begins with a cuspidal type over k . Such a type has a projective envelope with a fairly explicit description; compactly inducing this projective envelope then yields a projective $\mathrm{GL}_n(F)$ -module. We study the structure of such modules in sections 4 and 7. The general structure theory relies heavily on the characteristic zero notion of a “generic pseudo-type”; we introduce this notion in section 6, and compute the Hecke algebras attached to such types. These Hecke algebras turn out to be analogues, in some sense, of spherical Hecke algebras.

The projectives constructed in section 4 are not the faithfully projective objects we need to consider, however. The latter are obtained by parabolic induction from modules we have already constructed. To show that the resulting modules are projective, we invoke a version of Bernstein’s second adjointness proved for smooth $W(k)[\mathrm{GL}_n(F)]$ -modules by Dat [D2]. Although it would suffice to simply invoke this result for the main purposes of the paper, we also give an alternate proof of Bernstein’s second adjointness that uses the techniques of this paper in section 10. The type theoretic approach we take is very different from that of Dat, and one might hope that it will apply in other contexts where the theory of types is sufficiently developed.

Once we have constructed the faithfully projective objects in question, and shown them to be faithfully projective, it remains to compute their central endomorphisms. This is done in section 11; our computation relies heavily on our a priori understanding of the Bernstein center over fields of characteristic zero. The result is Corollary 11.11, which expresses the Bernstein center as a certain (explicitly defined) subalgebra of a multivariate Laurent polynomial ring over a certain finite rank $W(k)$ -algebra. This finite rank $W(k)$ -algebra arises from the representation theory of finite groups: it is a tensor product of endomorphism rings of projective envelopes of cuspidal representations of $\mathrm{GL}_r(\mathbb{F}_{q^s})$ over k , for various r and s . The structure of such endomorphism rings is currently under investigation by David Paige.

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2. FAITHFULLY PROJECTIVE MODULES AND THE BERNSTEIN CENTER

Definition 2.1. Let \mathcal{A} be an abelian category. The *center* of \mathcal{A} is the ring of endomorphisms of the identity functor $\text{Id} : \mathcal{A} \rightarrow \mathcal{A}$. More prosaically, an element of \mathcal{A} is a choice of element $f_M \in \text{End}(M)$ for every object M in \mathcal{A} , satisfying the condition $f_M \circ \phi = \phi \circ f_N$ for every morphism $\phi : N \rightarrow M$ in \mathcal{A} .

If \mathcal{A} is the category of right R -modules for some (not necessarily commutative) ring R , then it is easy to see that the center of \mathcal{A} is the center $Z(R)$ of the ring R . Indeed, $Z(R)$ acts on every object of \mathcal{A} ; this defines a map of $Z(R)$ into the center. Its inverse is constructed by considering the action of the Bernstein center on R , considered as a right R -module.

When \mathcal{A} is a more general abelian category, we can often describe its center by reducing to the case of a module category. We more or less follow the ideas of [R], section 1.1. The key is to find an object in \mathcal{A} that is faithfully projective, in the following sense:

Definition 2.2. Let \mathcal{A} be an abelian category with direct sums. An object P in \mathcal{A} is *faithfully projective* if:

- (1) P is a projective object of \mathcal{A} .
- (2) The functor $M \mapsto \text{Hom}(P, M)$ is faithful.
- (3) P is *small*; that is, one has an isomorphism:

$$\bigoplus_{i \in I} \text{Hom}(P, M_i) \cong \text{Hom}(P, \bigoplus_{i \in I} M_i)$$

for any family M_i of objects of \mathcal{A} indexed by a set I .

One checks easily that the condition that $M \mapsto \text{Hom}(P, M)$ is faithful is equivalent to the condition that $\text{Hom}(P, M)$ is nonzero for every object M of \mathcal{A} . If \mathcal{A} has the property that every object of \mathcal{A} has a simple subquotient, then it suffices to check that $\text{Hom}(P, M)$ is nonzero for every simple M .

If P is a faithfully projective object of \mathcal{A} , one has:

Proposition 2.3 ([R], Theorem 1.1). *Let P be faithfully projective. The functor $M \mapsto \text{Hom}(P, M)$ is an equivalence of categories from \mathcal{A} to the category of right $\text{End}(P)$ -modules. In particular, the center of \mathcal{A} is isomorphic to the center of $\text{End}(P)$.*

Idempotents of the center correspond to factorizations of \mathcal{A} as a product of categories. In practice we can obtain these factorizations by constructing suitable injective objects of \mathcal{A} .

Proposition 2.4. *Suppose that every object of \mathcal{A} has a simple subquotient, let S be a subset of the simple objects of \mathcal{A} , and let I_1, I_2 be injective objects of \mathcal{A} such that, up to isomorphism:*

- (1) every simple subquotient of I_1 is in S ,
- (2) every object in S is a subobject of I_1 ,
- (3) no simple subquotient of I_2 is in S , and
- (4) every simple object of \mathcal{A} that is not in S is a subobject of I_2 .

Then every object M of \mathcal{A} splits canonically as a product $M_1 \times M_2$, where every simple subquotient of M_1 is in S , and no simple subquotient of M_2 is in S . This gives a decomposition of \mathcal{A} as a product of the full subcategories \mathcal{A}_1 and \mathcal{A}_2 of \mathcal{A} , where the objects of \mathcal{A}_1 are those objects M_1 of \mathcal{A} such that every simple

subquotient of M_1 is in S , and the objects of \mathcal{A}_2 are those objects M_2 of \mathcal{A} such that no simple subquotient of M_2 is in S . Moreover, every object of \mathcal{A}_1 has an injective resolution by direct sums of copies of I_1 , and every object of \mathcal{A}_2 has an injective resolution by direct sums of copies of I_2 .

Proof. Let M_1 be the maximal quotient of M such that every simple subquotient of M_1 is in S , and let M_2 be the kernel of the map $M \rightarrow M_1$. We first show $\text{Hom}(M_2, I_1) = 0$. Suppose we have a nonzero map of M_2 into I_1 , with kernel N . Then the injection of M_2/N into I_1 would extend to an injection of M/N into I_1 , and thus M/N would be a quotient of M , dominating M_1 , all of whose simple subquotients were in S . It follows that no simple subquotient of M_2 lies in S , as such a subquotient would yield a nonzero map of M_2 to I_1 .

If we let M_3 be the maximal quotient of M such that no simple subquotient of M_3 is in S , then the same argument (with I_1 and I_2 reversed) shows that every simple subquotient of the kernel of the map $M \rightarrow M_3$ lies in S . In particular the projection of M_2 onto M_3 is injective. Suppose the image of M_2 were not all of M_3 . Then (as M surjects onto M_3), there is a simple subquotient of M_3 that is also a subquotient of M/M_2 ; such an object would have to be in both S and its complement. Thus M_2 is isomorphic to M_3 , and hence M splits, canonically, as a product $M_1 \times M_2$. The decomposition of \mathcal{A} as the product $\mathcal{A}' \times \mathcal{A}''$ is now immediate, as is the claim about resolutions. \square

Remark 2.5. It is easy to make a dual argument with projective objects; we state the proposition in terms of injectives because that is the form of the proposition we will use.

3. THE BERNSTEIN CENTER OF $\text{Rep}_{\overline{\mathcal{K}}}(G)$

Let $G = \text{GL}_n(F)$ be a general linear group over a p -adic field F , and let k be an algebraically closed field of characteristic ℓ not equal to p . Our goal is to study the Bernstein center of the category $\text{Rep}_{W(k)}(G)$ of smooth $W(k)[G]$ -modules. We assume throughout that ℓ is odd, so that $W(k)$ necessarily contains a square root of q , where q is the order of the residue field of F . We begin by studying the category $\text{Rep}_{\overline{\mathcal{K}}}(G)$ of smooth $\overline{\mathcal{K}}[G]$ -modules, where \mathcal{K} is the field of fractions of $W(k)$. Most of the results of this section are standard. We limit ourselves to the case of $\text{GL}_n(F)$, although the results of this section have analogues for a general reductive group.

The description of the center of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ depends heavily on the theory of parabolic induction, and particularly the notions of cuspidal and supercuspidal support, which we now recall. Let (M, π) be an ordered pair consisting of a Levi subgroup M of G and an absolutely irreducible cuspidal representation π of M .

Let P be a parabolic subgroup of G , with Levi subgroup M and unipotent radical U , and let $\pi = \pi_1 \otimes \dots \otimes \pi_r$ be a $W(k)[M]$ -module. We let i_P^G be the normalized parabolic induction functor of [BZ]; that is, $i_P^G \pi$ is the $W(k)[G]$ -module obtained by extending π by a trivial U -action to a representation of P , twisting by a square root of the modulus character of P , and inducing to G . (This depends on a choice of square root of q in $W(k)$; we fix such a choice once and for all.) Similarly, we denote by r_G^P the parabolic restriction functor from $W(k)[G]$ -modules to $W(k)[M]$ -modules.

Definition 3.1. Let M be a Levi subgroup of G , and let π be a supercuspidal representation of M over a field L . An absolutely irreducible representation Π

of G over L has *supercuspidal support* equal to (M, π) if π is supercuspidal, and there exists a parabolic subgroup P of G , with Levi subgroup M , such that Π is isomorphic to a Jordan–Hölder constituent of the normalized parabolic induction $i_P^G \pi$.

Definition 3.2. Let M be a Levi subgroup of G , and let π be a cuspidal representation of M over a field L . An absolutely irreducible representation Π of G has *cuspidal support* equal to (M, π) if there exists a parabolic subgroup P of G , with Levi subgroup M , such that Π is isomorphic to a *quotient* of $i_P^G \pi$.

Over a field L of characteristic zero, the notions of cuspidal and supercuspidal support are equivalent, but the notions differ over fields of finite characteristic.

Two pairs (M, π) and (M', π') are conjugate in G if there is an element g of G that conjugates M to M' and π to π' . This determines an equivalence relation on the set of pairs (M, π) . Both the cuspidal and supercuspidal support of an absolutely irreducible representation Π of G are uniquely determined up to conjugacy.

Definition 3.3. We say that two representations π, π' of G that differ by a twist by $\chi \circ \det$, where χ is an unramified character of F^\times , are *inertially equivalent*. More generally, if M is a Levi subgroup of G , two representations π and π' are inertially equivalent if they differ by a twist by an unramified character χ of M , that is, a character χ trivial on all compact open subgroups of M .

We are primarily interested in cuspidal and supercuspidal support up to inertial equivalence. Two pairs (M, π) and (M', π') are inertially equivalent if there is a representation π'' of M , inertially equivalent to π , such that (M, π'') is conjugate to (M', π') . The *inertial supercuspidal support* (resp. *inertial cuspidal support*) of an absolutely irreducible representation Π of G is the inertial equivalence class of its supercuspidal support (resp. cuspidal support).

Theorem 3.4 (Bernstein-Deligne, [BD], 2.13). *Let M be a Levi subgroup of G , and let π be an irreducible cuspidal representation of M . Let $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M, \pi}$ be the full subcategory of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ consisting of representations Π such that every simple subquotient of Π has inertial supercuspidal support (M, π) . Then $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M, \pi}$ is a direct factor of $\text{Rep}_{\overline{\mathcal{K}}}(G)$.*

There is thus an idempotent $e_{M, \pi, \overline{\mathcal{K}}}$ of the Bernstein center of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ that acts by the identity on all objects of $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M, \pi}$ and annihilates all of the other Bernstein components.

Moreover, it is possible to give a complete description of the center $A_{M, \pi}$ of $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M, \pi}$. Let $\Psi(M)$ denote the group of unramified characters of M . Then $\Psi(M)$ can be identified with the algebraic torus $\text{Spec } \overline{\mathcal{K}}[M/M_0]$, where M_0 is the subgroup of M generated by all compact open subgroups of M . The group $\Psi(M)$ acts transitively (by twisting) on the space of representations of M inertially equivalent to π , and the stabilizer of π is a finite subgroup H of $\Psi(M)$. Note that H depends only on the inertial equivalence class of π , not π itself. The group $\Psi(M)/H$ is a torus, isomorphic to $\text{Spec } \overline{\mathcal{K}}[M/M_0]^H$; a choice of π identifies $\Psi(M)/H$ with the space of representations of M inertially equivalent to π .

Let W_M be the subgroup of the Weyl group $W(G)$ of G (taken with respect to a maximal torus contained in M) consisting of elements w of $W(G)$ such that $wMw^{-1} = M$. Define a subgroup $W_M(\pi)$ of W_M consisting of all w in W_M such that π^w is inertially equivalent to π (this subgroup depends only on the inertial

equivalence class of π .) Then $W_M(\pi)$ acts on the space of representations of M inertially equivalent to π , and hence (via a choice of π) on the torus $\text{Spec } \overline{\mathcal{K}}[M/M_0]^H$. This action is in general a twist of the usual (permutation) action of W_M on M/M_0 , but if π is *invariant* under the action of $W_M(\pi)$, then the action of $W_M(\pi)$ on M/M_0 is untwisted.

We have:

Theorem 3.5 (Bernstein-Deligne). *A choice of π identifies $A_{M,\pi}$ with the ring $(\overline{\mathcal{K}}[M/M_0]^H)^{W_M(\pi)}$. More canonically, the space of representations of M inertially equivalent to π is naturally a Ψ/H -torsor with an action of $W_M(\pi)$, and the center of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ is the ring of $W_M(\pi)$ -invariant regular functions on this torsor. Moreover, if f is an element of $A_{M,\pi}$, and Π is an object of $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M,\pi}$ with supercuspidal support (M, π') , then f acts on Π by the scalar $f(\pi')$.*

If π is invariant under the action of $W_M(\pi)$, then the action of $W_M(\pi)$ on M/M_0 is the usual (untwisted) permutation action.

We conclude with a standard result describing the action of the Bernstein center on modules arising by parabolic induction. Let (M_i, π_i) be pairs consisting of a Levi subgroup M_i of $\text{GL}_{n_i}(F)$, and an irreducible cuspidal representation π_i of M_i such that π_i is invariant under the action of $W_{M_i}(\pi_i)$. Let M be the product of the M_i , considered as a subgroup of $\text{GL}_n(F)$, where n is the sum of the n_i . Let π be the tensor product of the π_i ; it is an irreducible cuspidal representation of M . We then have an action of $W_M(\pi)$ on the inertial equivalence class of (M, π) ; we assume that π is invariant under this action.

In this setting, the group $(M/M_0)^H$ is the product of the groups $(M_i/(M_i)_0)^{H_i}$, where H_i is the subgroup of characters fixing π_i under twist. The isomorphism:

$$\overline{\mathcal{K}}[M/M_0]^H \cong \bigotimes_i \overline{\mathcal{K}}[M_i/(M_i)_0]^{H_i}$$

then restricts to give an embedding:

$$\Phi : (\overline{\mathcal{K}}[M/M_0]^H)^{W_M(\pi)} \hookrightarrow \bigotimes_i (\overline{\mathcal{K}}[M_i/(M_i)_0]^{H_i})^{W_{M_i}(\pi_i)}$$

Proposition 3.6. *Let Π_i be a collection of representations of GL_{n_i} such that for each i , Π_i lies in $\text{Rep}_{\overline{\mathcal{K}}}(\text{GL}_{n_i}(F))_{M_i, \pi_i}$. Let $P = LU$ be a parabolic subgroup of $\text{GL}_n(F)$, with L isomorphic to the product of the $\text{GL}_{n_i}(F)$, and let Π be the tensor product of the Π_i , considered as a representation of L . Then $i_P^{\text{GL}_n(F)} \Pi$ lies in $\text{Rep}_{\overline{\mathcal{K}}}(\text{GL}_n(F))_{M,\pi}$.*

Moreover, under the identifications

$$A_{M,\pi} \cong (\overline{\mathcal{K}}[M/M_0]^H)^{W_M(\pi)},$$

$$A_{M_i, \pi_i} \cong (\overline{\mathcal{K}}[M_i/(M_i)_0]^{H_i})^{W_{M_i}(\pi_i)}$$

induced by (M, π) and (M_i, π_i) , if x lies in $A_{M,\pi}$, then the endomorphism of $i_P^{\text{GL}_n(F)} \Pi$ induced by x coincides with the endomorphism of $i_P^{\text{GL}_n(F)} \Pi$ arising from the action of $\Phi(x)$ on Π .

4. CONSTRUCTION OF PROJECTIVES

Our goal is to apply the theory of section 2 to the category $\text{Rep}_{W(k)}(G)$ of smooth $W(k)[G]$ -modules. In particular we will factor this category as a product of blocks, and construct an explicit faithfully projective module in each block. The

first step is to obtain a supply of suitable projective $W(k)[G]$ -modules, and study their properties.

Definition 4.1. Let R be a $W(k)$ -algebra. By an R -type of G , we mean a pair (K, τ) , where K is a compact open subgroup of G and τ is an $R[K]$ -module that is finitely generated as an R -module. The Hecke algebra $H(G, K, \tau)$ is the ring $\text{End}_{R[G]}(\text{c-Ind}_K^G \tau)$.

For the most part we will be concerned with R -types for $R = k$, or $R = \mathcal{K}$, where \mathcal{K} is the fraction field of $W(k)$. Note that for any $R[G]$ -module π , Frobenius reciprocity gives an isomorphism of $\text{Hom}_{R[K]}(\tau, \pi)$ with $\text{Hom}_{R[G]}(\text{c-Ind}_K^G \tau, \pi)$, and hence an action of $H(G, K, \tau)$ on $\text{Hom}_{R[K]}(\tau, \pi)$. Moreover, if V is the underlying R -module of τ , $H(G, K, \tau)$ can be identified with the convolution algebra of compactly supported smooth functions $f : G \rightarrow \text{End}_R(V)$ that are left and right K -invariant, in the sense that $f(kgk') = \tau(k)f(g)\tau(k')$. If g is in G , then we denote by $I_g(\tau)$ the space $\text{Hom}_{K \cap gKg^{-1}}(\tau, \tau^g)$, where τ^g is the representation of gKg^{-1} defined by $\tau^g(k) = \tau(g^{-1}kg)$. Then the map $f \mapsto f(g)$ is an isomorphism between the space $H(G, K, \tau)_{KgK}$ of functions in $H(G, K, \tau)$ supported on KgK and $I_g(\tau)$.

In this section, we will primarily be concerned with a certain class of types which are called maximal distinguished cuspidal types in [V2], IV.3.1B. We omit the precise definition of these types here; for our purposes it suffices to know certain specific properties of a maximal distinguished cuspidal R -type (K, τ) , where R is a field.

Such a type arises from a simple stratum $[\mathfrak{A}, n, 0, \beta]$, together with a character θ in the set $\mathcal{C}(\mathfrak{A}, 0, \beta)$ defined in [BK1], 3.2.1. Here \mathfrak{A} is a maximal order in $M_n(F)$, and β is an element of \mathfrak{A} such that $E = F[\beta]$ is a field. This allows us to identify E^\times with a subgroup of $\text{GL}_n(F)$. Let e and f denote the ramification index and residue class degree of E over F . One then has:

- K is the group $J(\beta, \mathfrak{A})$ of [BK1]. In particular, K contains a normal pro- p subgroup K^1 (called $J^1(\beta, \mathfrak{A})$ in [BK1],) such that the quotient K/K^1 is isomorphic to $\text{GL}_{\frac{n}{ef}}(\mathbb{F}_{q^f})$, where q is the order of the residue field of F .
- τ has the form $\kappa \otimes \sigma$, where σ is a the inflation of a cuspidal representation of K/K^1 over R , and κ is a representation of K that is a β -extension of the unique irreducible representation of K^1 containing θ .

Maximal distinguished cuspidal R -types have the following useful properties:

Theorem 4.2 ([V2], IV.1.1-IV.1.3). *Let R be a field, and let (K, τ) be a maximal distinguished cuspidal R -type arising from an extension E/F .*

- (1) *There is a unique embedding of $\text{GL}_{\frac{n}{ef}}(E)$ into G such that the center E^\times of $\text{GL}_{\frac{n}{ef}}(E)$ normalizes K and K^1 , and acts trivially on K/K^1 . We identify $\text{GL}_{\frac{n}{ef}}(E)$ and E^\times with their images under this embedding. The intersection of $\text{GL}_{\frac{n}{ef}}(E)$ with K is $\text{GL}_{\frac{n}{ef}}(\mathcal{O}_E)$.*
- (2) *The subgroup E^\times of G normalizes τ . In particular τ extends to a representation of $E^\times K$, and any two extensions of τ differ by a twist by a character of $E^\times K/K \cong \mathbb{Z}$.*
- (3) *The G -intertwining of (K, τ) is equal to $E^\times K$.*
- (4) *For any extension $\hat{\tau}$ of τ to a representation of $E^\times K$, there is an isomorphism of $H(G, K, \tau)$ with the polynomial ring $R[T, T^{-1}]$, that sends T to the*

element $f_{\hat{\tau}}$ of $H(G, K, \tau)$ supported on $K\varpi_E K$, where ϖ_E is a uniformizer of E , for which $f_{\text{inu}}(k\varpi_E k') = \tau(k)\hat{\tau}(\varpi_E)\tau(k')$.

- (5) For any extension $\hat{\tau}$ of τ to a representation of $E^\times K$, the representation $\text{c-Ind}_{E^\times K}^G \hat{\tau}$ is an irreducible cuspidal representation of G over R .
- (6) Every irreducible cuspidal representation of G over R arises in this fashion. Those irreducible cuspidal π that arise from a given (K, τ) are precisely those π whose restriction to K contains τ .

If R is a field, and π and π' are two irreducible cuspidal R -representations containing a maximal distinguished cuspidal type (K, τ) , then $\text{Hom}_{R[K]}(\tau, \pi)$ and $\text{Hom}_{R[K]}(\tau, \pi')$ are modules over $H(G, K, \tau) = R[T, T^{-1}]$ that are one-dimensional as R -vector spaces. In particular T acts via scalars c and c' on $\text{Hom}_{R[K]}(\tau, \pi)$ and $\text{Hom}_{R[K]}(\tau, \pi')$, respectively. Let χ be an unramified k -valued character of F^\times such that $\chi(\varpi_F)^{\frac{m}{e}} = c'c^{-1}$. As T is supported on $K\varpi_E K$, and $\det \varpi_E = \varpi_F^{\frac{m}{e}}$, the $H(G, K, \tau)$ -modules $\text{Hom}_{R[K]}(\tau, \pi \otimes \chi \circ \det)$ and $\text{Hom}_{R[K]}(\tau, \pi')$ are isomorphic, and so π' is a twist of π by an unramified character; that is, π and π' are inertially equivalent.

Remark 4.3. The isomorphism of $R[T, T^{-1}]$ with $H(G, K, \tau)$ depends on a choice of extension $\hat{\tau}$ of τ , and also a uniformizer ϖ_E . When R is a field, this isomorphism may be reinterpreted in the language Bernstein and Deligne use to describe the Bernstein center, and made independent of ϖ_E (but not of $\hat{\tau}$). Let π be the irreducible cuspidal representation of G whose restriction to $E^\times K$ contains $\hat{\tau}$. Then every representation of G inertially equivalent to π has the form $\pi \otimes \chi$ for some unramified character χ of G/G_0 , and we have $\pi = \pi \otimes \chi$ if, and only if, $\pi \otimes \chi$ contains $\hat{\tau}$. The latter holds precisely when $\hat{\tau} = \hat{\tau} \otimes \chi|_{E^\times K}$, which holds if and only if $\chi(\varpi_E) = 1$. Let Z be the subgroup of G generated by ϖ_E ; our choice of ϖ_E identifies $R[T, T^{-1}]$ with $R[Z]$. The map $Z \rightarrow G/G_0$ is injective (but not in general surjective), and induces a map $\text{Hom}(G/G_0, \mathbb{G}_m) \rightarrow \text{Hom}(Z, \mathbb{G}_m)$ by restriction. Let H be the kernel of this map; the induced map on rings of regular functions then identifies $R[Z]$ with the H -invariants $R[G/G_0]^H$. The identifications:

$$H(G, K, \tau) \cong R[T, T^{-1}] \cong R[Z] \cong R[G/G_0]^H$$

give an identification of $H(G, K, \tau)$ with $R[G/G_0]^H$ that does not depend on ϖ_E . The map $\chi \mapsto \pi \otimes \chi$ describes a bijection between $(\text{Spec } R[G/G_0])/H$ and the set of representations of G inertially equivalent to π . Under the above isomorphisms, the character of $H(G, K, \tau)$ that corresponds to a representation $\pi \otimes \chi$ of G is the character of $R[G/G_0]^H$ obtained by treating $R[G/G_0]$ as the ring of regular functions on the space of unramified characters of G and evaluating such functions at χ .

Let \mathcal{K}' be a finite extension of the field of fractions \mathcal{K} of $W(k)$, and let \mathcal{O} be its ring of integers. If π is an irreducible cuspidal integral representation of G over \mathcal{K}' , then π contains a unique homothety class of G -stable \mathcal{O} -lattices, and the reduction $r_\ell \pi$ of any such lattice modulo ℓ is an irreducible cuspidal representation of G over k . In this situation we have the following compatibilities between the types attached to π and $r_\ell \pi$, due to Vigneras:

Theorem 4.4 ([V2], IV.1.5). *Let π be an irreducible cuspidal representation of G over \mathcal{K}' , containing a maximal distinguished cuspidal \mathcal{K}' -type $(K, \hat{\tau})$.*

- (1) *There is an unramified character $\chi : F^\times \rightarrow (\mathcal{K}')^\times$, such that $\pi \otimes (\chi \circ \det)$ is integral.*
- (2) *The K -representation $\tilde{\tau}$ is defined over \mathcal{K} , and the mod ℓ reduction of $\tilde{\tau}$ is an irreducible k -representation τ of K , such that (K, τ) is a maximal distinguished cuspidal k -type contained in $r_\ell[\pi \otimes (\chi \circ \det)]$.*
- (3) *Every maximal distinguished cuspidal k -type arises from a maximal distinguished cuspidal \mathcal{K}' -type via “reduction mod ℓ ”, for some finite extension \mathcal{K}' of \mathcal{K} .*

We can use this theory of reduction mod ℓ to turn inertial equivalence into an equivalence relation on simple cuspidal smooth $W(k)[G]$ -modules π . Such π fall into two classes: either ℓ annihilates π , in which case π is an irreducible cuspidal k -representation of G , or ℓ is invertible on π , in which case π is an irreducible cuspidal representation of G over some finite extension \mathcal{K}' of \mathcal{K} .

Lemma 4.5. *Let π be a simple smooth $W(k)[G]$ -module on which ℓ is invertible, and let \mathcal{K}' be a finite extension of \mathcal{K} such that every $\mathcal{K}'[G]$ -simple subquotient of $\pi \otimes_{\mathcal{K}} \mathcal{K}'$ is absolutely simple. Then $\pi \otimes_{\mathcal{K}} \mathcal{K}'$ is a direct sum of absolutely simple $\mathcal{K}'[G]$ -modules, and $\text{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ acts transitively on these summands.*

Proof. Let π_0 be an absolutely simple $\mathcal{K}'[G]$ -submodule of $\pi \otimes_{\mathcal{K}} \mathcal{K}'$. Then the sum of the submodules π_0^g for g in $\text{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ is a Galois-stable $\mathcal{K}'[G]$ -submodule of $\pi \otimes_{\mathcal{K}} \mathcal{K}'$, and hence descends to a $\mathcal{K}[G]$ -submodule of π . This submodule must be all of π , and the result follows. \square

Definition 4.6. Let (K, τ) be a maximal distinguished cuspidal k -type, and let π be a simple cuspidal smooth $W(k)[G]$ -module. We say that π *belongs to the mod ℓ inertial equivalence class determined by (K, τ)* if either ℓ annihilates π and π contains (K, τ) , or if ℓ is invertible on π and there exists a finite extension \mathcal{K}' of \mathcal{K} such that one (equivalently, every) absolutely simple summand of $\pi \otimes_{\mathcal{K}} \mathcal{K}'$ is inertially equivalent to an integral representation of G over \mathcal{K}' whose mod ℓ reduction contains (K, τ) .

Fix a maximal distinguished cuspidal k -type (K, τ) , with $\tau = \kappa \otimes \sigma$, and let $\mathcal{P}_\sigma \rightarrow \sigma$ be the projective envelope of σ in the category of $W(k)[GL_{\frac{n}{e_f}}(\mathbb{F}_{q^f})]$ -modules. We then have:

Lemma 4.7. *The representation κ lifts to a representation $\tilde{\kappa}$ of K over $W(k)$.*

Proof. As K_1 is a pro- p -group, the restriction κ_1 of κ to K_1 lifts uniquely to a representation $\tilde{\kappa}_1$ of K_1 over $W(k)$, normalized by K . The obstruction to extending $\tilde{\kappa}_1$ to K is thus an element of $H^2(K, W(k)^\times)$. The first two paragraphs of [BK1], Proposition 5.2.4, show that this element can be represented by a cocycle taking values in the p -power roots of unity, and is thus a p -power torsion element α of $H^2(K, W(k)^\times)$. (In [BK1] the authors work over \mathbb{C} rather than $W(k)$, but their argument adapts without difficulty. Note that they denote by η_M the representation we call $\tilde{\kappa}_1$, by J_M the group we call K , and J_M^1 the group we call K_1 .)

Let U be a p -sylog subgroup of K containing K_1 . The restriction of κ to U lifts uniquely to a representation over $W(k)$, extending $\tilde{\kappa}_1$. It follows that the image of α in $H^2(U, W(k)^\times)$ under restriction vanishes. But the corestriction of this image to $H^2(K, W(k)^\times)$ is equal to $r\alpha$, where r is the index of U in K . Thus α is killed by a power of p and an integer prime to p , and must therefore vanish. \square

Lemma 4.8. *The tensor product $\tilde{\kappa} \otimes \mathcal{P}_\sigma$ is a projective envelope of $\kappa \otimes \sigma$ in the category of $W(k)[K]$ -modules.*

Proof. The restriction of κ to K_1 is irreducible, and the restriction of $\tilde{\kappa} \otimes \mathcal{P}_\sigma$ to K_1 is a direct sum of copies of $\tilde{\kappa}_1$. We thus have isomorphisms of $W(k)[K/K_1]$ -modules:

$$\mathrm{Hom}_{K_1}(\tilde{\kappa}, \tilde{\kappa} \otimes \mathcal{P}_\sigma) \cong \mathcal{P}_\sigma.$$

(Here $g \in K/K_1$ acts on $\mathrm{Hom}_{K_1}(\tilde{\kappa}, \tilde{\kappa} \otimes \mathcal{P}_\sigma)$ by $f \mapsto f^g$, where $f^g(x) = gf(g^{-1}x)$; note that this action depends on $\tilde{\kappa}$, not just its restriction to K_1 .)

Now suppose we have a surjection:

$$\theta' \rightarrow \theta$$

of $W(k)[K]$ -modules. We need to show that any map $\tilde{\kappa} \otimes \mathcal{P}_\sigma \rightarrow \theta$ lifts to a map to θ' . As we have identified \mathcal{P}_σ with $\mathrm{Hom}_{K_1}(\tilde{\kappa}, \tilde{\kappa} \otimes \mathcal{P}_\sigma)$, such a map induces a map $\mathcal{P}_\sigma \rightarrow \mathrm{Hom}_{K_1}(\tilde{\kappa}, \theta)$. This latter map is K/K_1 -equivariant.

As K_1 is a pro- p group, the surjection of $\theta' \rightarrow \theta$ induces a surjection

$$\mathrm{Hom}_{K_1}(\tilde{\kappa}, \theta') \rightarrow \mathrm{Hom}_{K_1}(\tilde{\kappa}, \theta)$$

of $W(k)[K/K_1]$ -modules. As \mathcal{P}_σ is projective, the map

$$\mathcal{P}_\sigma \rightarrow \mathrm{Hom}_{K_1}(\tilde{\kappa}, \theta)$$

lifts to a map

$$\mathcal{P}_\sigma \rightarrow \mathrm{Hom}_{K_1}(\tilde{\kappa}, \theta').$$

Tensoring with $\tilde{\kappa}$, we obtain the desired map $\tilde{\kappa} \otimes \mathcal{P}_\sigma \rightarrow \theta'$, so $\tilde{\kappa} \otimes \mathcal{P}_\sigma$ is projective.

On the other hand, $\tilde{\kappa} \otimes \mathcal{P}_\sigma$ is indecomposable over $W(k)[K]$, and is therefore a projective envelope of $\kappa \otimes \sigma$ in the category of $W(k)[K]$ -modules. \square

As the functor $\mathrm{c}\text{-Ind}_K^G$ is a left adjoint of an exact functor, it takes projectives to projectives. In particular the module $\mathcal{P}_{K,\tau}$ defined by $\mathcal{P}_{K,\tau} := \mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathcal{P}_\sigma$ is a projective object in $\mathrm{Rep}_{W(k)}(G)$.

Proposition 4.9. *Let π be a simple cuspidal smooth $W(k)[G]$ -module in the mod ℓ inertial equivalence class determined by (K, τ) . Then there exists a surjection $\mathcal{P}_{K,\tau} \rightarrow \pi$.*

Proof. First suppose that ℓ annihilates π . The surjection of \mathcal{P}_σ onto σ gives rise to a surjection $\mathcal{P}_{K,\tau} \rightarrow \mathrm{c}\text{-Ind}_K^G \tau$. As the restriction of π to K contains τ , we have a nonzero (thus surjective) map $\mathrm{c}\text{-Ind}_K^G \tau \rightarrow \pi$ as claimed.

On the other hand, if ℓ is invertible in π , then fix an absolutely simple summand π_0 of $\pi \otimes_{\mathcal{K}} \mathcal{K}'$ for some finite extension \mathcal{K}' of \mathcal{K} . As π_0 is in the mod ℓ inertial equivalence class determined by (K, τ) , there is a maximal distinguished cuspidal \mathcal{K}' -type $(K, \tilde{\tau})$ contained in π_0 ; its mod ℓ reduction is (K, τ) . If we regard $\tilde{\tau}$ as a representation of K over the ring of integers \mathcal{O}' of \mathcal{K}' , we have surjections

$$\tilde{\tau} \rightarrow \tau$$

$$\tilde{\kappa} \otimes \mathcal{P}_\sigma \rightarrow \tau,$$

and thus obtain a map $\tilde{\kappa} \otimes \mathcal{P}_\sigma \rightarrow \tilde{\tau}$ by projectivity of $\tilde{\kappa} \otimes \mathcal{P}_\sigma$. This map is necessarily surjective, so by applying the functor $\mathrm{c}\text{-Ind}_K^G$, we obtain a surjection

$$\mathcal{P}_{K,\tau} \otimes_{W(k)} \mathcal{O}' \rightarrow \mathrm{c}\text{-Ind}_K^G \tilde{\tau}.$$

Composing this surjection with the nonzero maps

$$\mathrm{c}\text{-Ind}_K^G \tilde{\tau} \rightarrow \mathrm{c}\text{-Ind}_K^G(\tilde{\tau}) \otimes_{W(k)} \mathcal{K} \rightarrow \pi_0$$

yields a nonzero map $\mathcal{P}_{K,\tau} \otimes_{\mathcal{K}} \mathcal{K}' \rightarrow \pi_0$, and hence a nonzero map

$$\mathcal{P}_{K,\tau} \otimes_{\mathcal{K}} \mathcal{K}' \rightarrow \pi \otimes_{\mathcal{K}} \mathcal{K}'.$$

As $\mathrm{Hom}_{\mathcal{K}'[G]}(\mathcal{P}_{K,\tau} \otimes_{\mathcal{K}} \mathcal{K}', \pi \otimes_{\mathcal{K}} \mathcal{K}')$ is isomorphic to $\mathrm{Hom}_{\mathcal{K}[G]}(\mathcal{P}_{K,\tau}, \pi) \otimes \mathcal{K}'$, there exists a nonzero map from $\mathcal{P}_{K,\tau}$ to π , which must be surjective by simplicity of π . \square

It will follow from results in section 6 that not every simple quotient of $\mathcal{P}_{K,\tau}$ has the above form.

Our next goal is to use the $\mathcal{P}_{K,\tau}$ to construct projectives that admit surjections onto representations with given cuspidal support. We must first introduce some additional language. As a maximal distinguished cuspidal k -type determines an inertial equivalence class of cuspidal representations, we will sometimes say that the supercuspidal or cuspidal support of a representation Π is given by a collection $\{(K_1, \tau_1), \dots, (K_r, \tau_r)\}$ of maximal distinguished cuspidal k -types; this means that Π has supercuspidal (or cuspidal) support (M, π) , where M is a ‘‘block diagonal’’ subgroup of the form $\mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_r}$, and π is a tensor product $\pi_1 \otimes \dots \otimes \pi_r$ where π_i is in the inertial equivalence class determined by (K_i, τ_i) for all i .

Definition 4.10. Let M be a Levi subgroup of G and let π be an irreducible cuspidal representation of M over k . We say that a simple smooth $W(k)[G]$ -module Π has *mod ℓ inertial cuspidal support* equal to (M, π) if either:

- (1) Π is killed by ℓ , and its cuspidal support is inertially equivalent to (M, π) ,
or
- (2) ℓ is invertible on Π , and there exists a finite extension \mathcal{K}' of \mathcal{K} , with ring of integers \mathcal{O}' , such that $\Pi \otimes_{\mathcal{K}} \mathcal{K}'$ is a direct sum of absolutely simple $\mathcal{K}'[G]$ -modules, and for some (equivalently every) absolutely simple summand Π_0 of $\Pi \otimes_{\mathcal{K}} \mathcal{K}'$, there exists a smooth \mathcal{O}' -integral representation $\tilde{\pi}$ of M lifting π , such that the cuspidal support of Π_0 is inertially equivalent to $(M, \tilde{\pi})$.

In this language, Proposition 4.9 says that every simple $W(k)[G]$ -module with mod ℓ cuspidal support given by (K, τ) is a quotient of $\mathcal{P}_{K,\tau}$.

We will also need a notion of mod ℓ inertial supercuspidal support. We first recall a standard result about the behavior of supercuspidal support under reduction mod ℓ :

Proposition 4.11. *Let $\tilde{\Pi}$ be an absolutely irreducible smooth integral representation of G over a finite extension \mathcal{K}' of \mathcal{K} , with supercuspidal support $(M, \tilde{\pi})$. Then $\tilde{\pi}$ is an integral representation of M . Moreover, let Π and π denote the mod ℓ reductions of $\tilde{\Pi}$ and $\tilde{\pi}$, respectively. Then π is irreducible and cuspidal (but not necessarily supercuspidal). Moreover, the supercuspidal support of any simple subquotient of Π is equal to the supercuspidal support of π .*

Definition 4.12. Let M be a Levi subgroup of G and let π be an irreducible supercuspidal representation of M over k . We say that a simple smooth $W(k)[G]$ -module Π has *mod ℓ inertial supercuspidal support* equal to (M, π) if there exists a Levi subgroup M' of G containing M , and an irreducible cuspidal representation π' of M' over k , such that Π has mod ℓ inertial cuspidal support (M', π') , and π' has supercuspidal support (M, π) .

Proposition 4.13. *Let Π be an irreducible smooth integral representation of G over a finite extension \mathcal{K}' of \mathcal{K} . The following are equivalent:*

- (1) Π has mod ℓ inertial supercuspidal support (M, π) .
- (2) Every simple subquotient of the mod ℓ reduction of Π has supercuspidal support inertially equivalent to (M, π) .

Proof. This is immediate from Proposition 4.11. \square

If M is a Levi subgroup of G , and π is an irreducible cuspidal representation of M over k , define $\mathcal{P}_{(M, \pi)}$ to be the normalized parabolic induction

$$\mathcal{P}_{(M, \pi)} := i_P^G[\mathcal{P}_{K_1, \tau_1} \otimes \cdots \otimes \mathcal{P}_{K_r, \tau_r}],$$

where P is a parabolic subgroup whose associate Levi subgroup is M , and the (K_i, τ_i) , are a sequence of maximal distinguished cuspidal k -types whose associated mod ℓ inertial equivalence class is (M, π) .

Remark 4.14. Strictly speaking, $\mathcal{P}_{(M, \pi)}$ may depend on the choice of P ; we suppress this dependence from the notation. In fact, it seems likely that different choices of P give rise to isomorphic modules $\mathcal{P}_{(M, \pi)}$, but we will not need this and do not attempt to prove it.

Lemma 4.15. *Let Π be an absolutely irreducible representation of G , and let π_1, \dots, π_r be a sequence of absolutely irreducible cuspidal representations such that π is a quotient of $i_P^G[\pi_1 \otimes \cdots \otimes \pi_r]$. Let s_i be the permutation of $1, \dots, r$ that interchanges i and $i+1$ and fixes all other integers. Then either Π is a quotient of $i_P^G[\pi_{s_i(1)} \otimes \cdots \otimes \pi_{s_i(r)}]$, or π_i and π_{i+1} are inertially equivalent.*

Proof. The parabolic induction $i_P^G[\pi_1 \otimes \cdots \otimes \pi_r]$ is isomorphic to $i_P^G[\pi_{s_i(1)} \otimes \cdots \otimes \pi_{s_i(r)}]$ unless $\pi_i = (|\ | \circ \det)^{\pm 1} \pi_{i+1}$. If this is the case then π_i is inertially equivalent to π_{i+1} . \square

Proposition 4.16. *let Π be a simple smooth $W(k)[G]$ -module with mod ℓ cuspidal support given by the inertial equivalence class (M, π) . Then Π is a quotient of $\mathcal{P}_{(M, \pi)}$.*

Proof. Either Π is defined and absolutely irreducible over k , or ℓ is invertible on Π and there exists a finite extension \mathcal{K}' of \mathcal{K} such that $\Pi \otimes_{\mathcal{K}} \mathcal{K}'$ is a direct sum of absolutely simple $\mathcal{K}'[G]$ -modules. Choose an absolutely irreducible cuspidal representation $\tilde{\pi}$ of M , defined over the appropriate field (k or \mathcal{K}') such that, over this field, Π admits a nonzero map from $i_P^G \tilde{\pi}$. Write $\pi = \pi_1 \otimes \cdots \otimes \pi_r$, and choose (K_i, τ_i) maximal distinguished cuspidal types such that π_i is in the inertial equivalence class determined by (K_i, τ_i) for all i . Then, for a suitable parabolic subgroup P , we have

$$\mathcal{P}_{M, \pi} = i_P^G[\mathcal{P}_{(K_1, \tau_1)} \otimes \cdots \otimes \mathcal{P}_{(K_r, \tau_r)}].$$

Reorder the $\tilde{\pi}_i$ such that for all i $\tilde{\pi}_i$ is in the mod ℓ inertial equivalence class determined by (K_i, τ_i) . By the previous lemma, we may do this and still assume that there is a nonzero map from $i_P^G \tilde{\pi}$ to Π . Now we have a nonzero map of $\mathcal{P}_{K_i, \tau_i}$ into $\tilde{\pi}_i$ for each i , that is surjective if $\tilde{\pi}$ is defined over k and that becomes surjective after inverting ℓ if $\tilde{\pi}$ is defined over some \mathcal{K}' . Hence, after induction, we obtain a nonzero map of $\mathcal{P}_{(M, \pi)}$ to $i_P^G \tilde{\pi}$ that is surjective if Π is defined over k , and becomes surjective after tensoring with \mathcal{K}' if ℓ is invertible on Π . If Π is defined over k , then composing this surjection with the surjection of $i_P^G \tilde{\pi}$ onto Π yields a nonzero map

of $\mathcal{P}_{(M,\pi)}$ to Π ; this map must be surjective as Π is simple. On the other hand, if ℓ is invertible on Π , we have a nonzero map

$$\mathcal{P}_{(M,\pi)} \otimes_{W(k)} \mathcal{K}' \rightarrow \Pi \otimes_{\mathcal{K}} \mathcal{K}'$$

obtained by composing the surjection of $\mathcal{P}_{(M,\pi)} \otimes \mathcal{K}'$ onto $i_{\bar{P}}^{\bar{G}} \tilde{\pi}$ with the map $i_{\bar{P}}^{\bar{G}} \tilde{\pi} \rightarrow \Pi \otimes_{\mathcal{K}} \mathcal{K}'$. We thus have a nonzero map of $\mathcal{P}_{(M,\pi)}$ onto Π , and the result follows. \square

We will see later that in fact the $W(k)[G]$ -modules $\mathcal{P}_{(M,\pi)}$ are projective, and once we have established this they will form the basic building blocks of our theory.

5. FINITE GROUP THEORY

Our first step in understanding the projectives $\mathcal{P}_{(M,\pi)}$ and $\mathcal{P}_{K,\tau}$ is to understand the projective envelope \mathcal{P}_{σ} of a representation σ of $\mathrm{GL}_n(\mathbb{F}_q)$. This mostly uses standard facts from the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ that we now recall. For conciseness, we let \bar{G} denote the group $\mathrm{GL}_n(\mathbb{F}_q)$, and fix an ℓ prime to q .

If $\bar{P} = \bar{M}\bar{U}$ is a split parabolic subgroup of \bar{G} , with Levi subgroup \bar{M} and unipotent radical \bar{U} , we have parabolic induction and restriction functors:

$$\begin{aligned} i_{\bar{P}}^{\bar{G}} : \mathrm{Rep}_{W(k)}(\bar{M}) &\rightarrow \mathrm{Rep}_{W(k)}(\bar{G}) \\ r_{\bar{G}}^{\bar{P}} : \mathrm{Rep}_{W(k)}(\bar{G}) &\rightarrow \mathrm{Rep}_{W(k)}(\bar{M}). \end{aligned}$$

Here $i_{\bar{P}}^{\bar{G}}$ takes a $W(k)[\bar{M}]$ -module, considers it as a $W(k)[\bar{P}]$ -module by letting \bar{U} act trivially, and induces to \bar{G} . Its adjoint $r_{\bar{G}}^{\bar{P}}$ (which is both a left and right adjoint in the finite group case) takes the \bar{U} -invariants of a $W(k)[\bar{G}]$ -module and considers the resulting space as a $W(k)[\bar{M}]$ -module.

Just as in the representation theory of GL_n over a local field, we can then define:

Definition 5.1. A $W(k)[\bar{G}]$ -module π is *cuspidal* if $r_{\bar{G}}^{\bar{P}}\pi = 0$ for all proper split parabolics \bar{P} of \bar{G} . An irreducible $k[\bar{G}]$ or $\bar{\mathcal{K}}[\bar{G}]$ -module π is *supercuspidal* if it does not arise as a quotient of $i_{\bar{P}}^{\bar{G}}\pi'$ for any proper split parabolic subgroup $\bar{P} = \bar{M}\bar{U}$ of \bar{G} and any representation π' of \bar{M} (over k or $\bar{\mathcal{K}}$, as appropriate).

Over $\bar{\mathcal{K}}$, an irreducible representation π is cuspidal if and only if it is supercuspidal; over k a supercuspidal representation is cuspidal but the converse need not hold. We also define:

Definition 5.2. Let $\bar{P} = \bar{M}\bar{U}$ be a split parabolic subgroup of \bar{G} , and let π' be an irreducible representation of \bar{M} .

- (1) An irreducible $k[\bar{G}]$ or $\bar{\mathcal{K}}[\bar{G}]$ -module π has *cuspidal support* (\bar{M}, π') if π' is cuspidal and π is a quotient of $i_{\bar{P}}^{\bar{G}}\pi'$.
- (2) An irreducible $k[\bar{G}]$ or $\bar{\mathcal{K}}[\bar{G}]$ -module π' has *supercuspidal support* (\bar{M}, π') if π' is supercuspidal and π is a subquotient of $i_{\bar{P}}^{\bar{G}}\pi'$.

The cuspidal and supercuspidal support of an irreducible π always exist, and are unique up to \bar{G} -conjugacy. Over $\bar{\mathcal{K}}$ the two notions coincide, but this is not true over k because of the existence of cuspidal representations that are not supercuspidal.

As the parabolic induction and restriction functors are defined on the level of $W(k)[\bar{G}]$ -modules, it is clear that the reduction mod ℓ of a cuspidal representation is cuspidal. The notion of supercuspidal support is compatible with reduction mod ℓ in the following sense: if $\tilde{\pi}$ is an irreducible representation of \bar{G} over $\bar{\mathcal{K}}$, and π is

any subquotient of its mod ℓ reduction, then the supercuspidal support of π is equal to the supercuspidal support of the mod ℓ reduction of the supercuspidal support of $\tilde{\pi}$.

Deligne-Lusztig theory provides a parameterization of the irreducible cuspidal representations of \overline{G} over $\overline{\mathcal{K}}$ in terms of semisimple elements s of \overline{G} whose characteristic polynomials are irreducible, up to conjugacy. To an arbitrary semisimple element s (up to conjugacy), we associate a subset of the irreducible representations of \overline{G} over $\overline{\mathcal{K}}$ as follows: let \overline{M}_s be the split Levi subgroup of \overline{G} minimal among those split Levi subgroups containing s . Then \overline{M}_s is a product of general linear groups \overline{G}_{n_i} , and the factors s_i of s under this decomposition all have irreducible characteristic polynomials. We call the s_i the “irreducible factors” of s , and refer to an s with only one irreducible factor as “irreducible.” Thus each s_i yields a cuspidal representation of \overline{G}_{n_i} , and hence s yields a cuspidal representation π' of \overline{M}_s . Let $\mathcal{I}(s)$ be the set of irreducible representations of \overline{G} over $\overline{\mathcal{K}}$ with cuspidal support (\overline{M}_s, π') . Note that if s and t are conjugate then the pair (\overline{M}_t, π'') attached to t is conjugate to the pair (\overline{M}_s, π') , so that $\mathcal{I}(s) = \mathcal{I}(t)$. More generally, if \overline{M} is a Levi subgroup of \overline{G} , and s is a semisimple conjugacy class in \overline{M} , we let $\mathcal{I}_{\overline{M}}(s)$ be the set of irreducible representations of \overline{M} over $\overline{\mathcal{K}}$ whose cuspidal support is \overline{M} -conjugate to (\overline{M}_s, π') . We let I_s denote the parabolic induction $i_{\overline{P}}^{\overline{G}}\pi'$; this depends only on the conjugacy class of s , and its irreducible summands are precisely the elements of $\mathcal{I}(s)$.

We now recall the concept of a generic representation of \overline{G} . Let \overline{U} be the unipotent radical of a Borel subgroup of \overline{G} , and let $\Psi : \overline{U} \rightarrow W(k)^\times$ be a generic character. (For instance, if U is the subgroup of upper triangular matrices with 1's on the diagonal, we can fix a nontrivial map ψ of \mathbb{F}_q^+ into $W(k)^\times$ and set $\Psi(u) = \psi(u_{12} + \dots + u_{n-1,n})$.) We say an irreducible representation of \overline{G} over k (resp. $\overline{\mathcal{K}}$) is *generic* if its restriction to \overline{U} contains a copy of $\Psi \otimes_{W(k)} k$ (resp. $\Psi \otimes_{W(k)} \overline{\mathcal{K}}$). By Frobenius reciprocity a representation is generic if and only if it admits a nontrivial map from $\text{c-Ind}_{\overline{U}}^{\overline{G}} \Psi$.

We summarize the relevant facts about generic representations that we will need below:

- (1) If π is an irreducible generic representation over k (resp. $\overline{\mathcal{K}}$) then its restriction to \overline{U} contains exactly one copy of $\Psi \otimes_{W(k)} k$ (resp. $\Psi \otimes_{W(k)} \overline{\mathcal{K}}$.) (Uniqueness of Whittaker models.)
- (2) Every cuspidal representation is generic.
- (3) If \overline{P} is a split parabolic subgroup of \overline{G} , with Levi subgroup \overline{M} , and π is an irreducible generic representation of \overline{M} , then $i_{\overline{P}}^{\overline{G}}\pi$ has a unique irreducible generic subquotient. (In particular there is, up to isomorphism, a unique generic irreducible representation with given supercuspidal support.) On the other hand, if π is irreducible but not generic, then $i_{\overline{P}}^{\overline{G}}\pi$ has no generic subquotient.

In light of these facts, for any semisimple element s of \overline{G} , we let St_s denote the unique irreducible generic representation of \overline{G} over $\overline{\mathcal{K}}$ that lies in $\mathcal{I}(s)$. (Of course, St_s only depends on s up to conjugacy.) Note that St_s is a direct summand of I_s . It will be necessary to understand the behavior of I_s and St_s under parabolic restriction.

Proposition 5.3. *We have a decomposition:*

$$r_{\overline{G}}^{\overline{P}} \text{St}_s \cong \bigoplus_t \text{St}_{\overline{M},t},$$

where t runs over a set of representatives for \overline{M} -conjugacy classes of semisimple elements of \overline{M} that are \overline{G} -conjugate to s , and $\text{St}_{\overline{M},t}$ is the unique irreducible generic representation of \overline{M} over $\overline{\mathcal{K}}$ that lies in $\mathcal{I}_{\overline{M}}(t)$.

Proof. By Frobenius Reciprocity, for any irreducible representation σ of \overline{M} , we have an isomorphism:

$$\text{Hom}_{\overline{M}}(\sigma, r_{\overline{G}}^{\overline{P}} \text{St}_s) = \text{Hom}_{\overline{G}}(i_{\overline{P}}^{\overline{G}} \sigma, \text{St}_s).$$

As St_s is generic and irreducible, the right hand side is zero unless $i_{\overline{P}}^{\overline{G}} \sigma$ has an irreducible generic summand; if this is the case then σ is irreducible and generic, $i_{\overline{P}}^{\overline{G}} \sigma$ has a unique irreducible generic summand, so the right hand side has dimension at most one. In particular, every irreducible summand of $r_{\overline{G}}^{\overline{P}} \text{St}_s$ is generic and occurs with multiplicity one. Moreover, if the right hand side is nonzero, then the cuspidal support of one (hence every) summand of σ is given by s , so the \overline{M} -cuspidal support of σ is given by a conjugacy class t of \overline{M} that is \overline{G} -conjugate to s . \square

We can rewrite this isomorphism as follows: Let \overline{M}_s be the minimal split Levi subgroup of \overline{G} containing s , so that $\text{St}_{\overline{M}_s,s}$ is cuspidal. Fix a maximal torus of \overline{M} ; then conjugating s appropriately we may assume that it is also a maximal torus of \overline{M}_s . Consider the set $W(\overline{M}_s, \overline{M})$ of elements w of $W(G)$ such that $w\overline{M}_s w^{-1}$ lies in \overline{M} . Then $W(\overline{M}_s, \overline{M})$ has a left action by $W(M)$ and a right action by the subgroup $W_{\overline{M}_s}(s)$ of $W(G)$ consisting of those w in $W(G)$ such that $wM_s w^{-1} = M_s$ and $ws w^{-1}$ is M_s -conjugate to s . Moreover, the map $s \mapsto w s w^{-1}$ then yields a bijection between $W(M) \backslash W(\overline{M}_s, \overline{M}) / W_{\overline{M}_s}(s)$ and the set of \overline{M} -conjugacy classes t of elements that are \overline{G} -conjugate to s . We thus obtain a decomposition:

Proposition 5.4. *We have a decomposition:*

$$r_{\overline{G}}^{\overline{P}} \text{St}_s \cong \bigoplus_w \text{St}_{\overline{M}, w s w^{-1}},$$

where w runs over a set of representatives for $W(M) \backslash W(\overline{M}_s, \overline{M}) / W_{\overline{M}_s}(s)$.

We will need to understand the compatibility of this decomposition with a decomposition of $r_{\overline{G}}^{\overline{P}} I_s$. We have:

Proposition 5.5. *There is a direct sum decomposition:*

$$r_{\overline{G}}^{\overline{P}} I_s \cong \bigoplus_w I_{\overline{M}, w s w^{-1}},$$

where w runs over a set of representatives for $W(M) \backslash W(\overline{M}_s, \overline{M})$, and $I_{\overline{M}, w s w^{-1}}$ is the parabolic induction: $i_{w\overline{P}_s w^{-1} \cap \overline{M}}^{\overline{M}} \text{St}_{\overline{M}_s,s}^w$. Moreover, on a summand $\text{St}_{\overline{M}, w' s (w')^{-1}}$ of $r_{\overline{G}}^{\overline{P}} \text{St}_s$, the map $r_{\overline{G}}^{\overline{P}} \text{St}_s \rightarrow r_{\overline{G}}^{\overline{P}} I_s$ induces an injective map

$$\text{St}_{\overline{M}, w' s (w')^{-1}} \rightarrow \bigoplus_w I_{\overline{M}, w s w^{-1}}.$$

Proof. This is a consequence of Mackey's induction-restriction formula, together with Proposition 5.3. \square

We now turn to considerations related to reduction modulo ℓ . Given an irreducible cuspidal representation π of \overline{G} over $\overline{\mathcal{K}}$, its mod ℓ reduction is irreducible and cuspidal. Every cuspidal representation of \overline{G} over k arises by mod ℓ reduction from some such π . If π corresponds to a semisimple conjugacy class s , then the reduction mod ℓ of π is supercuspidal if, and only if, the characteristic polynomial of the ℓ -regular part s^{reg} of s is irreducible. Moreover, if π and π' are irreducible cuspidal representations corresponding to semisimple conjugacy classes s and s' , then the mod ℓ reductions of π and π' coincide if, and only if, $s^{\text{reg}} = (s')^{\text{reg}}$. Thus the supercuspidal representations of \overline{G} over k are parameterized by ℓ -regular semisimple conjugacy classes in \overline{G} with irreducible characteristic polynomial, and the cuspidal representations of \overline{G} over k are parameterized by ℓ -regular semisimple conjugacy classes s' such that there exists a semisimple conjugacy class s , with irreducible characteristic polynomial, such that $s' = s^{\text{reg}}$.

Let π be a cuspidal but not supercuspidal representation of \overline{G} over k , and let s' be the corresponding semisimple element. Such an s' , factors into m identical irreducible factors s'_0 for some m dividing n . Moreover, the supercuspidal support of the π that corresponds to s' is the tensor product of m copies of the supercuspidal representation of $\overline{G}_{\frac{n}{m}}$ corresponding to s'_0 .

Fix an ℓ -regular semisimple element s' of \overline{G} , and let $\mathcal{E}(s')$ be the union of the sets $\mathcal{I}(s)$ for those semisimple s with $s^{\text{reg}} = s'$. Let $e_{s'}$ be the idempotent in $\overline{\mathcal{K}}[\overline{G}]$ that is the sum of the primitive idempotents e_π for all π in $\mathcal{E}(s')$. Then one has:

Theorem 5.6. *The element $e_{s'}$ lies in $W(k)[\overline{G}]$.*

Proof. This is an immediate consequence of [CE], Theorem 9.12. \square

Fix an irreducible cuspidal representation π of \overline{G} over k that is not supercuspidal, corresponding to an ℓ -regular semisimple element s' (up to conjugacy). The representation π arises as the mod ℓ reduction of an irreducible cuspidal representation $\tilde{\pi}$ of \overline{G} over $\overline{\mathcal{K}}$; there is thus a semisimple element s of \overline{G} , with irreducible characteristic polynomial, such that $s' = s^{\text{reg}}$. It is then easy to see that there exists an $m > 1$ dividing n such that, up to conjugacy, s' factors as a block matrix consisting of m irreducible factors s'_0 . Moreover, m lies in the set $\{1, e_q, \ell e_q, \ell^2 e_q, \dots\}$, where e_q is the order of q modulo ℓ . The supercuspidal support of π is the tensor product of m copies of the supercuspidal representation π_0 corresponding to s'_0 .

Proposition 5.7. *Suppose we have an irreducible representation $\tilde{\pi}$ of \overline{G} over $\overline{\mathcal{K}}$ whose mod ℓ reduction contains π as a subquotient. Let s be the semisimple conjugacy class corresponding to the supercuspidal support of $\tilde{\pi}$. Then $s^{\text{reg}} = s'$.*

Proof. The supercuspidal support of $\tilde{\pi}$ is given by the irreducible factors of s . Its mod ℓ reduction has a block factorization into the ℓ -regular parts of the irreducible factors of s ; these are products of irreducible factors of s^{reg} . So the supercuspidal support of the mod ℓ reduction of the supercuspidal support of $\tilde{\pi}$ is given by the irreducible factorization of s^{reg} . On the other hand, this coincides with the supercuspidal support of π , which is given by the irreducible factorization of s' . \square

Fix an irreducible cuspidal representation π of \overline{G} over k , and let s' be a representative of the corresponding ℓ -regular semisimple conjugacy class. Let \mathcal{P}_π be a projective envelope of π .

Proposition 5.8. *The $W(k)[\overline{G}]$ -module $e_{s'} \text{Ind}_{\overline{U}}^{\overline{G}} \Psi$ is a projective envelope of π , and hence is isomorphic to \mathcal{P}_π . In particular $\mathcal{P}_\pi \otimes_{W(k)} \overline{\mathcal{K}}$ is isomorphic to the direct sum*

$$\bigoplus_{s: s^{\text{reg}}=s'} \text{St}_s.$$

Moreover, the endomorphism ring $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$ is a reduced, commutative, free $W(k)$ -module of finite rank.

Proof. The module $e_{s'} \text{Ind}_{\overline{U}}^{\overline{G}} \Psi$ is projective, as induction takes projectives to projectives. Suppose π' is an irreducible representation of \overline{G} over k that admits a nonzero map from $e_{s'} \text{Ind}_{\overline{U}}^{\overline{G}} \Psi$. Then π' is generic and has supercuspidal support given by s' , so $\pi' = \pi$. Thus $e_{s'} \text{Ind}_{\overline{U}}^{\overline{G}} \Psi$ is a projective envelope of π .

As \mathcal{P}_π is free of finite rank over $W(k)$, so is $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$. Thus $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$ embeds into $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi) \otimes_{W(k)} \overline{\mathcal{K}}$. The latter is the endomorphism ring of $e_{s'} \text{Ind}_{\overline{U}}^{\overline{G}}(\Psi \otimes_{W(k)} \overline{\mathcal{K}})$. This module is a direct sum of the generic representations $\tilde{\pi}$ of \overline{G} over $\overline{\mathcal{K}}$ whose mod ℓ reduction contains π , each with multiplicity one. In particular its endomorphism ring is reduced and commutative. \square

Let \overline{P} be a parabolic subgroup of \overline{G} , let \overline{U} be its unipotent radical, and let \overline{M} be the corresponding Levi subgroup. We will need to understand the restriction $r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi$. The following lemma is an immediate consequence of Proposition 5.3:

Lemma 5.9. *We have an isomorphism:*

$$r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi \otimes \overline{\mathcal{K}} = \bigoplus_{s: s^{\text{reg}}=s'} \bigoplus_{t \sim s} \text{St}_{\overline{M}, t}.$$

In particular, the endomorphism ring $\text{End}_{W(k)[\overline{M}]}(r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi)$ is reduced and commutative.

Suppose that each block of \overline{M} has size divisible by $\frac{n}{m}$. Then s' is conjugate to an element of \overline{M} , and this element is unique up to \overline{M} -conjugacy. Then $\text{St}_{\overline{M}, s'}$ is the unique generic representation of \overline{M} determined by this conjugacy class in \overline{M} . Let $\mathcal{P}_{\overline{M}, s'}$ be a projective envelope of $\text{St}_{\overline{M}, s'}$.

Proposition 5.10. *The restriction $r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi$ is zero unless each block of \overline{M} has size divisible by $\frac{n}{m}$. When the latter occurs there is an isomorphism:*

$$r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi \cong \mathcal{P}_{\overline{M}, s'}.$$

Proof. Every Jordan-Hölder constituent of \mathcal{P}_π has supercuspidal support corresponding to a tensor product of m copies of the representation with supercuspidal support s'_0 , and thus $r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi = 0$ unless the condition on the block size of M holds. When this condition does hold, $r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi$ is projective, and hence a direct sum of indecomposable projectives. Moreover, $i_{\overline{P}}^{\overline{G}} \text{St}_{\overline{M}, s'}$ contains a unique generic Jordan-Hölder constituent; this constituent is necessarily isomorphic to π , and thus yields

a map $\mathcal{P}_\pi \rightarrow i_{\overline{P}}^{\overline{G}} \text{St}_{\overline{M}, s'}$. By Frobenius reciprocity we obtain a map:

$$r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi \rightarrow \text{St}_{\overline{M}, s'},$$

and therefore $r_{\overline{G}}^{\overline{P}} \mathcal{P}_\pi$ has a summand isomorphic to $\mathcal{P}_{\overline{M}, s'}$. It is then easy to see that this is the only summand, by tensoring with \overline{K} and applying Lemma 5.9. \square

Corollary 5.11. *The map*

$$e_{s'} Z(W(k)[\overline{G}]) \rightarrow \text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$$

is an isomorphism, and thus identifies $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$ with the center of the block containing π .

Proof. The $W(k)[\overline{G}]$ -module $e_{s'} W(k)[\overline{G}](\mathcal{P}_\pi)$ is a projective module, and is therefore a direct sum (with multiplicities) of projective envelopes of modules in the block containing π . Any such module π' has supercuspidal support corresponding to $(s'_0)^m$, and hence cuspidal support of the form $(\overline{M}, \pi_{\overline{M}, s'})$ for some Levi \overline{M} of G . It is easy to see that $i_{\overline{P}}^{\overline{G}} \mathcal{P}_{\overline{M}, s'}$ is then a projective envelope of π' . Thus $e_{s'} W(k)[\overline{G}](\mathcal{P}_\pi)$ is isomorphic to a direct sum of modules of the form $i_{\overline{P}}^{\overline{G}} \mathcal{P}_{\overline{M}, s'}$ for various \overline{M} .

As $\mathcal{P}_{\overline{M}, s'}$ is isomorphic to a parabolic restriction of \mathcal{P}_π , the endomorphisms of \mathcal{P}_π act naturally on $\mathcal{P}_{\overline{M}, s'}$, and hence on $i_{\overline{P}}^{\overline{G}} \mathcal{P}_{\overline{M}, s'}$. This gives an action of $\text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi)$ on $e_{s'} W(k)[\overline{G}]$, whose image lies in the center of $e_{s'} W(k)[\overline{G}]$.

Conversely, we have an inverse map

$$e_\pi Z(W(k)[\overline{G}]) \rightarrow \text{End}_{W(k)[\overline{G}]}(\mathcal{P}_\pi),$$

and the result follows. \square

We now obtain more precise results about \mathcal{P}_π under the additional hypothesis that $\ell > n$.

Let $\Phi_e(x)$ be the e th cyclotomic polynomial.

Lemma 5.12. *Let r be the order of $q \bmod \ell$, and suppose there exists an e not equal to r such that ℓ divides $\Phi_e(q)$. Then ℓ divides e .*

Proof. As q has exact order $r \bmod \ell$, it is clear that ℓ divides $\Phi_r(q)$, and also that r divides e . The polynomial $x^e - 1$ is divisible by $\Phi_r(x)\Phi_e(x)$; as q is a root of both of these polynomials mod ℓ we see that $x^e - 1$ has a double root mod ℓ , and so ℓ divides e as required. \square

It follows that for $\ell > n$, there exists at most one e dividing n with $\Phi_e(q)$ divisible by ℓ .

As a result, one has:

Lemma 5.13. *Let s' be an ℓ -regular semisimple element of $\text{GL}_n(\mathbb{F}_q)$, whose characteristic polynomial is reducible, and suppose that there exists a semisimple element s of $\text{GL}_n(\mathbb{F}_q)$ with irreducible characteristic polynomial such that $s' = s^{\text{reg}}$. Suppose also that $\ell > n$. Then for any s such that $s' = s^{\text{reg}}$, either $s = s'$ or the characteristic polynomial of s is irreducible.*

Proof. Fix a semisimple element s with irreducible characteristic polynomial such that $s' = s^{\text{reg}}$. Then s is contained in a subgroup of G_n isomorphic to $\mathbb{F}_{q^n}^\times$, and (when considered as an element of $\mathbb{F}_{q^n}^\times$), s generates \mathbb{F}_{q^n} over \mathbb{F}_q . We regard s and s' as elements of \mathbb{F}_{q^n} . Write $s = s' s_\ell$, where s_ℓ is ℓ -power torsion. Then there are integers e_ℓ and e' dividing n such that s' generates $\mathbb{F}_{q^{e'}}$ over \mathbb{F}_q and s_ℓ generates $\mathbb{F}_{q^{e_\ell}}$ over \mathbb{F}_q . The least common multiple of e' and e_ℓ is equal to n . Note that e' is not equal to n , as this would imply that the characteristic polynomial of s' was irreducible.

In particular, e_ℓ cannot be equal to 1, so e_ℓ is a nontrivial root of unity. Thus ℓ divides $\Phi_{e_\ell}(q)$, and hence e_ℓ is the unique $m < n$ such that ℓ divides $\Phi_m(q)$.

Now if s is an arbitrary element with $s^{\text{reg}} = s$, we can write s as $s' s_\ell$ for some ℓ -power root of unity s_ℓ . Either s_ℓ is trivial or s_ℓ lies in $\mathbb{F}_{q^{e_\ell}}$; as we know that the least common multiple of e' and e_ℓ is n the latter case implies that s generates \mathbb{F}_{q^n} over \mathbb{F}_q , as required. \square

In particular, if $\ell > n$, and σ is a representation that is cuspidal but not supercuspidal, corresponding to a semisimple element s' , then every irreducible representation $\tilde{\sigma}$ of G_n over $\overline{\mathcal{K}}$ whose mod ℓ reduction contains σ is either a supercuspidal lift of σ , or has supercuspidal support given by s' . We thus have:

Corollary 5.14. *For $\ell > n$, and π cuspidal but not supercuspidal, we have:*

$$\mathcal{P}_\sigma \otimes \overline{\mathcal{K}} \cong \text{St}_{s'} \oplus \bigoplus_{\tilde{\sigma}} \tilde{\sigma},$$

where $\tilde{\sigma}$ runs over the supercuspidal representations lifting σ .

6. GENERIC PSEUDO-TYPES

The goal of the next two sections will be to apply the finite group theory of section 5 to understand the structure of $\mathcal{P}_{K,\tau}$. Recall that, by definition, we have

$$\mathcal{P}_{K,\tau} = \text{c-Ind}_K^G \tilde{\kappa} \otimes \mathcal{P}_\sigma.$$

The representation σ is inflated from a cuspidal representation of $\text{GL}_{\frac{n}{e_f}}(\mathbb{F}_{q^f})$; such a representation is of the form $\text{St}_{s'}$ for some semisimple ℓ -regular element s' of $\text{GL}_{\frac{n}{e_f}}(\mathbb{F}_{q^f})$. For conciseness we abbreviate $\text{GL}_{\frac{n}{e_f}}(\mathbb{F}_{q^f})$ by \overline{G} for this section.

The decomposition

$$\mathcal{P}_\sigma \otimes \overline{\mathcal{K}} \cong \bigoplus_s \text{St}_s$$

of Proposition 5.8 gives rise to a decomposition:

$$\mathcal{P}_{K,\tau} \otimes \overline{\mathcal{K}} \cong \bigoplus_s \text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s,$$

where s runs over a set of representatives for the \overline{G} -conjugacy classes of semisimple elements s whose ℓ -regular part is conjugate to s' .

Definition 6.1. Let $(K, \kappa \otimes \sigma)$ be a maximal distinguished cuspidal k -type. A *generic pseudo-type* attached to $(K, \kappa \otimes \sigma)$ is a $\overline{\mathcal{K}}$ -type of the form $(K, \tilde{\kappa} \otimes \text{St}_s)$, where St_s is a generic representation of $\text{GL}_{\frac{n}{e_f}}(\mathbb{F}_q^f)$ in the same block as σ .

If St_s is cuspidal, then the generic pseudo-type $(K, \tilde{\kappa} \otimes \text{St}_s)$ is a maximal distinguished $\overline{\mathcal{K}}$ -type, whose mod ℓ inertial cuspidal support is given by $(K, \kappa \otimes \sigma)$. In general a generic pseudo-type is not a type in the usual sense of the word; we will see that in some sense the Hecke algebras attached to generic pseudo-types are analogues of spherical Hecke algebras. (In particular, they are the centers of Hecke algebras attached to types.)

In this section we will prove several basic results about the structure of representations induced from generic pseudo-types, as well as certain natural $W(k)[G]$ -submodules of these representations.

Our approach makes use of the theory of G -covers, which can be found in [BK1] for a field of characteristic zero, and [V2] over a general base ring. We largely follow the presentation of [V2], section II. Let $P = MU$ be a parabolic subgroup of G , with Levi subgroup M , unipotent radical U , and opposite parabolic subgroup $P^\circ = MU^\circ$. Let (K, τ) be an R -type of G , and let K_M, K^+, K^- denote the intersections $K \cap M, K \cap U, K \cap U^\circ$, respectively. Let τ_M be the restriction of τ to K_M .

Definition 6.2. The pair (K, τ) is *decomposed with respect to P* if $K = K^- K_M K^+$, and τ is trivial on K^+ and K^- . If τ is the trivial representation we will sometimes say that K is *decomposed with respect to P* .

Now suppose that (K, τ) is decomposed with respect to P , and let λ be an element of M . Following [V2], II.4, we say that λ is *positive* if $\lambda K^+ \lambda^{-1}$ is contained in K^+ , and $\lambda^{-1} K^- \lambda$ is contained in K^- . We say λ is *negative* if λ^{-1} is positive.

We say that λ is *strictly positive* if the following two conditions hold:

- For any pair of open compact subgroups U_1, U_2 of U , there exists a positive m such that $\lambda^m U_1 \lambda^{-m}$ is contained in U_2 .
- For any pair of open compact subgroups U_1°, U_2° of U° , there exists a positive m such that $\lambda^{-m} U_1^\circ \lambda^m$ is contained in U_2° .

The discussion of [V2], II.3 shows that when (K, τ) is decomposed with respect to P , there is a natural R -linear injection: $T' : H(M, K_M, \tau_M) \rightarrow H(G, K, \tau)$, that takes every element of $H(M, K_M, \tau_M)$ supported on $K_M \lambda K_M$ to an element supported on $K \lambda K$. Moreover, let $H(M, K_M, \tau_M)^+$ be the subalgebra of $H(M, K_M, \tau_M)$ consisting of elements supported on double cosets $K_M \lambda K_M$ with λ positive. Then the restriction T^+ of T' to $H(M, K_M, \tau_M)^+$ is an algebra homomorphism.

We now have the following result:

Proposition 6.3 ([V2], II.6). *Suppose there exists a central, strictly positive element λ of M such that $T^+(1_{K_M \lambda K_M})$ is an invertible element of $H(G, K, \tau)$, where $1_{K_M \lambda K_M}$ is the characteristic function of $K_M \lambda K_M$, considered as an element of $H(M, K_M, \tau_M)$. Then T^+ extends uniquely to an algebra map:*

$$T : H(M, K_M, \tau_M) \rightarrow H(G, K, \tau).$$

If (K, τ) is decomposed with respect to P , and the hypothesis of Proposition 6.3 is satisfied, we say that (K, τ) is a G -cover of (K_M, τ_M) . We then have:

Theorem 6.4 ([V2], II.10.1, II.10.2). *Suppose that (K, τ) is a G -cover of (K_M, τ_M) . Then:*

- (1) For any representation Π of G , we have an isomorphism of $H(M, K_M, \tau_M)$ -modules:

$$\mathrm{Hom}_K(\tau, \Pi) \rightarrow \mathrm{Hom}_{K_M}(\tau_M, r_G^P \Pi),$$

where $H(M, K_M, \tau_M)$ acts on the left hand side via the map T of Proposition 6.3.

- (2) For any representation Π of M , we have an isomorphism of $H(G, K, \tau)$ -modules:

$$\mathrm{Hom}_K(\tau, i_P^G \Pi) \rightarrow \mathrm{Hom}_{K_M}(\tau_M, \Pi) \otimes_{H(M, K_M, \tau_M)} H(G, K, \tau).$$

We now return to the study of the $\overline{K}[G]$ -module $\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathrm{St}_s$. Fix a (split) Levi subgroup \overline{M} of \overline{G} minimal among those split Levi subgroups containing s (such an \overline{M} is uniquely determined by s .) We will make a choice of Levi subgroup M of G depending on \overline{M} , as follows: let V be an n -dimensional F -vector space on which G acts. The distinguished subgroup $\mathrm{GL}_{\frac{n}{e_f}}(E)$ of G coming from the type (K, τ) gives V the structure of an E -vector space. Let L be an \mathcal{O}_E -lattice in V stable under $\mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$; then we have a map: $\mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E) \rightarrow \overline{G}$ coming from the isomorphism: $\overline{G} \cong \mathrm{GL}_{\mathbb{F}_q}(L/\varpi_E L)$. The Levi subgroup \overline{M} of \overline{G} then gives a direct sum decomposition:

$$L/\varpi_E L = \bigoplus_i \overline{L}_i,$$

where the \overline{L}_i are the minimal subspaces of $L/\varpi_E L$ stable under \overline{M} . Choose a lift of this to a direct sum decomposition:

$$L = \bigoplus_i L_i$$

of \mathcal{O}_E -modules, and hence a decomposition:

$$V = \bigoplus_i V_i$$

of E -vector spaces. Let M be the corresponding Levi subgroup of G consisting of matrices that preserve each of the V_i . Then M is a product of linear groups $M_i = \mathrm{Aut}_F(V_i)$. Note that the image of $M \cap \mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ in \overline{G} is precisely \overline{M} .

Let P be the parabolic subgroup of G that preserves the subspaces $V_1, V_1 + V_2$, etc., and let U be its unipotent radical. Let \overline{P} and \overline{U} be the images of $P \cap \mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ and $U \cap \mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ in \overline{G} . Then \overline{P} is a parabolic subgroup of \overline{G} with Levi \overline{M} and unipotent radical \overline{U} .

We now come to the key construction of this section. The pair $(K, \tilde{\kappa} \otimes \mathrm{St}_s)$ is not a type in the traditional sense (unless St_s is cuspidal). We will show, however, that this pair is closely related to a G -cover of a certain cuspidal M -type (K_M, τ_M) . Our construction of the pair (K_M, τ_M) closely parallels sections 7.1 and 7.2 of [BK1], and is more or less the “reverse” of the construction of section 7 of [BK3].

Recall that the maximal distinguished cuspidal type (K, τ) arises from a simple stratum $[\mathfrak{A}, n, 0, \beta]$, together with a character θ in $\mathcal{C}(\mathfrak{A}, 0, \beta)$. Given this data, the group K is the group $J(\beta, \mathfrak{A})$ in [BK1], and the representation $\tilde{\kappa}$ of K is a β -extension of the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ whose restriction to the subgroup $H^1(\beta, \mathfrak{A})$ contains the character θ .

Let \mathfrak{A}_i be order in $\mathrm{End}_F(V_i)$ induced by \mathfrak{A} ; that is, the image of the subring of \mathfrak{A} that preserves V_i in $\mathrm{End}_F(V_i)$. Then conjugation by E^\times stabilizes \mathfrak{A}_i and $\mathfrak{A}_i \cap \mathrm{End}_E(V_i) = \mathrm{End}_{\mathcal{O}_E}(L_i)$. Given these orders, the procedure at the beginning of [BK3], 7.2 constructs an order \mathfrak{A}' in $\mathrm{End}_F(V)$ (this is the order denoted by \mathfrak{A}

in section 7 of [BK3].) The order \mathfrak{A}' is contained in the maximal order \mathfrak{A} . Set $K' = J(\beta, \mathfrak{A}')$.

Let $K_{\overline{P}}$ be the preimage of \overline{P} in K , via our identification of K/K_1 with \overline{G} . Then by Theorem 5.2.3 (ii) of [BK1], there is a unique $W(k)[K']$ -module $\tilde{\kappa}'$ of K' such that we have:

$$\mathrm{Ind}_{K_{\overline{P}}}^{(\mathfrak{A}')^\times} \tilde{\kappa}|_{K_{\overline{P}}} \cong \mathrm{Ind}_{K'}^{(\mathfrak{A}')^\times} \tilde{\kappa}'.$$

(Strictly speaking, this is proved in [BK1] over an algebraically closed field of characteristic zero, rather than over $W(k)$; in the proof of [V1], III.4.21 Vigneras observes that the same result holds for the $W(k)$ -representations we use here.) Over \mathcal{K} , the representation $\tilde{\kappa}'$ can alternatively be described, up to twist, in the following way: the character θ gives rise to an endo-class of ps-characters $(\Theta, 0, \beta)$ in the sense of [BK3], section 4. Then $(\Theta, 0, \beta)$ in particular give rise to a character θ' in $\mathcal{C}(\mathfrak{A}', 0, \beta)$. From this perspective $\tilde{\kappa}'$ is a β -extension of the unique irreducible representation of $J^1(\beta, \mathfrak{A}')$ whose restriction to $H^1(\beta, \mathfrak{A}')$ contains the character θ' .

We now apply the construction of [BK1], 7.2 to the representation $\tilde{\kappa}'$. That is, let K'' be the subset $(J(\beta, \mathfrak{A}') \cap P)H^1(\beta, \mathfrak{A}')$; it is shown in [BK1], 7.1 and 7.2 that K'' is a group, and that the $K' \cap U$ -fixed vectors in $\tilde{\kappa}'$ are stable under K'' . Thus these fixed vectors give a representation $\tilde{\kappa}''$ of K'' . Let K_M be the intersection $K'' \cap M$, and let $\tilde{\kappa}_M$ be the restriction of $\tilde{\kappa}''$ to K_M . We then have the following:

Lemma 6.5. *The pair $(K'', \tilde{\kappa}'')$ satisfies the conditions (7.2.1) of [BK3]. Explicitly:*

- (1) *The restriction of $\tilde{\kappa}''$ to $H^1(\beta, \mathfrak{A}')$ is a multiple of θ' .*
- (2) *The representation $\tilde{\kappa}''$ is trivial on $K'' \cap U$ and $K'' \cap U^\circ$.*
- (3) *The group K_M is the product of the groups $K_i = K \cap M_i$, and the representation $\tilde{\kappa}_M$ is a tensor product of irreducible representations $\tilde{\kappa}_i$ of K_i for each i . Moreover, the subgroup K_i of M_i is equal to $J(\beta, \mathfrak{A}_i)$, and each $\tilde{\kappa}_i$ is a β -extension of the unique representation of $J^1(\beta, \mathfrak{A}_i)$ whose restriction to $H^1(\beta, \mathfrak{A}_i)$ contains the element θ_i of $\mathcal{C}(\mathfrak{A}_i, 0, \beta)$ determined by the ps-character $(\Theta, 0, \beta)$.*

Moreover, the map:

$$\mathrm{Ind}_{K''}^{K'} \tilde{\kappa}'' \rightarrow \tilde{\kappa}'$$

(obtained by Frobenius reciprocity from the realization of $\tilde{\kappa}''$ as the $U \cap K''$ -invariants of $\tilde{\kappa}'$) is an isomorphism.

Proof. The first claim follows from [BK1], 5.1.1 and our description of $\tilde{\kappa}''$ as a β -extension. The second is clear from the construction of $\tilde{\kappa}''$. The decomposition of $\tilde{\kappa}$ as a tensor product of $\tilde{\kappa}_i$ is [BK1], 7.2.14, as is the fact that each $\tilde{\kappa}_i$ is a β -extension (the necessary intertwining property on the $\tilde{\kappa}_i$ is verified as part of [BK1], 7.2.15.) The fact that the characters θ_i are the ones claimed in the theorem follows from [BK1], 7.1.19. The isomorphism is [BK1], 7.2.15. \square

Let s_i be the projection of s to \overline{M}_i ; then s_i is a semisimple element of \overline{M}_i with irreducible characteristic polynomial. As K_i contains $J^1(\beta, \mathfrak{A}_i)$ as a normal subgroup, and the quotient is naturally isomorphic to \overline{M}_i , we may regard the cuspidal representation St_{s_i} of \overline{M}_i as a representation of K_i . We set $\tau_i = \tilde{\kappa}_i \otimes \mathrm{St}_{s_i}$; the pair (K_i, τ_i) is then a maximal distinguished cuspidal type in M_i . Let τ_M be the tensor product of the τ_i ; it is then a representation of K_M , and we have $\tau_M = \tilde{\kappa}_M \otimes \mathrm{St}_{\overline{M}, s}$, where $\mathrm{St}_{\overline{M}, s}$ is the tensor product of the cuspidal representations St_{s_i} . The pair (K_M, τ_M) is a maximal distinguished cuspidal M -type.

On the other hand, by construction, the quotient of K'' by $J^1(\beta, \mathfrak{A}')$ is naturally isomorphic to \overline{M} . We can thus regard $\text{St}_{\overline{M},s}$ as a representation of K'' , and form the representation $\tau'' = \tilde{\kappa}'' \otimes \text{St}_{\overline{M},s}$. Consider also the representation $\tau_{\overline{P}} = \tilde{\kappa}|_{K_{\overline{P}}} \otimes \text{St}_{\overline{M},s}$, where we regard $\text{St}_{\overline{M},s}$ as a representation of $K_{\overline{P}}$ via the surjection

$$K_{\overline{P}} \rightarrow \overline{P} \rightarrow \overline{M}.$$

We then have:

Theorem 6.6. *The pair (K'', τ'') is a G -cover of (K_M, τ_M) . Moreover, there are natural isomorphisms:*

$$\begin{aligned} \text{c-Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'' &\cong \text{c-Ind}_{K_{\overline{P}}}^{(\mathfrak{A}')^\times} \tau_{\overline{P}} \\ \text{c-Ind}_{K''}^{\mathfrak{A}^\times} \tau'' &\cong \text{c-Ind}_K^{\mathfrak{A}^\times} \tilde{\kappa} \otimes I_s. \end{aligned}$$

where we regard I_s as a representation of K via the surjection of K onto \overline{G} .

Proof. We have verified that $(K'', \tilde{\kappa}'')$ satisfies the list of properties in [BK3], 7.2.1. In particular, if one applies the procedure of [BK3], section 7.2 to the type (K_M, τ_M) , one arrives at the representation (K'', τ'') . Thus (K'', τ'') is a G -cover of (K_M, τ_M) by Theorem 7.2 of [BK3]. (Notice that the procedure given there in particular applies to the type (K_M, τ_M) because we have verified that each of the types (K_i, τ_i) arises from the same endo-class $(\Theta, 0, \beta)$ of ps-character.)

Whenever we have H' a subgroup of H , and representations A of H and A' of H' , we have a general identity:

$$\text{Ind}_{H'}^H A|_{H'} \otimes B \cong A \otimes \text{Ind}_H^{H'} B.$$

It follows from this and the isomorphism:

$$\tilde{\kappa}' \cong \text{Ind}_{K''}^{K'} \tilde{\kappa}'$$

that we have an isomorphism:

$$\text{c-Ind}_{K''}^{\mathfrak{A}^\times} \tau'' \cong \text{c-Ind}_{K'}^{\mathfrak{A}^\times} \tau',$$

where $\tau' = \tilde{\kappa}' \otimes \text{St}_{\overline{M},s}$. (Recall that the surjections of K' and K'' onto \overline{M} are compatible with the inclusion of K'' in K' , so we can regard $\text{St}_{\overline{M},s}$ as a representation of K' here.)

Next, the isomorphism:

$$\text{Ind}_{K'}^{(\mathfrak{A}')^\times} \kappa' \cong \text{Ind}_{K_{\overline{P}}}^{(\mathfrak{A}')^\times} \kappa|_{K_{\overline{P}}}$$

induces an isomorphism:

$$\text{Ind}_{K'}^{(\mathfrak{A}')^\times} \tau' \cong \text{Ind}_{K_{\overline{P}}}^{(\mathfrak{A}')^\times} \tau_{\overline{P}}$$

Finally, we have an isomorphism:

$$\text{Ind}_{K_{\overline{P}}}^{\mathfrak{A}^\times} \kappa|_{K_{\overline{P}}} \otimes \text{St}_{\overline{M},s} \cong \text{Ind}_K^{\mathfrak{A}^\times} \kappa \otimes I_s$$

obtained by inducing the previous isomorphism from $(\mathfrak{A}')^\times$ to \mathfrak{A}^\times and applying the tensor product identity. \square

The maximal distinguished cuspidal type (K_M, τ_M) gives rise to a unique inertial equivalence class of cuspidal representations of M ; let π be an irreducible representation of M over $\overline{\mathcal{K}}$ that lies in this inertial equivalence class (or equivalently, that contains the type (K_M, τ_M) .) We then have:

Corollary 6.7. *The $\overline{\mathcal{K}}[G]$ -modules $\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes I_s$ and $\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathrm{St}_s$ are objects of $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)_{(M,\pi)}$.*

Proof. Theorem 6.6 implies that we have an isomorphism:

$$\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes I_s \cong \mathrm{c}\text{-Ind}_{K''}^G \tau''.$$

As (K'', τ'') is a G -cover of the maximal distinguished cuspidal type (K_M, τ_M) , it follows immediately from [BK2], Theorem 8.3 that $\mathrm{c}\text{-Ind}_{K''}^G \tau''$ lies in $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)_{(M,\pi)}$. As for $\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathrm{St}_s$, note that, by definition, St_s is the (unique) generic summand of I_s , and so this $\overline{\mathcal{K}}[G]$ -module is a direct summand of the first. \square

We now turn to the question of understanding the Hecke algebra $H(G, K, \tilde{\kappa} \otimes \mathrm{St}_s)$, or equivalently the endomorphism ring $\mathrm{End}_{\overline{\mathcal{K}}[G]}(\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathrm{St}_s)$. The isomorphisms of Theorem 6.6 induce isomorphisms:

$$\begin{aligned} H(G, K'', \tau'') &\cong H(G, (\mathfrak{A}')^\times, \mathrm{Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'') \\ H(G, K_{\overline{P}}, \tau_{\overline{P}}) &\cong H(G, (\mathfrak{A}')^\times, \mathrm{Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'') \end{aligned}$$

These isomorphisms are compatible with support in the sense that an element of one of the left hand Hecke algebras supported on a double coset $K''gK''$ or $K_{\overline{P}}gK_{\overline{P}}$ gets set to an element of the right hand Hecke algebra supported on $(\mathfrak{A}')^\times g (\mathfrak{A}')^\times$.

We would like to use this observation to compare the spaces $H(G, K'', \tau'')_{K''gK''}$ and $H(G, K_{\overline{P}}, \tau_{\overline{P}})_{K_{\overline{P}}gK_{\overline{P}}}$ for various g in G .

On the one hand, the space $H(G, K'', \tau'')$ is well-understood; the discussion in section 1 of [BK3] shows that it is a tensor product of affine Hecke algebras. More precisely, recall that M is the subgroup of G consisting of endomorphisms of V that preserve each summand V_i of V , and let Z be the subgroup of M consisting of elements that act by a power of ϖ_E on each V_i .

Choose a maximal torus T_E of $\mathrm{GL}_{\frac{n}{e_f}}(E)$, and let \overline{T} be its reduction mod ϖ_E . Then \overline{T} is a maximal torus of \overline{G} ; we assume it is contained in \overline{M} . Let $W(\overline{G})$ be the Weyl group of \overline{G} with respect to \overline{T} . The choices of T_E and \overline{T} give an isomorphism of $W(\overline{G})$ with the Weyl group of $\mathrm{GL}_{\frac{n}{e_f}}(E)$.

Now choose a maximal torus T_F of G . We will say that T_F is *compatible with T_E* if every T_E stable line in F^n is a union of T_F -stable lines. A choice of T_F compatible with T_E identifies the Weyl group of $\mathrm{GL}_{\frac{n}{e_f}}(E)$ with a subgroup of $W(G)$, and thus lets us consider $W(\overline{G})$ as a subgroup of $W(G)$. In what follows, whenever we have a trio of groups $G, \mathrm{GL}_{\frac{n}{e_f}}(E), \overline{G}$, we will choose maximal tori of these groups related in the sense described above, and implicitly make the corresponding identifications on Weyl groups.

Let $W_{\overline{M}}$ be the subgroup of $W(\overline{G})$ normalizing \overline{M} , and let $W_{\overline{M}}(s)$ be the subgroup of $W_{\overline{M}}$ consisting of w such that $ws w^{-1}$ is M -conjugate to s . If W_M is the subgroup of $W(G)$ normalizing M , we can identify $W_{\overline{M}}$ with a subgroup of W_M . Then $H(G, K'', \tau'')$ is supported on the double cosets $K''gK''$ for g in $W_{\overline{M}}(s)Z$, and each $H(G, K'', \tau'')_{K''gK''}$ is a one-dimensional $\overline{\mathcal{K}}$ -vector space. (Observe that if w, w' are in $W_{\overline{M}}(s)$, and z, z' lie in Z , then $K''wzK'' = K''w'z'K''$ if, and only if, $z = z'$ and $w^{-1}w'$ lies in $W(M)$.)

Moreover, if we write the characteristic polynomial of s as a product of irreducible polynomials $f_1^{m_1} \dots f_r^{m_r}$, with $\deg f_j = d_j$, then the quotient of $W_{\overline{M}}(s)$ by the subgroup $W(M)$ of $W_{\overline{M}}(s)$ is a product of permutation groups $W_j \cong S_{m_j}$, where

W_j permutes the “blocks” of s with characteristic polynomial f_j . If we let Z_j be the subgroup of Z consisting of elements that are the identity away from those blocks, then Z_j is a subgroup of Z invariant under the conjugation action of $W_{\overline{M}}(s)$, and the conjugation action of $W_{\overline{M}}(s)$ on Z_j factors through S_{m_j} . Moreover, $W_j Z_j$ is a subgroup of GL_n , and the subspace H_j of $H(G, K'', \tau'')$ supported on cosets of the form KgK for g in $W_j Z_j$ is a subalgebra of $H(G, K'', \tau'')$ isomorphic to the affine Hecke algebra $H(q^{f_j d_j}, m_j)$. (This isomorphism depends on certain choices and is therefore not canonical; we refer the reader to [BK1], 5.6, for its construction.)

The algebra $H(G, K'', \tau'')$ is then the tensor product of the H_j . Moreover, the map from $H(q^{f_j d_j}, m_j)$ to H_j is compatible with supports in a certain sense. Specifically, we have a natural isomorphism of $H(q^{f_j d_j}, m_j)$ with $H(\mathrm{GL}_{m_j}(E'), I)$, where E' is the unramified extension of E of degree d_j , and I is the standard Iwahori subgroup of $\mathrm{GL}_{m_j}(E')$. We may embed $\mathrm{GL}_{m_j}(E')$ in M_j in such a way that the image of $\mathrm{GL}_{m_j}(\mathcal{O}_{E'})$ is equal to the intersection of \mathfrak{A} with the image of $\mathrm{GL}_{m_j}(E')$, and so that the maximal tori of M_j and $M_j \cap \mathrm{GL}_{\frac{n}{e_j}}(E)$ arising from T_F and T_E are compatible with the standard maximal torus of $\mathrm{GL}_{m_j}(E')$. Then the reduction mod ϖ_E of $\mathrm{GL}_{m_j}(\mathcal{O}_{E'})$ is a subgroup of \overline{M}_j isomorphic to $\mathrm{GL}_{m_j}(\mathbb{F}_{q^{d_j}})$; we assume we have chosen our embedding so that the standard maximal torus of $\mathrm{GL}_{m_j}(\mathbb{F}_{q^{d_j}})$ is contained in the Levi M_{s_j} . (This makes M_{s_j} the minimal split Levi containing the standard maximal torus of $\mathrm{GL}_{m_j}(\mathbb{F}_{q^{d_j}})$.)

This embedding allows us to identify W_j with the (standard) Weyl group W'_j of $\mathrm{GL}_{m_j}(E')$. Our choices identify Z_j with a subgroup Z'_j of the diagonal matrices in $\mathrm{GL}_{m_j}(E')$, and then $\mathrm{GL}_{m_j}(E')$ is a union of double cosets $Iw'z'I$, with w' in W'_j and z' in Z'_j . The identification of $H(q^{f_j d_j}, m_j)$ with H_j then takes the subspace $H(\mathrm{GL}_{m_j}(E'), I)_{Iw'z'I}$ to $H(G, K'', \tau'')_{KwzK}$, where w and z are the elements of W'_j and Z'_j corresponding to w and z .

For each j , the Hecke algebra $H(q^{f_j d_j}, m_j)$ contains a subalgebra isomorphic to $\overline{\mathcal{K}}[Z'_j]$, via the construction of section 3 of [Lu]. By [Lu], 3.11, this isomorphism identifies $\overline{\mathcal{K}}[Z'_j]^{W'_j}$ with the center of $H(q^{f_j d_j}, m_j)$. Taking the tensor product over all j , and composing with the isomorphisms of $H(q^{f_j d_j}, m_j)$ with H_j gives an isomorphism of $\overline{\mathcal{K}}[Z]^{W_{\overline{M}}(s)}$ with the center of $H(G, K'', \tau'')$.

As (K'', τ'') is a G -cover of (K_M, τ_M) , we have a map

$$T : H(M, K_M, \tau_M) \rightarrow H(G, K'', \tau'')$$

that is not (in general) support preserving, but that fits nicely into the above picture. In particular, as (K_M, τ_M) is a maximal distinguished cuspidal M -type, our choice of irreducible representation π of M containing τ_M gives rise to an isomorphism:

$$(\overline{\mathcal{K}}[M/M_0])^H \cong H(M, K_M, \tau_M).$$

Here, as in Remark 4.3, we view $\overline{\mathcal{K}}[M/M_0]$ as the ring of regular functions on the torus $\mathrm{Hom}(M/M_0, \overline{\mathcal{K}})$, and H is the subgroup of this torus consisting of characters χ of M/M_0 such that $\pi \otimes \chi$ is isomorphic to π . The action of an element f of $H(M, K_M, \tau_M)$ on an irreducible representation π' inertially equivalent to π is then given by choosing a character χ such that $\pi' = \pi \otimes \chi$, and evaluating the element of $\overline{\mathcal{K}}[M/M_0]^H$ corresponding to f at χ . The inclusion of Z in M defines an isomorphism of Z onto $(M/M_0)^H$, so we have an isomorphism of $H(M, K_M, \tau_M)$ with $\overline{\mathcal{K}}[Z]$.

Conjugating by elements of $W_{\overline{M}}(s)$ permutes the space of representations of M inertially equivalent to π , and thus defines an action of $W_{\overline{M}}(s)$ on $\overline{\mathcal{K}}[Z]$. This action depends on the choice of π ; we may choose π so that π is invariant under conjugation by $W_{\overline{M}}(s)$. For such a π , the action of $W_{\overline{M}}(s)$ on $\overline{\mathcal{K}}[Z]$ is simply by permuting the factors Z_i .

A choice of such a π yields an embedding of $\overline{\mathcal{K}}[Z]$ into $H(G, K'', \tau'')$. Moreover, any such π determines, for all j , a unique support-preserving isomorphism $H(q^{fd_j}, m_j) \cong H_j$ of the type described above, such that the diagram:

$$\begin{array}{ccc} \overline{\mathcal{K}}[Z] & = & \overline{\mathcal{K}}[Z] \\ \downarrow & & \downarrow \\ \otimes_j H(q^{fd_j}, m_j) & \rightarrow & H(G, K'', \tau'') \end{array}$$

commutes. (Conversely, for any such collection of isomorphisms there is a corresponding $W_{\overline{M}}(s)$ -invariant π .)

Our next step is to translate this detailed structure theory for $H(G, K'', \tau'')$ into a corresponding theory for $H(G, K_{\overline{P}}, \tau_{\overline{P}})$, via the isomorphisms of Theorem 6.6. To do so we must study the intertwining of $\tau_{\overline{P}}$ and of τ .

Lemma 6.8. *Let \tilde{P} be the preimage of \overline{P} under the map*

$$\mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E) \rightarrow \mathrm{GL}_{\frac{n}{e_f}}(\mathbb{F}_{q^f}).$$

- (1) *Let ξ be a $W(k)$ -representation of K trivial on K_1 , and let g be an element of G that intertwines $\tilde{\kappa} \otimes \xi$. Then g lies in $K \mathrm{GL}_{\frac{n}{e_f}}(E)K$. Moreover, if g lies in $\mathrm{GL}_{\frac{n}{e_f}}(E)$, then g intertwines ξ when ξ is considered as a representation of $\mathrm{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ inflated from $\mathrm{GL}_{\frac{n}{e_f}}(\mathbb{F}_{q^f})$, and there is a natural isomorphism:*

$$I_g(\tilde{\kappa} \otimes \xi) \cong I_g(\tilde{\kappa}) \otimes I_g(\xi).$$

- (2) *Let ξ be a $W(k)$ -representation of $K_{\overline{P}}$ trivial on K_1 , and let g be an element of G that intertwines $\tilde{\kappa}|_{K_{\overline{P}}} \otimes \xi$. Then g lies in $K_{\overline{P}} \mathrm{GL}_{\frac{n}{e_f}}(E)K_{\overline{P}}$. Moreover, if g lies in $\mathrm{GL}_{\frac{n}{e_f}}(E)$, and then g intertwines ξ when ξ is considered as a representation of \tilde{P} inflated from \overline{P} , and there is a natural isomorphism:*

$$I_g(\tilde{\kappa}|_{K_{\overline{P}}} \otimes \xi) \cong I_g(\tilde{\kappa}) \otimes I_g(\xi).$$

Proof. Case (1) is almost precisely [BK1], Proposition 5.3.2, except that the coefficient space here is $W(k)$ rather than \mathbb{C} . In spite of this the argument of [BK1] adapts without difficulty, and the argument in case (2) is identical. \square

Corollary 6.9. *Let g be an element of G that intertwines $\tau_{\overline{P}}$. Then g lies in the double coset $K_{\overline{P}} \mathrm{GL}_n(E)K_{\overline{P}}$. Moreover, if g lies in $K_{\overline{P}} W_{\overline{M}}(s) Z K_{\overline{P}}$, then $I_g(\tau_{\overline{P}})$ is one-dimensional.*

Proof. The previous lemma shows that g lies in $K_{\overline{P}} \mathrm{GL}_{\frac{n}{e_f}}(E)K_{\overline{P}}$, so the first statement is clear. For the second statement, it suffices to consider g in $W_{\overline{M}}(s)Z$. For such g , $I_g(\tau_{\overline{P}})$ is equal to $I_g(\mathrm{St}_{\overline{M},s})$, where $\mathrm{St}_{\overline{M},s}$ is considered as a representation of the parahoric subgroup \tilde{P} of $\mathrm{GL}_{\frac{n}{e_f}}(E)$ inflated from \overline{M} . We thus have

$$I_g(\mathrm{St}_{\overline{M},s}) = \mathrm{Hom}_{\tilde{P} \cap g \tilde{P} g^{-1}}(\mathrm{St}_{\overline{M},s}, \mathrm{St}_{\overline{M},s}^g).$$

It is easy to see that for g in $W_{\overline{M}}(s)Z$, we have a surjection

$$\tilde{P} \cap g \tilde{P} g^{-1} \rightarrow \overline{M}$$

induced by the surjection of \tilde{P} onto \overline{M} , and that the representations $\text{St}_{\overline{M},s}$ and $\text{St}_{\overline{M},s}^g$, when considered as representations of $\tilde{P} \cap g\tilde{P}g^{-1}$, are both inflated from $\text{St}_{\overline{M}}$ via this surjection. The result follows. \square

As a result we obtain:

Proposition 6.10. *The isomorphism:*

$$H(G, K'', \tau'') \cong H(G, K_{\overline{P}}, \tau_{\overline{P}})$$

of Theorem 6.6 is support-preserving, in the sense that for all g in $\text{GL}_{\frac{n}{ef}}(E)$, it induces an isomorphism:

$$H(G, K'', \tau'')_{K''gK''} \cong H(G, K_{\overline{P}}, \tau_{\overline{P}})_{K_{\overline{P}}gK_{\overline{P}}}.$$

Proof. For any g in G , we have an isomorphism:

$$\bigoplus_{g'} H(G, K'', \tau'')_{K''g'K''} \cong H(G, (\mathfrak{A}')^\times, \text{c-Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'')_{(\mathfrak{A}')^\times g (\mathfrak{A}')^\times}$$

where g' lies in a set of representatives for the double cosets $K''g'K''$ in $(\mathfrak{A}')^\times g (\mathfrak{A}')^\times$. An easy calculation shows that if g and g' lie in $W_{\overline{M}}(s)Z$, and

$$(\mathfrak{A}')^\times g (\mathfrak{A}')^\times = (\mathfrak{A}')^\times g' (\mathfrak{A}')^\times,$$

then $K''gK'' = K''g'K''$ and $K_{\overline{P}}gK_{\overline{P}} = K_{\overline{P}}g'K_{\overline{P}}$. As $H(G, K'', \tau'')$ is supported on $K''W_{\overline{M}}(s)ZK''$, it follows that $H(G, (\mathfrak{A}')^\times, \text{c-Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'')$ is supported on the double cosets $(\mathfrak{A}')^\times W_{\overline{M}}(s)Z(\mathfrak{A}')^\times$. Moreover, each $H(G, K'', \tau'')_{K''gK''}$ is one-dimensional for g in $W_{\overline{M}}(s)Z$. It follows that $H(G, (\mathfrak{A}')^\times, \text{c-Ind}_{K''}^{(\mathfrak{A}')^\times} \tau'')_{(\mathfrak{A}')^\times g (\mathfrak{A}')^\times}$ is as well.

On the other hand, for g in G we also have an isomorphism

$$\bigoplus_{g'} H(G, K_{\overline{P}}, \tau_{\overline{P}})_{K_{\overline{P}}g'K_{\overline{P}}} \cong H(G, (\mathfrak{A}')^\times, \text{c-Ind}_{K_{\overline{P}}}^{(\mathfrak{A}')^\times} \tau_{\overline{P}})_{(\mathfrak{A}')^\times g (\mathfrak{A}')^\times},$$

where g' runs over a set of representatives for the cosets $K_{\overline{P}}g'K_{\overline{P}}$ in $(\mathfrak{A}')^\times g (\mathfrak{A}')^\times$. For g in $W_{\overline{M}}(s)Z$, the right-hand side is one-dimensional, and the summand on the left corresponding to $g' = g$ is also one-dimensional. On the other hand, if g doesn't lie in $(\mathfrak{A}')^\times W_{\overline{M}}(s)Z(\mathfrak{A}')^\times$ then the right hand side is zero, so all of the summands on the left vanish as well. It follows that $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ is supported on $K_{\overline{P}}W_{\overline{M}}(s)ZK_{\overline{P}}$, and that for g in $W_{\overline{M}}(s)Z$, the isomorphisms above identify $H(G, K_{\overline{P}}, \tau_{\overline{P}})_{K_{\overline{P}}gK_{\overline{P}}}$ with $H(G, K'', \tau'')_{K''gK''}$ as required. \square

Finally, we want to understand the relationship between $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ and $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$. As St_s is a direct summand of I_s , $\text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$ is a direct summand of $\text{c-Ind}_K^G \tilde{\kappa} \otimes I_s$. In particular any central endomorphism of the latter commutes with the projection onto $\text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$, and thus induces an element of $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$. On the other hand, such central endomorphisms are central elements of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$, and hence of $H(G, K'', \tau'')$ via the isomorphisms of Theorem 6.6. Our description of the center of $H(G, K'', \tau'')$ as $\overline{\mathcal{K}}[Z]^{W_{\overline{M}}(s)}$ thus yields a support-preserving map:

$$\overline{\mathcal{K}}[Z]^{W_{\overline{M}}(s)} \rightarrow H(G, K, \tilde{\kappa} \otimes \text{St}_s).$$

(Indeed, we know that $\overline{\mathcal{K}}[Z]^{W_{\overline{M}}(s)}$ is the center of the category $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M,\pi}$, so this map is simply the map that gives the action of the Bernstein center on the space

c- $\text{Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$.) The support and intertwining calculations we have done, together with analogous calculations for $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$, will allow us to show that this map is an isomorphism. Injectivity is straightforward:

Proposition 6.11. *The map $Z(H(G, K, \tilde{\kappa} \otimes I_s)) \rightarrow H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ is injective.*

Proof. As $\text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$ is a direct summand of $\text{c-Ind}_K^G \tilde{\kappa} \otimes I_s$, any endomorphism of the former extends by zero to an endomorphism of the latter. This allows us to view $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ as a submodule of $H(G, K, \tilde{\kappa} \otimes I_s)$; from this point of view the claim is that this submodule is not annihilated by any nonzero element of $Z(H(G, K, \tilde{\kappa} \otimes I_s))$. But $H(G, K, \tilde{\kappa} \otimes I_s)$ is a tensor product of affine Hecke algebras; in particular (for instance, by Bernstein's presentation of $H(q, n)$ [Lu]), it is free over its center and its center is a domain. Thus *no* element of $H(G, K, \tilde{\kappa} \otimes I_s)$ is annihilated by any element of the center of $H(G, K, \tilde{\kappa} \otimes I_s)$. \square

In light of this injectivity, we can prove that the center of $H(G, K, \tilde{\kappa} \otimes I_s)$ is isomorphic to $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ by comparing the dimensions of $H(G, K, \tilde{\kappa} \otimes \text{St}_s)_{K_g K}$ and $Z(H(G, K, \tilde{\kappa} \otimes I_s))_{K_g K}$ for a suitable set of g . Fix an element g of $M \cap \text{GL}_{\frac{n}{e_f}}(E)$. For such a g , let \bar{P}_g be the image of $g \text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E) g^{-1} \cap \text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ in \bar{G} . Then \bar{P}_g is a parabolic subgroup with unipotent radical \bar{U}_g , and Levi \bar{M}_g .

Let \tilde{P}_g be the parahoric subgroup $g \text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E) g^{-1} \cap \text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$, and observe that $g^{-1} \tilde{P}_g g$ is the preimage of the opposite parabolic \bar{P}_g° in $\text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$. Let \tilde{U}_g be the preimage of \bar{U}_g in \tilde{P}_g ; then for any u in \tilde{U}_g , the conjugate $g^{-1} u g$ reduces to the identity in \bar{G} . The conjugation map $\tilde{P}_g \rightarrow g^{-1} \tilde{P}_g g$ thus descends to a map $\bar{M}_g \rightarrow \bar{M}_g$; we say g is \bar{M}_g -central if the resulting automorphism of \bar{M}_g is trivial. It is clear that for any g , there exists a k in $\text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)$ such that gk is \bar{M}_g -central. We then have:

Lemma 6.12. *Let g be an element of $M \cap \text{GL}_{\frac{n}{e_f}}(E)$, and let ξ be a $W(k)[\bar{G}]$ -module considered as a module over $W(k)[\text{GL}_{\frac{n}{e_f}}(\mathcal{O}_E)]$ by inflation. Then $I_g(\xi)$ is free of rank one over $\text{End}_{W(k)[\bar{G}]}(r_{\bar{G}}^{\bar{P}_g} \xi)$. Moreover, if g is \bar{M}_g -central, we have a natural isomorphism:*

$$I_g(\xi) \cong \text{End}_{W(k)[\bar{G}]}(r_{\bar{G}}^{\bar{P}_g} \xi).$$

Proof. By definition, $I_g(\xi) = \text{Hom}_{W(k)[\tilde{P}_g]}(\xi, \xi^g)$. Note that for u in \tilde{U}_g , the element $g^{-1} u g$ acts trivially on ξ . Thus u acts trivially on ξ^g . In particular any element of $I_g(\xi)$ gives rise to a map:

$$\text{Hom}_{W(k)[\tilde{P}_g]}(r_{\bar{G}}^{\bar{P}_g} \xi, r_{\bar{G}}^{\bar{P}_g} \xi^g).$$

Conversely, any such map gives rise to an element of $I_g(\xi)$. (Here we are identifying the \bar{U}_g -invariants of ξ with the \bar{U}_g -coinvariants via the natural map from invariants to coinvariants. Moreover, the representation $r_{\bar{G}}^{\bar{P}_g} \xi$ is a representation of \bar{M}_g considered as a representation of \tilde{P}_g by inflation.) As conjugation by g descends to an inner automorphism of \bar{M}_g , $r_{\bar{G}}^{\bar{P}_g} \xi^g$ is isomorphic to $r_{\bar{G}}^{\bar{P}_g} \xi$, and we can take this isomorphism to be the identity when g is \bar{M}_g -central. The result is then clear. \square

Let z be an element of Z . Then our fixed maximal torus of \overline{G} is a maximal torus of \overline{M}_g . Recall from the previous section that we have defined $W(\overline{M}, \overline{M}_z)$ as the set of w in $W(\overline{G})$ such that $w\overline{M}w^{-1}$ is contained in \overline{M}_z .

Proposition 6.13. *The spaces $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ and $H(G, K, \tilde{\kappa} \otimes i_{\overline{P}}^{\overline{G}} \text{St}_{\overline{M}, s})$ are supported on KZK . Moreover, for z in Z , we have:*

$$H(G, K, \tilde{\kappa} \otimes I_s)_{KzK} \cong I_z(\tilde{\kappa}) \otimes \text{End}_{\overline{\mathcal{K}}[\overline{M}_z]}(\oplus_w I_{\overline{M}_z, wsw^{-1}})$$

$$H(G, K, \tilde{\kappa} \otimes \text{St}_s)_{KzK} \cong I_z(\tilde{\kappa}) \otimes \text{End}_{\overline{\mathcal{K}}[\overline{M}_z]}(\oplus_{w'} \text{St}_{\overline{M}_z, w's(w')^{-1}})$$

where w runs over a set of representatives for $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z)$, and w' runs over a set of representatives for $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$,

Proof. By Lemma 6.8, it suffices to compute the spaces $I_z(I_s)$ and $I_z(\text{St}_s)$. Note that for any z in Z , it is clear that z is \overline{M}_z -central. The result is thus immediate from Lemma 6.12, together with Propositions 5.3 and 5.5. \square

Corollary 6.14. *For $z \in Z$, the dimension of $H(G, K, \tilde{\kappa} \otimes \text{St}_s)_{KzK}$ is equal to the cardinality of $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$.*

Proof. The modules $\text{St}_{\overline{M}_z, wsw^{-1}}$ and $\text{St}_{\overline{M}_z, (w')s(w')^{-1}}$ are isomorphic if, and only if, w and w' represent the same class in $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$. It follows that

$$\dim_{\overline{\mathcal{K}}} \text{End}_{\overline{\mathcal{K}}[\overline{M}_z]}(\oplus_w \text{St}_{\overline{M}_z, wsw^{-1}}) = \#W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$$

and the result follows by Proposition 6.13. \square

It remains to compute the dimension of the subspace of the center of $H(G, K, \tilde{\kappa} \otimes I_s)$ supported on KgK . As this Hecke algebra is a tensor product of Iwahori Hecke algebras, we first observe:

Lemma 6.15. *Let $H(m, q^{\text{df}})$ be the Iwahori Hecke algebra $H(\text{GL}_m(E'), I)$, where E' is an unramified extension of E of degree D and I is an Iwahori subgroup of $\text{GL}_m(E')$. Then for any $g \in \text{GL}_m(E')$, the subspace of central elements of $H(m, q^{\text{df}})$ supported on the union of the double cosets $Iwgw'I$, for w, w' in the Weyl group of $\text{GL}_m(E')$, is one-dimensional. Moreover, the sum of these spaces as g varies is the entire center of $H(m, q^{\text{df}})$.*

Proof. This is presumably well-known. Let J be a maximal compact of $\text{GL}_m(E)$ containing I . Then the induction $\text{c-Ind}_I^J 1$ contains the trivial character of J , and so $\text{c-Ind}_J^{\text{GL}_m(E)} 1$ is a direct summand of $\text{c-Ind}_I^{\text{GL}_m(E)} 1 = \text{c-Ind}_J^{\text{GL}_m(E)} \text{c-Ind}_I^J 1$. In particular the center of $H(\text{GL}_m(E), I)$ preserves the summand $\text{c-Ind}_J^{\text{GL}_m(E)} 1$; this gives a support-preserving map from the center of $H(\text{GL}_m(E), I)$ to the spherical Hecke algebra $H(\text{GL}_m(E), J)$. (This map simply gives the action of the center of the unipotent block on $\text{c-Ind}_J^{\text{GL}_m(E)} 1$.) One verifies easily that the center of $H(\text{GL}_m(E), I)$ acts faithfully on $\text{c-Ind}_J^{\text{GL}_m(E)} 1$ and that every endomorphism of $\text{c-Ind}_J^{\text{GL}_m(E)} 1$ arises from the action of the center. We thus have a support preserving isomorphism of this center with $H(\text{GL}_m(E), 1)$, and the result is immediate. \square

Fixing an isomorphism of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ with a tensor product of Iwahori Hecke algebras (as in the discussion preceding lemma 6.8,) we obtain:

Corollary 6.16. *For any z in Z , the space of central elements of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ supported on the union of double cosets of the form $K_{\overline{P}}wzw'K_{\overline{P}}$ with w, w' in $W_{\overline{M}}(s)$ is one-dimensional. Moreover, the sum of these spaces as z varies is the entire center of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$.*

Theorem 6.17. *The map:*

$$\overline{\mathcal{K}}[Z]^{W_{\overline{M}}(s)} = Z(H(G, K_{\overline{P}}, \tau_{\overline{P}})) \rightarrow H(G, K, \tilde{\kappa} \otimes \text{St}_s)$$

is an isomorphism.

Proof. We have already shown that this map is injective; in light of the previous lemma it thus suffices to show that for z in Z , the space of central elements of $H(G, K, \tilde{\kappa} \otimes I_s)$ (or equivalently, $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ supported on a double coset KzK) has dimension equal to the cardinality of $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$.

The support-preserving isomorphism of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ with the tensor product of the spaces $H(m_j, q^f)$ shows that, for z in Z , there is a one-dimensional subspace of $Z(H(G, K_{\overline{P}}, \tau_{\overline{P}}))$ supported on double cosets $K_{\overline{P}}wzw'K_{\overline{P}}$, for w, w' in $W_{\overline{M}}(s)$. If z and z' lie in Z , the collection of double cosets $\{K_{\overline{P}}wzw'K_{\overline{P}}\}$ coincides with the collection $\{K_{\overline{P}}wz'w'K_{\overline{P}}\}$ if, and only if, $z = wz'w^{-1}$ for some w in $W_{\overline{M}}(s)$.

The central elements of $H(G, K_{\overline{P}}, \tau_{\overline{P}})$ supported on KzK are the (direct) sum of those supported on the collections $\{K_{\overline{P}}wz'w'K_{\overline{P}}\}$ for those z' in Z such that $KzK = Kz'K$.

Let z and z' lie in Z , and suppose that $KzK = Kz'K$. Then there exists an element w' of $W(\text{GL}_{\frac{n}{e^f}}(E)) = W(\overline{G})$ such that $z' = (w')^{-1}zw'$. As both z and z' lie in Z , the group \overline{M} is contained in both \overline{M}_z and $\overline{M}_{z'}$. We have $\overline{M}_{z'} = (w')^{-1}\overline{M}_zw'$, and thus it follows that $w'\overline{M}(w')^{-1}$ is contained in \overline{M}_z . In particular w' lies in $W(\overline{M}, \overline{M}_z)$. Note that w' is determined by z and z' up to left multiplication by an element of $W(\overline{M}_z)$.

It follows that the distinct collections of cosets $\{K_{\overline{P}}wz'w'K_{\overline{P}}\}$ for z' such that $KzK = Kz'K$ are in bijection with $W(\overline{M}_z) \backslash W(\overline{M}, \overline{M}_z) / W_{\overline{M}}(s)$, and each such collection contributes a one-dimensional space to the subspace of $Z(H(G, K_{\overline{P}}, \tau_{\overline{P}}))$ supported on KzK . The result follows. \square

7. STRUCTURE OF $\text{End}_G(\mathcal{P}_{K,\tau})$

Our next step will be to understand the endomorphism ring $\text{End}_{W(k)[G]}(\mathcal{P}_{K,\tau}) = H(G, K, \mathcal{P}_{\kappa \otimes \sigma})$.

To simplify notation, let E_σ be the ring $\text{End}_{W(k)[K]}(\mathcal{P}_\sigma)$, and let $E_{K,\tau}$ be the ring $\text{End}_{W(k)[G]}(\mathcal{P}_{K,\tau})$. By Proposition 5.8, E_σ is a reduced commutative $W(k)$ -algebra that is free of finite rank over $W(k)$.

Fix a finite extension \mathcal{K}' of \mathcal{K} sufficiently large that every map $E_\sigma \rightarrow \overline{\mathcal{K}}$ factors through \mathcal{K}' . We then have a decomposition

$$\mathcal{P}_\sigma \otimes \mathcal{K}' \cong \bigoplus_{s: s^{\text{reg}}=s'} \text{St}_s,$$

where s' is the ℓ -regular semisimple element of \overline{G} corresponding to the representation σ . This yields a decomposition:

$$\mathcal{P}_{K,\tau} \otimes \mathcal{K}' = \bigoplus_{s: s^{\text{reg}}=s'} \text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s.$$

Proposition 7.1. *Let g be an element of G that intertwines $\mathcal{P}_{K,\tau}$. Then g lies in $K \mathrm{GL}_{\frac{n}{e_f}}(E)K$, and we have a direct sum decomposition:*

$$I_g(\mathcal{P}_{K,\tau}) \otimes \mathcal{K}' = \bigoplus_{s:s^{\mathrm{reg}}=s'} I_g(\tilde{\kappa} \otimes \mathrm{St}_s).$$

Moreover, if g lies in $\mathrm{GL}_{\frac{n}{e_f}}$ and is \overline{M}_g -central, then there is a natural isomorphism:

$$I_g(\mathcal{P}_{K,\tau}) \cong I_g(\tilde{\kappa}) \otimes \mathrm{End}_{W(k)[\overline{M}_g]}(r_{\overline{G}}^{\overline{P}g} \mathcal{P}_\sigma).$$

Proof. Every statement other than the decomposition is a direct consequence of Lemma 6.12. As for the direct sum decomposition, the decomposition:

$$P_\sigma \otimes \mathcal{K}' = \bigoplus_{s:s^{\mathrm{reg}}=s'} \mathrm{St}_s,$$

yields a decomposition

$$r_{\overline{G}}^{\overline{P}g} \mathcal{P}_\sigma \otimes \mathcal{K}' = \bigoplus_{s:s^{\mathrm{reg}}=s'} r_{\overline{G}}^{\overline{P}g} \mathrm{St}_s.$$

From Proposition 5.3 we see that no two irreducible summands of the right hand side are isomorphic to each other, so any endomorphism of the right hand side preserves each of the summands. We thus have a decomposition:

$$\mathrm{End}_{W(k)[\overline{M}_g]}(r_{\overline{G}}^{\overline{P}g} \mathcal{P}_\sigma) = \bigoplus_{s:s^{\mathrm{reg}}=s'} \mathrm{End}_{W(k)[\overline{M}_g]}(r_{\overline{G}}^{\overline{P}g} \mathrm{St}_s).$$

Tensoring both sides with $I_g(\tilde{\kappa})$ yields the desired result. \square

Corollary 7.2. *The action of $E_{K,\tau} \otimes \mathcal{K}'$ on the direct sum decomposition*

$$\mathcal{P}_{K,\tau} \otimes \mathcal{K}' = \bigoplus_{s:s^{\mathrm{reg}}=s'} \mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes \mathrm{St}_s$$

preserves each summand, and thus yields an isomorphism:

$$E_{K,\tau} \otimes \overline{\mathcal{K}} \cong \prod_{s:s^{\mathrm{reg}}=s'} H(G, K, \tilde{\kappa} \otimes \mathrm{St}_s).$$

In particular $E_{K,\tau}$ is reduced, commutative, and ℓ -torsion free.

Proof. The direct sum decomposition follows immediately from Proposition 7.1 by summing over the elements supported on KgK for each g . The embedding follows from the absence of ℓ -torsion in $E_{K,\tau}$ (which is immediate from the fact that $\mathcal{P}_{K,\tau}$ is projective.) Reducedness and commutativity of $E_{K,\tau}$ then follow from the corresponding results for $H(G, K, \tilde{\kappa} \otimes \mathrm{St}_s)$. \square

In what follows, it will be necessary to understand families of cuspidal representations of $\mathrm{GL}_i(F)$ for various i , all of which have supercuspidal support inertially equivalent to some power of a fixed supercuspidal representation over k . Fix an integer n_1 , and let (K_1, τ_1) be a maximal distinguished *supercuspidal* k -type for $\mathrm{GL}_{n_1}(F)$. We have $\tau_1 = \kappa_1 \otimes \sigma_1$, where σ_1 is the supercuspidal representation of $\mathrm{GL}_{\frac{n_1}{e_f}}(\mathbb{F}_{q^f})$ attached to an ℓ -regular element s'_1 of $\mathrm{GL}_{\frac{n_1}{e_f}}(\mathbb{F}_{q^f})$ with irreducible characteristic polynomial.

Recall that e_{q^f} is the order of q^f modulo ℓ , and let m lie in $\{1, e_{q^f}, \ell e_{q^f}, \ell^2 e_{q^f}, \dots\}$. For precisely such m , the generic representation σ_m of $\mathrm{GL}_{\frac{n_0 m}{e_f}}(\mathbb{F}_{q^f})$ corresponding to a block matrix consisting of m copies of s'_1 is cuspidal.

For such m we may also define a representation κ_m of a suitable compact open subgroup of $\mathrm{GL}_{n_1 m}(F)$. The data giving rise to the type (K_1, τ_1) consist of an extension E of F of ramification index e and residue class degree f , a stratum $[\mathfrak{A}_1, n, 0, \beta]$, with $E = F[\beta]$, and a character θ in $\mathcal{C}(\mathfrak{A}_1, 0, \beta)$. Then θ gives rise to an endo-equivalence class $(\Theta, 0, \beta)$. Let \mathfrak{A}_m be the maximal order $M_m(\mathfrak{A}_1)$ of $M_{n_1 m}(F)$; then the endo-class $(\Theta, 0, \beta)$ gives rise to a compact open subgroup K_m of $\mathrm{GL}_{n_1 m}(F)$ and a representation κ_m of K_m . Let $\tau_m = \kappa_m \otimes \sigma_m$; then (K_m, κ_m) is a maximal distinguished cuspidal k -type in $\mathrm{GL}_{n_1 m}(F)$. (Note that κ_m is only well-defined up to certain twists; we will pin these twists down precisely later on.)

Moreover, if π_1 is an irreducible representation of $\mathrm{GL}_{n_1}(F)$ containing (K_1, τ_1) , then for any m , and any irreducible representation π of $\mathrm{GL}_{n_1 m}(F)$ containing (K_m, τ_m) , the inertial supercuspidal support of π is given by $\pi_1^{\otimes m}$. Conversely, if π is any irreducible cuspidal representation of $\mathrm{GL}_{n_1 m}(F)$ with inertial supercuspidal support $\pi_1^{\otimes m}$, then m lies in the set $\{1, e_{q^f}, \ell e_{q^f}, \dots\}$, and π contains (K_m, τ_m) . (This follows from the theory of Zelevinski parameters in [V2], section V).

We now define a partial order \preceq on the collection of all irreducible cuspidal k -representations of $\mathrm{GL}_i(F)$ (for all i). Let π be an irreducible cuspidal representation of $\mathrm{GL}_{ij}(F)$ over k , and let π' be an irreducible cuspidal k -representation of $\mathrm{GL}_i(F)$. We say that $\pi' \preceq \pi$ if there exists a parabolic subgroup $P = MU$ of $\mathrm{GL}_{ij}(F)$, and an irreducible cuspidal representation π'_M of M that is inertially equivalent to $(\pi')^{\otimes j}$, such that π is isomorphic to a Jordan-Hölder constituent of $i_P^G \pi'_M$. Note that if $\pi' \preceq \pi$, then the inertial supercuspidal supports of π' and π are given by (different) tensor powers of the same supercuspidal representation. In particular, if π contains (K_m, τ_m) , and π' is any cuspidal representation comparable to π , then π' contains $(K_{m'}, \tau_{m'})$ for some m' . Moreover, it will follow from later work (particularly, Theorem 7.4 below) that in this case $\pi' \preceq \pi$ if, and only if, $m' \leq m$. We will say that π' (resp. m') *immediately precedes* π (resp. m) if $m' < m$, and there are no elements of $\{1, e_{q^f}, \ell e_{q^f}, \dots\}$ strictly between m' and m .

For each m , let G_m be the group $\mathrm{GL}_{n_1 m}(F)$ and let \overline{G}_m be the group $\mathrm{GL}_{\frac{n_1 m}{e^f}}(\mathbb{F}_{q^f})$. Our next goal is to give a set of isomorphisms of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathrm{St}_s)$ with appropriate rings of invariants that is normalized across all the types (K_m, τ_m) in a systematic way. For each semisimple conjugacy class s in \overline{G}_m with ℓ -regular part $(s'_1)^m$, let \overline{M}_s be the minimal split Levi subgroup of \overline{G}_m containing s ; let M_s be the corresponding Levi of G_m , and let Z_s be the subgroup of the center of $M_s \cap \mathrm{GL}_{\frac{n_1 m}{e^f}}(E)$ consisting of diagonal matrices whose entries are powers of ϖ_E . For $i = 1, \dots, m$, we have elements $\theta_{i,s}$ of $W(k)[Z_s]^{W_{\overline{M}_s}(s)}$, defined by taking $\theta_{i,s}$ to be the sum of the elements of Z_s whose characteristic polynomial (as an element of $\mathrm{GL}_{\frac{n_1 m}{e^f}}(E)$) has the form $(t - \varpi_E)^{\frac{n_1 i}{e^f}} (t - 1)^{\frac{n_1(m-i)}{e^f}}$. Note that $\theta_{m,s}$ is invertible.

Of particular interest to us will be the subgroups corresponding to $s = (s'_1)^m$; we denote these by $\overline{M}_{0,m}$, $M_{0,m}$, and $Z_{0,m}$. For i between 1 and m , let $z_{i,m}$ be an element of $Z_{0,m}$ with characteristic polynomial $(t - \varpi_E)^{\frac{n_1 i}{e^f}} (t - 1)^{\frac{n_1(m-i)}{e^f}}$.

Fix, once and for all, an absolutely irreducible integral supercuspidal representation $\pi_{s'_1}$ of $\mathrm{GL}_{n_1}(F)$ containing $(K_1, \tilde{\kappa}_1 \otimes \mathrm{St}_{s'_1})$. Our first goal is to construct, for each $m \in \{1, e_{q^f}, \ell e_{q^f}, \dots\}$, and each irreducible conjugacy class s in \overline{G}_m such that $s^{\mathrm{reg}} = (s'_1)^m$, an absolutely irreducible cuspidal representation π_s of $\mathrm{GL}_{n_1 m}(F)$ containing $(K_m, \tilde{\kappa}_m \otimes \mathrm{St}_s)$.

We proceed as follows: note that $(K_m, \tilde{\kappa}_m \otimes \text{St}_s)$ is a maximal distinguished cuspidal $\overline{\mathcal{K}}$ -type. In particular $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_s)$ is isomorphic to $\overline{\mathcal{K}}[\Theta^{\pm 1}]$, where Θ is an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_s)$ supported on $K_m z_{m,m} K_m$. We have an isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_s)_{K_m z_{m,m} K_m} \cong I_{z_{m,m}}(\tilde{\kappa}_m) \otimes \text{End}_{\overline{\mathcal{K}}[\overline{G}_m]}(\text{St}_s).$$

We also have an isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_{(s'_1)^m})_{K_m z_{m,m} K_m} \cong I_{z_{m,m}}(\tilde{\kappa}_m) \otimes \text{End}_{\overline{\mathcal{K}}[\overline{G}_m]}(\text{St}_{(s'_1)^m}).$$

Let $\pi_{(s'_1)^m}$ be the cuspidal representation $\pi_{s'_1}^{\otimes m}$ of $M_{(s'_1)^m}$. Such a choice gives an isomorphism of $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_{(s'_1)^m})$ with $\overline{\mathcal{K}}[Z_{0,m}]^{W_{\overline{M}_{0,m}}((s'_1)^m)}$; the element $z_{m,m}$ of $\overline{\mathcal{K}}[Z_{0,m}]$ maps to an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_{(s'_1)^m})_{K_m z_{m,m} K_m}$. This element has the form $\phi \otimes 1$ for some $\phi \in I_{z_{m,m}}(\tilde{\kappa}_m)$, where we regard 1 as the identity endomorphism of $\text{St}_{(s'_1)^m}$.

We may then consider the element $\phi \otimes 1$ of $I_{z_{m,m}}(\tilde{\kappa}) \otimes \text{End}_{\overline{\mathcal{K}}[\overline{G}_m]}(\text{St}_s)$. This gives us an element Θ of $H(G_m, K_m, \tilde{\kappa} \otimes \text{St}_s)_{K_m z_{m,m} K_m}$. There is then a unique irreducible cuspidal representation π_s of G_m over $\overline{\mathcal{K}}$ that contains $(K_m, \tilde{\kappa} \otimes \text{St}_s)$, on which Θ acts via the identity. Note that π_s is integral.

We will refer to the family of cuspidal representations π_s as the *compatible family of cuspids* attached to our choice of $\pi_{s'_1}$. The dependence on the choice of $\pi_{(s'_1)^m}$ is mild; indeed, replacing $\pi_{(s'_1)^m}$ by an unramified twist $\pi_{(s'_1)^m} \otimes (\chi \circ \det)$ simply twists each π_s by $\chi \circ \det$ as well.

For each m , and each reducible conjugacy class s in \overline{G}_m , we define an irreducible cuspidal representation π_s of M_s , by taking π_s to be the tensor product of the cuspidal representations π_{s_i} defined above, where the s_i are the irreducible factors of s .

Our next goal is to construct elements of E_{K_m, τ_m} whose action on each summand of $\mathcal{P}_{K_m, \tau_m} \otimes \overline{\mathcal{K}}$ can be understood explicitly. We will show:

Theorem 7.3. *For each m , there exists a subalgebra C_{K_m, τ_m} of E_{K_m, τ_m} , generated over $W(k)$ by elements $\Theta_{1,m}, \dots, \Theta_{m,m}$ and $(\Theta_{m,m})^{-1}$, such that for any s with $s^{\text{reg}} = (s'_1)^m$, the composed map:*

$$C_{K_m, \tau_m} \rightarrow A_{M_s, \pi_s} \cong \overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)}$$

takes $\Theta_{i,m}$ to $\theta_{i,s}$. (Here the left-hand map is the map giving the action of E_{K_m, τ_m} on the summand $(\mathcal{P}_{K_m, \tau_m} \otimes \overline{\mathcal{K}})_{M_s, \pi_s}$ of $\mathcal{P}_{K_m, \tau_m} \otimes \overline{\mathcal{K}}$, and the right-hand isomorphism is normalized by the choice of pair (M_s, π_s) .)

Note that this property characterizes the $\Theta_{i,m}$ uniquely. We will construct the $\Theta_{i,m}$ via an inductive argument. The case $m = 1$ is easy: we have an isomorphism

$$H(G_1, K_1, \tilde{\kappa}_1 \otimes \mathcal{P}_{\sigma_1})_{K_1 z_{1,1} K_1} \cong I_{z_{1,1}}(\tilde{\kappa}_1) \otimes \text{End}_{W(k)[\overline{G}_1]}(\mathcal{P}_{\sigma_1}).$$

We define $\Theta_{1,1}$ to be the unique element of the form $\phi \otimes 1$, for some ϕ in $I_{z_{1,1}}(\tilde{\kappa}_1)$, that acts on the quotient $\pi_{(s'_1)}$ of $\mathcal{P}_{K_1, \tau_1} \otimes \overline{\mathcal{K}}$ via the identity. It is clear from our construction of the π_s that this $\Theta_{1,1}$ has the desired property.

We now turn to the inductive part of the argument. This will proceed via a G -cover argument, along the lines of the construction in section 6. Let m' immediately precede m , and set $j = \frac{m}{m'}$. Let $V_{m'}$ be the F -vector space on which $G_{m'}$ acts, and identify V_m with the direct sum of j copies $V_{m',1}, \dots, V_{m',j}$ of $V_{m'}$. Let M be the

Levi subgroup of G_m preserving this direct sum decomposition, and let P be the parabolic preserving the flag

$$V_{m',1} \subset V_{m',1} + V_{m',2} \subset \cdots \subset V_{m',1} + \cdots + V_{m',j}.$$

The groups $P = MU$ give rise to subgroups $\overline{P} = \overline{MU}$ of \overline{G}_m in the usual way.

We have maximal orders \mathfrak{A}_m of G_m and $\mathfrak{A}_{m'}$ of $G_{m'}$ attached to the types (K_m, τ_m) and $(K_{m'}, \tau_{m'})$. The procedure of [BK3], 7.2 also yields an order \mathfrak{A}'_m contained in \mathfrak{A}_m attached to the flag defined above. Set $K'_m = J(\beta, \mathfrak{A}'_m)$, let K''_m be the subgroup $(J(\beta, \mathfrak{A}'_m) \cap P)H^1(\beta, \mathfrak{A}'_m)$ of K'_m , and let $K_{m,\overline{P}}$ be the preimage of \overline{P} in K_m under the map from K_m to \overline{G}_m . Then, just as in section 6 we have representations $\tilde{\kappa}'_m$ of K'_m , $\tilde{\kappa}''_m$ of K''_m , and $\tilde{\kappa}_{m,\overline{P}}$ of $K_{m,\overline{P}}$, satisfying:

$$\begin{aligned} \tilde{\kappa}'_{m,\overline{P}} &= (\tilde{\kappa}'_m)|_{K_{m,\overline{P}}} \\ \mathrm{Ind}_{K''_m}^{K'_m} \tilde{\kappa}''_m &\cong \tilde{\kappa}'_m \\ \mathrm{Ind}_{K_{m,\overline{P}}}^{(\mathfrak{A}'_m)^\times} \tilde{\kappa}'_{m,\overline{P}} &\cong \mathrm{Ind}_{K'_m}^{(\mathfrak{A}'_m)^\times} \tilde{\kappa}'_m. \end{aligned}$$

Moreover, the intersection K_M of K''_m with M is (under the identification of M with a product of j copies of $G_{m'}$) equal to the product of j copies of $K_{m'}$. The restriction $\tilde{\kappa}_M$ of $\tilde{\kappa}''_m$ to K_M factors as a product of j copies of a representation $\tilde{\kappa}'_{m'}$ containing a character attached to \mathfrak{A}_m via the endo-class $(\Theta, 0, \beta)$. Thus $\tilde{\kappa}'_{m'}$ differs from $\tilde{\kappa}_{m'}$ by a twist; since the $\tilde{\kappa}_{m'}$ were only defined up to twist to begin with we can choose them such that all these twists are trivial. (This, finally, pins down $\tilde{\kappa}_m$, for all m , up to a choice of $\tilde{\kappa}_1$.)

Finally, let $\mathcal{P}_{\overline{M}}$ denote the inflation of $\mathcal{P}_{\sigma_m}^{\otimes j}$ from \overline{M} to a representation of K_M , and also (somewhat abusively) the inflation of $\mathcal{P}_{\sigma_{m'}}^{\otimes j}$ to a representation of K''_m (via the surjection of K''_m onto \overline{P} .)

Exactly as in section 6, we obtain isomorphisms:

$$\mathrm{c}\text{-Ind}_{K''_m}^{K'_m} \tilde{\kappa}''_m \otimes \mathcal{P}_{\overline{M}} \cong \mathrm{c}\text{-Ind}_{K_{m,\overline{P}}}^{K'_m} \tilde{\kappa}_{m,\overline{P}} \otimes \mathcal{P}_{\overline{M}} \cong \mathrm{c}\text{-Ind}_{K_m}^{K'_m} \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}$$

$$H(G, K''_m, \tilde{\kappa}''_m \otimes \mathcal{P}_{\overline{M}}) \cong H(G_m, K_{m,\overline{P}}, \tilde{\kappa}_{m,\overline{P}} \otimes \mathcal{P}_{\overline{M}}) \cong H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}).$$

Moreover, the discussion in section 6 following Theorem 6.6 shows that these maps are compatible with supports.

Theorem 7.4. *The pair $(K''_m, \tilde{\kappa}''_m \otimes \mathcal{P}_{\overline{M}})$ is a G -cover of $(K_M, \tilde{\kappa}_M \otimes \mathcal{P}_{\overline{M}})$.*

The only non-obvious condition that must be satisfied is the existence of a strictly positive central element λ of M such that the image of $1_{K_M \lambda K_M}$ under T^+ is invertible in $H(G_m, K''_m, \tilde{\kappa}''_m \otimes \mathcal{P}_{\overline{M}})$. In fact, a weaker condition suffices. Suppose we have an invertible element x of $H(M, K_M, \tilde{\kappa}_M \otimes \mathcal{P}_{\overline{M}})$ that corresponds under the isomorphism:

$$H(M, K_M, \tilde{\kappa}_M \otimes \mathcal{P}_{\overline{M}}) \cong E_{K_{m'}, \tau_{m'}}^{\otimes j}$$

of the form $\otimes_i x_i$, where for each i , x_i is an invertible element of $E_{K_{m'}, \tau_{m'}}$ supported on $K_{m'} z_{m,m}^{r_i} K_{m'}$, and the r_i form a decreasing sequence of integers. Then x has strictly positive support. It is easy to see that if $T^+ x$ is invertible, then so is $T^+ 1_{K_M \lambda K_M}$: for some r , $x^r 1_{K_M \lambda^{-1} K_M}$ has positive support, and then

$$(T^+ 1_{K_M \lambda K_M})(T^+ x^r 1_{K_M \lambda^{-1} K_M}) = T^+ x^r,$$

and the latter is invertible. We will show the existence of such an x in several steps.

Proposition 7.5. *The pair $(K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}} \otimes \mathcal{K})$ is a G -cover of $(K_M, \tilde{\kappa}_M \otimes \mathcal{P}_{\overline{M}} \otimes \mathcal{K})$.*

Proof. It suffices to show this after tensoring with $\overline{\mathcal{K}}$, since $\overline{\mathcal{K}}$ is faithfully flat over \mathcal{K} . To do this we will use the direct sum decomposition of $\mathcal{P}_{\overline{M}}$. Let s_1, \dots, s_j be a sequence such that for all i , $s_i^{\text{reg}} = (s_1')^{m'}$. Then St_{s_i} is a direct summand of $\mathcal{P}_{\sigma_{m'}} \otimes \overline{\mathcal{K}}$ for all i , and it suffices to show that for each such sequence, the pair $(K_m'', \tilde{\kappa}_m'' \otimes \bigotimes_i \text{St}_{s_i})$ is a G -cover of $(K_M, \tilde{\kappa}_M \otimes \bigotimes_i \text{St}_{s_i})$. We will reduce this claim to results from section 6.

For each s_j , let \overline{M}_{s_j} and M_{s_j} be the Levi subgroups of $\overline{G}_{m'}$ and $G_{m'}$ attached to the type $(K_{m'}, \tau_{m'})$ and the conjugacy class s_j via the construction of section 6. We then have a cuspidal distinguished M_{s_j} -type $(K_{M_{s_j}}, \tilde{\kappa}_{M_{s_j}} \otimes \text{St}_{\overline{M}_{s_j}, s_j})$, and the results of section 6 give rise to a map:

$$T_{s_j} : H(M_{s_j}, K_{M_{s_j}}, \tilde{\kappa}_{M_{s_j}} \otimes \text{St}_{\overline{M}_{s_j}, s_j}) \rightarrow H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes i_{\overline{P}_{s_j}}^{\overline{G}_m} \text{St}_{\overline{M}_{s_j}, s_j})$$

for each j .

Similarly, let s represent the conjugacy class in \overline{G}_m given by the block matrix with blocks s_1, \dots, s_j . Then the construction of section 6 yields a map:

$$T_s : H(M_s, K_{M_s}, \tilde{\kappa}_{M_s} \otimes \text{St}_{\overline{M}_s, s}) \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}_s}^{\overline{G}_m} \text{St}_{\overline{M}_s, s}).$$

Note that we have a decomposition:

$$H(M_s, K_{M_s}, \tilde{\kappa}_{M_s} \otimes \text{St}_{\overline{M}_s, s}) \cong \bigotimes_j H(M_{s_j}, K_{M_{s_j}}, \tilde{\kappa}_{M_{s_j}} \otimes \text{St}_{\overline{M}_{s_j}, s_j}).$$

Let r_1, \dots, r_j be a decreasing sequence of integers, and, for each i , let x_i be an invertible element of $H(M_{s_i}, K_{M_{s_i}}, \tilde{\kappa}_{M_{s_i}} \otimes \text{St}_{\overline{M}_{s_i}, s_i})$ supported on $K_{M_{s_i}} z_{m', m'}^{r_i} K_{M_{s_i}}$. The image of each x_i in $H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes i_{\overline{P}_{s_i}}^{\overline{G}_m} \text{St}_{\overline{M}_{s_i}, s_i})$ is central and thus yields an element of $H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i})$ via the map

$$i_{\overline{P}_{s_i}}^{\overline{G}_m} \text{St}_{\overline{M}_{s_i}, s_i} \rightarrow \text{St}_{s_i}.$$

Let x be the tensor product, over i , of the elements x_i . The above maps allow us to consider x as an element of the summand $\bigotimes_i H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i})$ of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$. When considered in this way x has strictly positive support. It thus suffices to show that T^+x is invertible in the summand the summand $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_s)$ of $E_{K_m, \tau_m} \otimes \overline{\mathcal{K}}$.

On the other hand, $T_s x$ acts on $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{P}_s}^{\overline{G}_m} \text{St}_{\overline{M}_s, s}$, and preserves the summand $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \text{St}_s$. The action of $T_s x$ on this summand coincides with that of T^+x . As x is invertible and T_s is an algebra homomorphism, it follows that T^+x is invertible as required. \square

As a consequence, we obtain an embedding:

$$T : E_{K_{m'}, \tau_{m'}}^{\otimes j} \hookrightarrow H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}}) \otimes \mathcal{K}.$$

Next, note that $\text{St}_{\overline{M}, (s_1')^m}$ is a direct summand of $\mathcal{P}_{\overline{M}} \otimes \mathcal{K}$, so we obtain a projection map of $\mathcal{P}_{\overline{M}} \otimes \mathcal{K}$ onto $\text{St}_{\overline{M}, (s_1')^m}$. Let L' be the image of $\mathcal{P}_{\overline{M}}$ in $\text{St}_{\overline{M}, (s_1')^m}$.

Proposition 7.6. *The pair $(K_m'', \tilde{\kappa}_m'' \otimes L')$ is a G -cover of $(K_M, \tilde{\kappa}_M \otimes L')$. In particular any central unit of $H(M, K_M, \tilde{\kappa}_M \otimes L')$ with totally positive support lifts to an invertible element of $H(G, K_m'', \tilde{\kappa}_m'' \otimes L')$.*

Proof. This argument follows a similar approach to that of Proposition 7.5. The representation $\text{St}_{\overline{M}_{0,m},(s'_1)^m}$ is absolutely irreducible and cuspidal; in particular its reduction mod ℓ is absolutely irreducible. There is thus a unique $\overline{M}_{0,m}$ -stable lattice L_0 in $\text{St}_{\overline{M}_{0,m},(s'_1)^m}$ up to homothety.

The construction of section 6 gives a maximal distinguished cuspidal $M_{0,m}$ -type (K_{M_0}, τ_{M_0}) , with $\tau_{M_0} = \kappa_{M_0} \otimes \text{St}_{\overline{M}_{0,m},(s'_1)^m}$, and an M -type (K''_M, τ''_M) with $\tau''_M = \tilde{\kappa}''_M \otimes \text{St}_{\overline{M}_{0,m},(s'_1)^m}$, that is an M -cover of (K_{M_0}, τ_{M_0}) . Vigneras [V2] shows that in this situation, the pair $(K''_M, \tilde{\kappa}''_M \otimes L_0)$ is an M -cover of $(K_{M_0}, \tilde{\kappa}_{M_0} \otimes L_0)$. (This follows from [V2], IV.2.5 and IV.2.6).

Moreover, we have an isomorphism:

$$H(M, K''_M, \tilde{\kappa}''_M \otimes L_0) \cong H(M, K_M, \tilde{\kappa}_M \otimes i_{\overline{P}_0}^{\overline{M}} L_0).$$

(Here \overline{P}_0 is a parabolic in \overline{M} with Levi subgroup \overline{M}_0 .) These Hecke algebras are each j -fold tensor products of (integral) Iwahori Hecke algebras $H(m', q^f)$.

Note that $r_{\overline{M}}^{\overline{P}_0} L'$ is a lattice in $\text{St}_{\overline{M}_{0,m},(s'_1)^m}$ and is thus isomorphic to L_0 . This isomorphism yields a map

$$i_{\overline{P}_0}^{\overline{M}} L_0 \rightarrow L'$$

whose image is a sublattice L of L' . After inverting ℓ , L becomes a direct summand of $i_{\overline{P}_0}^{\overline{M}} L_0$, and therefore any central element of $H(M, K_M, \tilde{\kappa}_M \otimes i_{\overline{P}_0}^{\overline{M}} L_0)$ preserves the kernel of the surjection:

$$\text{c-Ind}_{K_M}^M \tilde{\kappa}_M \otimes i_{\overline{P}_0}^{\overline{M}} L_0 \rightarrow \text{c-Ind}_{K_M}^M \tilde{\kappa}_M \otimes L,$$

and thus descends to an element of $H(M, K_M, \tilde{\kappa}_M \otimes L)$.

We now compare $H(M, K_M, \tilde{\kappa}_M \otimes L)$ and $H(M, K_M, \tilde{\kappa}_M \otimes L')$. Note that for any $g \in M$ that intertwines $\tilde{\kappa}_M$, both $\text{End}_{W(k)[\overline{M}_g]}(r_{\overline{M}}^{\overline{P}_g} L)$ and $\text{End}_{W(k)[\overline{M}_g]}(r_{\overline{M}}^{\overline{P}_g} L')$ are free of rank one over $W(k)$. It follows that any element of $I_g(\tilde{\kappa} \otimes L')$ also intertwines $\tilde{\kappa} \otimes L$, and vice versa. We thus have an isomorphism of $H(M, K_M, \tilde{\kappa}_M \otimes L)$ and $H(M, K_M, \tilde{\kappa}_M \otimes L')$ induced by the inclusion of L in L' .

Let g be an element of $M_{0,m}$ that is central in M and strictly positive with respect to P . Fix a nonzero element x of $H(M_{0,m}, K_{M_0}, \tilde{\kappa}_{M_0} \otimes L_0)$ supported on $K_{M_0} g K_{M_0}$; then x is invertible, and so are the images of x in $H(M, K''_M, \tilde{\kappa}''_M \otimes L_0)$ and $H(M, K_M, \tilde{\kappa}_M \otimes L')$. It suffices to show that x maps to an invertible element of $H(G_m, K''_m, \tilde{\kappa}''_m \otimes L')$.

To do so we invoke Vigneras' G -cover result again, this time relative to G_m rather than M . Explicitly, there is a G_m -type $(K'''_m, \tau'''_m := \tilde{\kappa}'''_m \otimes L_0)$ that is a G_m -cover of $(K_{M_0}, \tilde{\kappa}_{M_0} \otimes L_0)$, and $H(G_m, K'''_m, \tilde{\kappa}'''_m \otimes L_0)$ is isomorphic to $H(m, q^f)$. Moreover, as $\tilde{\kappa}'''_m$ and $\tilde{\kappa}''_m$ arise from the same endo-class, we have an isomorphism:

$$H(G_m, K'''_m, \tilde{\kappa}'''_m \otimes L_0) \cong H(G_m, K''_m, \tilde{\kappa}''_m \otimes i_{\overline{P}_{0,m}}^{\overline{M}} L_0).$$

The image of x in $H(G_m, K''_m, \tilde{\kappa}''_m \otimes i_{\overline{P}_{0,m}}^{\overline{M}} L_0)$ preserves $[\text{c-Ind}_{K''_m}^{G_m} \tilde{\kappa}''_m \otimes L] \otimes \mathcal{K}$ as a summand of $[\text{c-Ind}_{K''_m}^{G_m} (\tilde{\kappa}''_m \otimes i_{\overline{P}_{0,m}}^{\overline{M}} L_0)] \otimes \mathcal{K}$, and thus preserves the kernel of the surjection

$$\text{c-Ind}_{K''_m}^{G_m} \tilde{\kappa}''_m \otimes (i_{\overline{P}_{0,m}}^{\overline{M}} L_0) \rightarrow \text{c-Ind}_{K''_m}^{G_m} \tilde{\kappa}''_m \otimes L.$$

In particular the image of x is a unit in the latter. On the other hand, as when we were working over M , there is an isomorphism of $H(G_m, K''_m, \tilde{\kappa}''_m \otimes L)$ with

$H(G_m, K_m'', \tilde{\kappa}_m'' \otimes L')$. It follows that the image of x in $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes L')$ is a unit, but this coincides with T^+ applied to the image of x in $H(M, K_M, \tilde{\kappa}_M \otimes L')$. \square

We have an isomorphism of $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_M)$ with $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M)$, and the center of the block of $\text{Rep}_{W(k)}(\overline{G}_m)$ containing $i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M$ is isomorphic to E_{σ_m} . Thus E_{σ_m} is naturally a central subalgebra of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M)$. Let I be the annihilator of $i_{\overline{\mathcal{P}}}^{\overline{G}_m} L'$ in E_{σ_m} . Then we have an isomorphism:

$$[i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M] \otimes_{E_{\sigma_m}} E_{\sigma_m} / I \cong i_{\overline{\mathcal{P}}}^{\overline{G}_m} L'.$$

This gives rise to an isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M) \otimes_{E_{\sigma_m}} E_{\sigma_m} / I \cong H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} L').$$

Theorem 7.4 is now straightforward. Let x be an invertible element of $E_{K_m', \tau_m'}^{\otimes j}$ with strictly positive support. Then T^+x is an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M)$ that is invertible after inverting ℓ by Proposition 7.5, and is invertible modulo I by Proposition 7.6. It suffices to show that T^+x is a unit, and this follows from:

Lemma 7.7. *Let E be a finite rank commutative local $W(k)$ -algebra, with maximal ideal m_E , and let R be a finitely generated non-commutative central E -algebra. Let x be an element of R such that x is invertible in $R[\frac{1}{\ell}]$ and R/m_ER . Then x is a unit.*

Proof. The left ideal $R[\frac{1}{\ell}]x$ of $R[\frac{1}{\ell}]$ is the unit ideal, so the left ideal Rx contains ℓ^a for some a . On the other hand, x is a unit in R/m_ER , so the ideal Rx contains an element congruent to 1 modulo m_ER . But then Rx contains elements congruent to 1 modulo $m_E^b R$ for all positive integers b . Since for b sufficiently large, m_E^b is contained in the ideal of E generated by ℓ^a , it follows that Rx contains an element of $1 + \ell^a R$, and hence that Rx is the unit ideal. \square

We have thus completed the proof of Theorem 7.4. The upshot is that we obtain a map:

$$E_{K_m', \tau_m'}^{\otimes j} \rightarrow H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_M) \cong H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M).$$

Our next step is to relate the latter to $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m})$. By Proposition 5.10 we may fix an isomorphism of $r_{\overline{G}}^{\overline{P}} \mathcal{P}_{\sigma_m}$ with \mathcal{P}_M . Frobenius reciprocity then gives us a map

$$i_{\overline{G}}^{\overline{P}} \mathcal{P}_M \rightarrow \mathcal{P}_{\sigma_m}.$$

Let \mathcal{P}'_{σ_m} be the image of this map in \mathcal{P}_{σ_m} . We have:

Lemma 7.8. *The quotient $\mathcal{P}_{\sigma_m} / \mathcal{P}'_{\sigma_m}$ is cuspidal.*

Proof. Note that the composition:

$$\mathcal{P}_M \rightarrow r_{\overline{G}_m}^{\overline{P}} i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M \rightarrow r_{\overline{G}_m}^{\overline{P}} \mathcal{P}_{\sigma_m}$$

is an isomorphism by construction. In particular the map

$$r_{\overline{G}_m}^{\overline{P}} i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_M \rightarrow r_{\overline{G}_m}^{\overline{P}} \mathcal{P}_{\sigma_m}$$

is surjective, and factors through $r_{\overline{G}_m}^{\overline{P}} \mathcal{P}'_{\sigma_m}$. We thus have an isomorphism

$$r_{\overline{G}_m}^{\overline{P}} \mathcal{P}'_{\sigma_m} \cong r_{\overline{G}_m}^{\overline{P}} \mathcal{P}_{\sigma_m}.$$

Now let π be a Jordan-Hölder constituent of $\mathcal{P}_{\sigma_m}/\mathcal{P}'_{\sigma_m}$. Then $r_{\overline{G}_m}^{\overline{P}} \pi = 0$ by the above calculation. Suppose π were not cuspidal. Then it would have cuspidal support (\overline{M}', π') for some proper Levi subgroup \overline{M}' of \overline{G}_m , with π' a product of cuspidal representations $\sigma_{m''}$, with $m'' \preceq m$. In particular \overline{M}' would then be conjugate to a subgroup of \overline{M} , and $r_{\overline{G}_m}^{\overline{P}} \pi$ could not be trivial. \square

Note that every endomorphism of \mathcal{P}_{σ_m} preserves the submodule \mathcal{P}'_{σ_m} . (If not, this endomorphism yields a nonzero map from \mathcal{P}'_{σ_m} to $\mathcal{P}_{\sigma_m}/\mathcal{P}'_{\sigma_m}$, and hence a nonzero map of $i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}$ to $\mathcal{P}_{\sigma_m}/\mathcal{P}'_{\sigma_m}$. This is impossible as every subquotient of the latter is cuspidal.) The kernel of the map $E_{\sigma_m} \rightarrow \text{End}_{W(k)[\overline{G}_m]}(\mathcal{P}'_{\sigma_m})$ is the ideal I that annihilates St_s for all reducible s .

Let I^{cusp} be the ideal of E_{K_m, τ_m} generated by the image of I under the map $E_{\sigma_m} \rightarrow E_{K_m, \tau_m}$. Note that for any proper parabolic subgroup \overline{P}_g of \overline{G}_m , I annihilates $r_{\overline{G}_m}^{\overline{P}_g} \mathcal{P}_{\sigma_m}$; it follows that I^{cusp} is supported on double cosets of the form $K_m z_{m,m}^r K_m$.

Proposition 7.9. *Every endomorphism of $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa} \otimes_m \mathcal{P}_{\sigma_m}$ preserves the submodule $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa} \otimes \mathcal{P}'_{\sigma_m}$. The resulting restriction map:*

$$E_{K_m, \tau_m}/I^{\text{cusp}} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$$

is injective. Moreover, for every g such that $K_m g K_m$ is not a double coset of the form $K_m z_{m,m}^r K_m$, the resulting map:

$$(E_{K_m, \tau_m})_{K_m g K_m} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})_{K_m g K_m}$$

is an isomorphism.

Proof. Fix an integer r . The subset of E_{K_m, τ_m} supported on $K_m z_{m,m}^r K_m$ is isomorphic to $I_{z_{m,m}^r}(\tilde{\kappa}_m) \otimes E_{\sigma_m}$, whereas the subset of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$ supported on this double coset is $I_{z_{m,m}^r}(\tilde{\kappa}_m) \otimes \text{End}_{W(k)[\overline{G}_m]}(\mathcal{P}'_{\sigma_m})$. The former maps naturally into the latter, so every element of E_{K_m, τ_m} supported on $K_m z_{m,m}^r K_m$ preserves $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$. Such an element annihilates this submodule if, and only if, it lies in $I_{z_{m,m}^r}(\tilde{\kappa}_m) \otimes I$; note that this is exactly the subspace of I^{cusp} supported on $K_m z_{m,m}^r K_m$.

Next, note that for any parabolic subgroup \overline{P}' of \overline{G}_m , the inclusion of \mathcal{P}'_{σ_m} into \mathcal{P}_{σ_m} has cuspidal cokernel, and thus induces an isomorphism:

$$r_{\overline{G}_m}^{\overline{P}'} \mathcal{P}'_{\sigma_m} \cong r_{\overline{G}_m}^{\overline{P}'} \mathcal{P}_{\sigma_m},$$

and hence by Proposition 7.1, an isomorphism

$$H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})_{K_m g K_m} \cong H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m})_{K_m g K_m}$$

for all double cosets $K_m g K_m$ that are not of the form $K_m z_{m,m}^r K_m$. Since Proposition 7.1 identifies $I_g(\tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m})$ with $I_g(\tilde{\kappa}_m) \otimes \text{End}(r_{\overline{G}_m}^{\overline{P}_g} \mathcal{P}_{\sigma})$, when g is \overline{M}_g -central, and a similar statement holds for $I_g(\tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$, we find that every element of

$I_g(\tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m})$ also intertwines $\tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$, and thus every element of E_{K_m, τ_m} supported on $K_m g K_m$ preserves $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$.

Putting together the results for all double cosets, we find that $E_{K_m, \tau_m} / I^{\text{cusp}}$ acts faithfully on $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$ as claimed. The previous paragraph shows that the corresponding map

$$(E_{K_m, \tau_m})_{K_m g K_m} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})_{K_m g K_m}$$

is an isomorphism except when $K_m g K_m$ has the form $K_m z_{m, m}^r K_m$. \square

This gives a relationship between E_{K_m, τ_m} and $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$. The next step is to relate the latter to $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}})$. The key point is:

Lemma 7.10. *Let c be a central element of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}})$. Then c preserves the kernel of the surjection:*

$$\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}} \rightarrow \text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m},$$

and thus descends to an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$.

Proof. After inverting ℓ , \mathcal{P}'_{σ_m} becomes a direct summand of $i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}}$, and so the module $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m} \otimes \mathcal{K}$ is a direct summand of $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}} \otimes \mathcal{K}$. The element c commutes with projection onto this summand, and so preserves the kernel of this projection. On the other hand, $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$ is ℓ -torsion free, and so the kernel of the map from $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}}$ onto $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}$ consists of those elements of the kernel of the projection that lie in $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}}$. If x is such an element, it is clear that cx is as well. \square

With these results in hand we return to the inductive construction. Fix m' immediately preceding m , and suppose that we have constructed elements $\Theta_{i, m'}$ as in Theorem 7.3. Suppose further that for $1 \leq i < m'$, the element $\Theta_{i, m'}$ is supported away from double cosets of the form $K_{m'} z_{m', m'}^r K_{m'}$, and that $\Theta_{m', m'}$ is an element of $E_{K_{m'}, \tau_{m'}}$ supported on $K_{m'} z_{m', m'} K_{m'}$ of the form $\phi \otimes 1$, where ϕ lies in $I_{z_{m', m'}}(\tilde{\kappa}_{m'})$. (These stipulations hold when $m' = 1$ by construction, and we will show that the inductive construction of the $\Theta_{i, m'}$ implies these conditions for each larger m' as well.) We now turn to constructing elements $\Theta_{i, m}$.

Note that Lemma 7.10 allows us to construct elements of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$ from central elements of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}})$. One way to obtain central elements of the latter is to consider the action of the center of $\text{Rep}_{\overline{\mathcal{K}}}(G_m)$ on the tensor product $\overline{\mathcal{K}} \otimes \text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}}$. An element of the center that preserves $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}}$ yields a central element of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{G_m} \mathcal{P}_{\overline{M}})$; we will call endomorphisms that arise in this way *strongly central*.

It is thus worthwhile to find a characterization of such strongly central endomorphisms. Fix elements s_1, \dots, s_j in $\overline{G}_{m'}$ such that $s_i^{\text{reg}} = (s'_1)^{m'}$. For each i , $\text{c-Ind}_{K_{m'}}^{G_{m'}} \tilde{\kappa}_{m'} \otimes \text{St}_{s_i}$ is a summand of $\mathcal{P}_{K_{m'}, \tau_{m'}} \otimes \overline{\mathcal{K}}$, and $E_{K_{m'}, \tau_{m'}}$ acts on this summand via the isomorphisms:

$$E_{K_{m'}, \tau_{m'}} \otimes \overline{\mathcal{K}} \cong H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i}) \cong \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)}.$$

(The last isomorphism is normalized by the pair (M_{s_i}, π_{s_i}) .)

As $\otimes_i \text{St}_{s_i}$ is a summand of $\mathcal{P}_{\overline{M}} \otimes \overline{\mathcal{K}}$, the induction $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes (i_{\overline{\mathcal{P}}}^{\overline{G}_m} \otimes_i \text{St}_{s_i})$ is a summand of $\text{c-Ind}_{K_m}^{G_m} (\tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}) \otimes \overline{\mathcal{K}}$. Moreover, we have isomorphisms:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes (i_{\overline{\mathcal{P}}}^{\overline{G}_m} \otimes_i \text{St}_{s_i})) \cong H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \otimes_i \text{St}_{s_i}),$$

and the pair $(K_m'', \tilde{\kappa}_m'' \otimes \otimes_i \text{St}_{s_i})$ is a G -cover of the pair $(K_M, \tilde{\kappa}_M \otimes \otimes_i \text{St}_{s_i})$. It follows that the action of $E_{K_m'', \tau_{m'}}^{\otimes j}$ on $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes (i_{\overline{\mathcal{P}}}^{\overline{G}_m} \otimes_i \text{St}_{s_i})$ factors through the map:

$$E_{K_m'', \tau_{m'}}^{\otimes j} \rightarrow \otimes_{i=1}^j H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i}).$$

In particular we have a commutative diagram:

$$\begin{array}{ccc} E_{K_m'', \tau_{m'}}^{\otimes j} & \rightarrow & \otimes_{i=1}^j H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i}) \\ \downarrow & & \downarrow \\ H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}) & \rightarrow & H(G_m, K_m, \tilde{\kappa}_m \otimes \otimes_i \text{St}_{s_i}) \end{array}$$

Moreover, we have:

Proposition 7.11. *Let s be the “block matrix” in \overline{G}_m whose blocks are s_1, \dots, s_j . The representation $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \otimes_i \text{St}_{s_i}$ lies in $\text{Rep}_{\overline{\mathcal{K}}}(G_m)_{M_s, \pi_s}$. Moreover, it identifies:*

$$\begin{aligned} A_{M_s, \pi_s} &\cong \overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)} \\ H(G_{m'}, K_{m'}, \tilde{\kappa}_{m'} \otimes \text{St}_{s_i}) &\cong \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)} \end{aligned}$$

via the pairs (M_s, π_s) and (M_{s_i}, π_{s_i}) , then the action of an element x of $\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)}$ on $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \otimes_i \text{St}_{s_i}$ coincides with that of the image of x under

$$\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)} \rightarrow \otimes_i \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)}.$$

In particular x acts via an element of $H(G_m, K_m, \tilde{\kappa} \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})$ that lies in the image of $E_{K_m'', \tau_{m'}}^{\otimes j}$.

Proof. This is immediate from the discussion in the above paragraph, together with Proposition 3.6. \square

For $1 \leq i \leq m$, we can define an element $\tilde{\Theta}_{i,m}$ of $E_{K_m'', \tau_{m'}}^{\otimes j}$ by the formula:

$$\tilde{\Theta}_{i,m} = \sum_{r_1, \dots, r_j} \Theta_{r_1, m'} \otimes \dots \otimes \Theta_{r_j, m'}$$

where the sum is over sequences r_1, \dots, r_j that sum to i . Note that for all s_1, \dots, s_j , the image of $\tilde{\Theta}_{i,m}$ under the map:

$$E_{K_m'', \tau_{m'}}^{\otimes j} \rightarrow \otimes_{i=1}^j \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)}$$

coincides with the image of $\theta_{i,s}$ under the map:

$$\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)} \rightarrow \otimes_i \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)}.$$

We therefore have:

Proposition 7.12. *The elements $\tilde{\Theta}_{i,m}$ of $E_{K_{m'},\tau_{m'}}^{\otimes j}$ map to strongly central elements of $H(G_m, K_m, \tilde{\kappa} \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M)$ under the map*

$$T : E_{K_{m'},\tau_{m'}}^{\otimes j} \rightarrow H(G_m, K_m, \tilde{\kappa} \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M).$$

Moreover, for $i < n$, the image of $\tilde{\Theta}_{i,m}$ is supported away from double cosets of the form $K_m z_{m,m}^r K_m$, whereas the image of $\tilde{\Theta}_{m,m}$ is invertible, supported on $K_m z_{m,m} K_m$ and has the form $\phi \otimes 1$ for an element ϕ of $I_{z_{m,m}}(\tilde{\kappa})$.

Proof. It suffices to show this is true after inverting ℓ , or even after tensoring with $\overline{\mathcal{K}}$. Indeed, we will show that the action of $\tilde{\Theta}_{i,m}$ on $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M$ coincides with that of an element x of the center of $\text{Rep}_{\overline{\mathcal{K}}}(G_m)$. Specifying such an x amounts to specifying, for each s_1, \dots, s_j , and corresponding s , an element x_s of A_{M_s, π_s} , such that, whenever we have (M_s, π_s) inertially equivalent to (M_t, π_t) , the elements x_s and x_t coincide.

For each s , let x_s be the element of A_{M_s, π_s} that corresponds to $\theta_{i,s}$ under the isomorphism of A_{M_s, π_s} with $\overline{\mathcal{K}}[Z_s]^{W_{M_s}(s)}$. It is then clear that for all s , the action of $\tilde{\Theta}_{i,m}$ on the summand $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes (i_{\overline{\mathcal{P}}}^{\overline{G}} \otimes_i \text{St}_{s_i})$ of $(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M) \otimes \overline{\mathcal{K}}$ coincides with that of x_s . It thus suffices to show that x_s coincides with x_t whenever (M_s, π_s) is inertially equivalent to (M_t, π_t) . For such a pair s, t , there exists a w in $W(\overline{G}_m)$ such that $t = wsw^{-1}$; then $M_t = wM_s w^{-1}$, $Z_t = wZ_s w^{-1}$, and conjugation by w induces an isomorphism of $\overline{\mathcal{K}}[Z_t]$ with $\overline{\mathcal{K}}[Z_s]$ that takes $\theta_{i,t}$ to $\theta_{i,s}$. It follows that the images of the $\tilde{\Theta}_{i,m}$ are strongly central.

As for the support of the images of $\tilde{\Theta}_{i,m}$ for $i < m$, note that when considered as an element of $H(M, K_M, \tilde{\kappa}_M \otimes \overline{\mathcal{P}}_M)$, each $\tilde{\Theta}_{i,m}$ with $i < m$ is supported away from double cosets of the form $K_m z_{m,m}^r K_m$. The map from $H(M, K_M, \tilde{\kappa}_M \otimes \overline{\mathcal{P}}_M)$ to $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M)$ is not support-preserving, but it suffices to show that it takes elements supported away from $K_M z_{m,m}^r K_M$ to elements supported away from $K_m z_{m,m}^r K_m$. This is an immediate consequence of the fact that the elements $z_{m,m}^r$ are positive and normalize K_M .

Finally, $\tilde{\Theta}_{m,m}$ is the tensor product $\Theta_{m',m'}^{\otimes j}$; in particular, it has support on the positive element $z_{m,m}$ of M . By the inductive nature of our construction $\Theta_{m',m'}$ has the form $\phi' \otimes 1$; it follows that the image of $\Theta_{m',m'}^{\otimes j}$ in $H(G_m, K_m, \tilde{\kappa}_m \otimes \overline{\mathcal{P}}_M)$ has the form $\psi \otimes 1$ for some ψ intertwining $\tilde{\kappa}_m$. Applying the isomorphism of this Hecke algebra with $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}} \overline{\mathcal{P}}_M)$ shows that the image of $\tilde{\Theta}_{m,m}$ in this subalgebra has the form $\phi \otimes 1$ as claimed. \square

The construction of the elements $\Theta_{i,m}$ as in Theorem 7.3 is now straightforward. The elements $\tilde{\Theta}_{i,m}$ are strongly central and we can thus consider their images $\overline{\Theta}_{i,m}$ in $H(G_m, K_m, \tilde{\kappa}_m \otimes \overline{\mathcal{P}}_M)$. Our results on the support of $\tilde{\Theta}_{i,m}$ imply that each $\overline{\Theta}_{i,m}$ is in the image of $E_{K_m, \tau_m} / I^{\text{cusp}}$ in $H(G_m, K_m, \tilde{\kappa}_m \otimes \overline{\mathcal{P}}_M)$. We can thus consider each $\overline{\Theta}_{i,m}$ as an element of $E_{K_m, \tau_m} / I^{\text{cusp}}$. Also note that the elements $\overline{\Theta}_{i,m}$ have the property demanded by Theorem 7.3 for all *reducible* s , because the corresponding property holds for the elements $\tilde{\Theta}_{i,m}$. We are thus reduced to finding suitable lifts of the $\overline{\Theta}_{i,m}$ to elements of E_{K_m, τ_m} .

The element $\overline{\Theta}_{m,m}$ has the form $\phi \otimes 1$, for an intertwiner ϕ in $I_{z_{m,m}}(\tilde{\kappa})$, where 1 is the identity endomorphism of \mathcal{P}'_m ; it lifts to the element $\phi \otimes 1$ of $I_{z_{m,m}}(\tilde{\kappa}) \otimes \text{End}_{W^{(k)}[\overline{G}_m]}(\mathcal{P})$, and this is the unique lift satisfying the demands of Theorem 7.3.

For $i < m$, the element $\overline{\Theta}_{i,m}$ is supported away from $K_m z_{m,m}^r K_m$; as I^{cusp} is supported only on double cosets of this form, there is a unique $\Theta_{i,m}$ in E_{K_m, τ_m} supported away from $K_m z_{m,m}^r K_m$ that lifts $\overline{\Theta}_{i,m}$. Then $\Theta_{i,m}$ maps to zero in $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_s)$ for any irreducible s , and so the commutative diagram of Theorem 7.3 holds for such s . The proof of Theorem 7.3 is thus complete. \square

We can also use the above construction to establish a relationship between E_{K_m, τ_m} and $E_{K_{m'}, \tau_{m'}}$. In particular, we have the following:

Theorem 7.13. *Let $m' = \frac{m}{j}$ strictly precede m . Then there exists a unique map:*

$$f_m : E_{K_m, \tau_m} / I_m^{\text{cusp}} \rightarrow E_{K_{m'}, \tau_{m'}}^{\otimes j},$$

such that, for each sequence s_1, \dots, s_j with $s_i^{\text{reg}} = (s'_1)^{m'}$ for $1 \leq i \leq j$, the diagram:

$$\begin{array}{ccc} E_{K_m, \tau_m} & \rightarrow & E_{K_{m'}, \tau_{m'}}^{\otimes j} \\ \downarrow & & \downarrow \\ \overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)} & \rightarrow & \bigotimes_{i=1}^j \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)} \end{array}$$

commutes. (As usual, s is the conjugacy class in $\text{GL}_{\frac{n_0 m}{e_f}}(\mathbb{F}_{q^f})$ that decomposes into blocks s_1, \dots, s_j , the left hand vertical map is normalized by (M_s, π_s) , and the right hand vertical map is normalized by the collection of pairs (M_{s_i}, π_{s_i}) .) The bottom horizontal map is induced by the isomorphism of Z_s with the product of the Z_{s_i} .

The proof of Theorem 7.13 will proceed in several steps. We begin by establishing more precise results on strongly central elements of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \overline{\mathcal{P}}_M)$.

Lemma 7.14. *Let x be an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \overline{\mathcal{P}}_M)$, and suppose that for some $a \geq 0$, $\ell^a x$ lies in the image of*

$$T : E_{K_{m'}, \tau_{m'}}^{\otimes j} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \overline{\mathcal{P}}_M).$$

Then x also lies in the image of T .

Proof. We consider x and $\ell^a x$ as elements of $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \overline{\mathcal{P}}_M)$. If $\ell^a x = T(y)$, then, for some invertible z in $E_{K_{m'}, \tau_{m'}}^{\otimes j}$ with strictly positive support, we have $T(y) = (T^+ z)^{-r} T^+(z^r y)$, for any r such that $z^r y$ has strictly positive support. As $T^+(z)$ is a unit in $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \overline{\mathcal{P}}_M)$, the divisibility of $T(y)$ by ℓ^a implies the divisibility of $T^+(z^r y)$ by ℓ^a . But for any positive element g of M , the map T^+ is an isomorphism of $H(M, K_M, \tilde{\kappa}_M \otimes \overline{\mathcal{P}}_M)_{K_M g K_M}$ with $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \overline{\mathcal{P}}_M)_{K_M g K_M}$. In particular $z^r y$ is divisible by ℓ^a in $H(M, K_M, \tilde{\kappa}_M \otimes \overline{\mathcal{P}}_M)$ (which is equal to $E_{K_{m'}, \tau_{m'}}^{\otimes j}$). As z is invertible, y is also divisible by ℓ^a . \square

Proposition 7.15. *Any strongly central element x of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{\mathcal{P}}}^{\overline{G}_m} \overline{\mathcal{P}}_M)$ is contained in the image of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$.*

Proof. By Lemma 7.14 it suffices to prove this after inverting ℓ , or even after tensoring with $\overline{\mathcal{K}}$. After doing so, the claim is an immediate consequence of Proposition 7.11. \square

In light of this, it suffices to show that the map

$$Z(H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})) \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$$

is injective on strongly central elements, and that image under this map of the strongly central elements contains the image of the map:

$$E_{K_m, \tau_m} / I_m^{\text{cusp}} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}).$$

Then the map f_m simply takes an element of $E_{K_m, \tau_m} / I_m^{\text{cusp}}$ to the corresponding element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\overline{M}})$, when the latter is considered as a strongly central element of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})$. Such an element can then be considered as an element of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$ by the above proposition.

Once we have constructed the map f_m in this way, the commutativity of the diagram of Theorem 7.13 is easy to verify: first observe that given an x in E_{K_m, τ_m} , there exists an element x_s of A_{M_s, π_s} such that the action of x on the summand of $(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}) \otimes \overline{\mathcal{K}}$ corresponding to s coincides with that of x_s . Then $f_m(x)$ acts on the summand of $(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}) \otimes \overline{\mathcal{K}}$ corresponding to x via x_s , since the map

$$(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}) \otimes \overline{\mathcal{K}} \rightarrow (\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m}) \otimes \overline{\mathcal{K}}$$

is equivariant for the action of the Bernstein center. It then follows from Proposition 7.11 that, when we consider $f_m(x)$ as an element of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$, the action of $f_m(x)$ on the summand $(\text{c-Ind}_{K_M}^M \tilde{\kappa}_M \otimes \mathcal{P}_{\overline{M}}) \otimes \overline{\mathcal{K}}$ corresponding to s_1, \dots, s_j is given by the image of x_s in $\otimes_i \overline{\mathcal{K}}[Z_{s_i}]$. This is equivalent to the commutativity of the diagram.

The injectivity of the strongly central elements into $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$ is straightforward. It suffices to check this after tensoring with $\overline{\mathcal{K}}$. A strongly central element that annihilates $(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}) \otimes \overline{\mathcal{K}}$ arises from a sum of elements of A_{M_s, π_s} as s runs over reducible conjugacy classes. But for all such s , the corresponding summand of $(\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m}) \otimes \overline{\mathcal{K}}$ is a faithful A_{M_s, π_s} -module.

To understand the subalgebra of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_{\sigma_m})$ generated by the images of the strongly central elements, we begin by recalling some results from section 6. In particular, for each s , we have support-preserving isomorphisms:

$$H(G_m, K_{m, \overline{P}_s}, \tilde{\kappa}_{m, \overline{P}_s} \otimes \text{St}_{\overline{M}_s, s}) \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}_s}^{G_m} \text{St}_{\overline{M}_s, s}),$$

and an isomorphism of the former with a tensor product of Iwahori Hecke algebras that is also support preserving in a sense made precise in section 6 (see in particular the discussion preceding Lemma 6.8). We choose the isomorphism of $H(G_m, K_{m, \overline{P}_s}, \tilde{\kappa}_{m, \overline{P}_s} \otimes \text{St}_{\overline{M}_s, s})$ with a tensor product of affine Hecke algebras to be the one corresponding to the representation π_s of M_s . By Corollary 6.16, for each z in Z_s , the space V_z of central elements of $H(G_m, K_{m, \overline{P}_s}, \tilde{\kappa}_{m, \overline{P}_s} \otimes \text{St}_{\overline{M}_s, s})$ supported on the union of the double cosets $K_{m, \overline{P}_s} w z w' K_{m, \overline{P}_s}$ is one-dimensional. The isomorphism of the center of $H(G_m, K_{m, \overline{P}_s}, \tilde{\kappa}_{m, \overline{P}_s} \otimes \text{St}_{\overline{M}_s, s})$ with $\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)}$ allows us to identify V_z with a subspace of the latter.

Now if t is another element of \overline{G}_m conjugate to s by an element $w_{s,t}$ of $W(\overline{G}_m)$, then conjugation by $w_{s,t}$ induces an isomorphism of $\overline{\mathcal{K}}[Z_s]$ with $\overline{\mathcal{K}}[Z_t]$; under this

isomorphism the subspace V_z of $\overline{\mathcal{K}}[Z_s]$ is identified with the subspace $V_{w_{s,t}z w_{s,t}^{-1}}$ of $\overline{\mathcal{K}}[Z_t]$.

With these observations in hand, we are in a position to prove:

Proposition 7.16. *Let x be a strongly central element of $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}})$, and let y be the part of x supported on $K_m z K_m$, for some $z \in G_m$. Then y is also strongly central.*

Proof. It suffices to check this after tensoring with $\overline{\mathcal{K}}$. For each s , the action of x on the summand $\text{c-Ind}_{K_m''}^{G_m} \tilde{\kappa}_m'' \otimes \text{St}_{\overline{M},s}$ of $(\text{c-Ind}_{K_m''}^{G_m} \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}}) \otimes \overline{\mathcal{K}}$ is via an element x_s , which we may consider as an element of $\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)}$.

Write x_s as a sum of elements x_{s,z_i} in V_{z_i} for various z_i in Z_s , and let y_s be the sum of x_{s,z_i} for those i such that z_i lies in $K_m z K_m$. It suffices to show that the action of y on $\text{c-Ind}_{K_m''}^{G_m} \tilde{\kappa}_m'' \otimes \text{St}_{\overline{M},s}$ agrees with that of y_s for all s , and that conjugation by $w_{s,t}$ sends y_s to y_t for all conjugate pairs s, t . The latter is immediate from the fact that $w_{s,t}$ lies in K_m .

For the former, note that we have a surjection:

$$\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa} \otimes i_{\overline{P}_s}^{\overline{G}_m} \text{St}_{\overline{M},s} \rightarrow \text{c-Ind}_{K_m}^{G_m} \tilde{\kappa} \otimes i_{\overline{P}}^{\overline{G}_m} \text{St}_{\overline{M},s} \cong \text{c-Ind}_{K_m''}^{G_m} \tilde{\kappa} \otimes \text{St}_{\overline{M},s}.$$

This surjection is support-preserving and equivariant for the action of $\overline{\mathcal{K}}[Z_s]^{W_{\overline{M}_s}(s)}$. Moreover, when considered as an endomorphism of the left-hand side, y_s is the part of x_s that is supported on $K_m z K_m$. Since the actions of x_s and x coincide on the right hand side, and y_s is the part of x_s supported on $K_m z K_m$, it follows that the actions of y_s and y on the right hand side coincide. \square

Our next step will be to define an action of a certain finite rank local $W(k)$ -algebra on the subspace of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})$ supported on a given double coset $K_m z K_m$. For any $z \in Z_0$, we have an isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m} \cong I_z(\tilde{\kappa}_m) \otimes \text{End}_{W(k)[\overline{M}_z]}(r_{\overline{G}_m}^{\overline{P}_z} i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}).$$

For each z , let e_z be the primitive central idempotent of $W(k)[\overline{M}_z]$ corresponding to the block of $\text{Rep}_{W(k)}(\overline{M}_z)$ containing $r_{\overline{G}_m}^{\overline{P}_z} \mathcal{P}_{\sigma_m}$. Then e_z acts via the identity on $r_{\overline{G}_m}^{\overline{P}_z} i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}$, and this yields an action of the algebra E_z defined by $E_z = e_z Z(W(k)[\overline{M}_z])$ on $r_{\overline{G}_m}^{\overline{P}_z} i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}$. The isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m} \cong I_z(\tilde{\kappa}_m) \otimes \text{End}_{W(k)[\overline{M}_z]}(r_{\overline{G}_m}^{\overline{P}_z} i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})$$

then gives an action of E_z on $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m}$.

We can reinterpret this action via the direct sum decomposition:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m} \cong \bigoplus_{w, w'} H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}})_{K_m'' w z w' K_m''}$$

(where w, w' run over elements of $W(\overline{G}_m)$ such that the double cosets $K_m'' w z w' K_m''$ partition $K_m z K_m$) as follows: for each such w, w' , we have an isomorphism:

$$H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}})_{K_m'' w z w' K_m''} \cong I_{w z w'}(\tilde{\kappa}) \otimes I_{w z w'}(\mathcal{P}_{\overline{M}}).$$

Moreover, $I_{w z w'}(\mathcal{P}_{\overline{M}})$ is, by definition, the space

$$\text{Hom}_{\tilde{P} \cap (w z w')^{-1} \tilde{P}(w z w')}(\mathcal{P}_{\overline{M}}, \mathcal{P}_{\overline{M}}^{w z w'}).$$

Note that $\mathcal{P}_{\overline{M}}$ is inflated from a representation of \overline{M} . Let u be an element of $(w')^{-1}\tilde{U}_z w'$, where \tilde{U}_z is the preimage of the unipotent radical \overline{U}_z of \overline{P}_z under “reduction mod ϖ_E ”. Then $zw'u(zw')^{-1}$ reduces to the identity mod ϖ_E , and therefore so does $(wzw'u)(wzw')^{-1}$. It follows that any map from $\mathcal{P}_{\overline{M}}$ to $\mathcal{P}_{\overline{M}}^{wzw'}$ factors through the $(w')^{-1}\overline{U}_z w' \cap \overline{M}$ -invariants of $\mathcal{P}_{\overline{M}}$. This space of invariants may be considered as a representation of the Levi subgroup $\overline{M} \cap (w')^{-1}\overline{M}_z w'$ of \overline{G}_m .

We next observe:

Lemma 7.17. *Let \overline{M}_1 be a Levi subgroup of \overline{G}_m , and $\overline{P}_2 = \overline{M}_2 \overline{U}_2$ be a parabolic subgroup of \overline{M}_1 . There exists a unique map $Z(W(k)[\overline{M}_1]) \rightarrow Z(W(k)[\overline{M}_2])$ such that for all $W(k)[\overline{M}_1]$ -modules Π , the diagram:*

$$\begin{array}{ccc} Z(W(k)[\overline{M}_1]) & \rightarrow & Z(W(k)[\overline{M}_2]) \\ \downarrow & & \downarrow \\ \text{End}_{W(k)[\overline{M}_1]}(\Pi) & \rightarrow & \text{End}_{W(k)[\overline{M}_2]}(r_{\overline{M}_1}^{\overline{P}_2} \Pi) \end{array}$$

commutes.

Proof. Note first that if Π is faithfully projective over \overline{M}_1 , then $r_{\overline{M}_1}^{\overline{P}_2} \Pi$ is faithfully projective over \overline{M}_2 , and so the vertical maps in the above diagram are isomorphisms. We may thus define the map in question to be the unique map that makes this diagram commute for that particular choice of Π . We can then verify that the diagram commutes for an arbitrary Π' by resolving Π' by direct sums of copies of Π . \square

We then have a sequence of maps:

$$e_z Z(W(k)[M_z]) \cong e_{(w')^{-1}zw'} Z(W(k)[(w')^{-1}M_z w']) \rightarrow Z(W(k)[M \cap (w')^{-1}M_z w'])$$

where the first map is conjugation by w' and the second is from the lemma. The right-hand group algebra acts on the $(w')^{-1}\overline{U}_z w' \cap \overline{M}$ -invariants of $\mathcal{P}_{\overline{M}}$, and thus on $I_{wzw'}(\tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}})$. Moreover, this action makes the inclusion

$$H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \mathcal{P}_{\overline{M}})_{K_m'' wzw' K_m''} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m}$$

equivariant for E_z .

Proposition 7.18. *Let x be strongly central in $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}})_{K_m z K_m}$. Then so is αx for any α in E_z .*

Proof. For all s , let x_s be the element of A_{M_s, π_s} whose action on the summand of $[\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes i_{\overline{P}}^{\overline{G}_m} \mathcal{P}_{\overline{M}}] \otimes \overline{\mathcal{K}}$ corresponding to s coincides with x . We may assume that x_s lies in V_z , and that x_t is zero for all t not conjugate to s in \overline{G}_m . (Any x supported on $K_m z K_m$ is a sum of elements that have this property for various s and z' with $K_m z' K_m = K_m z K_m$.)

The element x_s corresponds to an element of $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \text{St}_{\overline{M}, s})$ supported on double cosets of the form $K_m'' wzw' K_m''$, where w and w' lie in $W_{\overline{M}_s}(s)$. For any such w, w' , the element α of E_z acts on $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \text{St}_{\overline{M}, s})_{K_m'' wzw' K_m''}$ via the scalar $c \in \overline{\mathcal{K}}$ such that the element $(w')^{-1}\alpha(w')$ of $E_{(w')z(w')^{-1}}$ acts on $r_{\overline{M}}^{(w')^{-1}\overline{P}_z w' \cap M} \text{St}_{\overline{M}, s}$ via c .

Now if t is conjugate to s , via $w_{s,t} \in W(\overline{G}_m)$, then x_t lies in the subspace $V_{w_{s,t}zw_{s,t}^{-1}}$. For any w, w' in $W_{\overline{M}_t}(t)$, the argument of the previous paragraph shows that α acts on $H(G_m, K_m'', \tilde{\kappa}_m'' \otimes \text{St}_{\overline{M}_t})_{K_m'' w w_{s,t} z w_{s,t}^{-1} w' K_m''}$ via c . It follows that for x satisfying our assumptions, $\alpha x = cx$ and is thus strongly central. \square

With this in hand it is straightforward to understand the image of the strongly central elements in $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$. If we let $R_{K_m z K_m}$ be the space of strongly central elements of $H(G_m, K_m, \tilde{\kappa}_m \otimes i_{\overline{P}} \overline{G}_m)$ supported on $K_m z K_m$, then the map:

$$R_{K_m z K_m} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$$

is E_z -equivariant. Moreover, the subspace of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$ consisting of elements in the image of E_{K_m, τ_m} is a cyclic E_z -module. (This image is all of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$, which is free of rank one over E_z , if z is not a power of $z_{m,m}$; if z does have the form $z_{m,m}^r$ then the image is generated over E_z by $\Theta_{m,m}^r$.)

It is thus clear that when z is a power of $z_{m,m}$, the image of $R_{K_m z K_m}$ contains the part of the image of E_{K_m, τ_m} supported on $K_m z K_m$. When z is not such a power, it suffices to show that the image of $R_{K_m z K_m}$ contains an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$ that generates this space over E_z .

Let L be the image of \mathcal{P}'_m in $\text{St}_{(s')^m}$; the map from \mathcal{P}' to L gives a map:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m) \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes L).$$

The isomorphism of $L \otimes \mathcal{K}$ with $\text{St}_{(s')^m}$ gives us a map:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes L) \rightarrow \mathcal{K}[Z_0]^{W_{\overline{M}}((s')^m)},$$

and the image of this map contains the subalgebra $W(k)[Z_0]^{W_{\overline{M}}((s')^m)}$ generated by the $\Theta_{i,m}$. Conversely, we have:

Lemma 7.19. *The image of $H(G_m, K_m, \tilde{\kappa}_m \otimes L)$ in $\mathcal{K}[Z_0]^{W_{\overline{M}}((s')^m)}$ is contained in $W(k)[Z_0]^{W_{\overline{M}}((s')^m)}$.*

Proof. Suppose there is an α in $H(G_m, K_m, \tilde{\kappa}_m \otimes L)$ whose image does not lie in $W(k)[Z_0]^{W_{\overline{M}}((s')^m)}$. Then there exists a finite extension \mathcal{K}' of \mathcal{K} , and a map

$$\phi : W(k)[Z_0]^{W_{\overline{M}}((s')^m)} \rightarrow \mathcal{O}'$$

whose tensor product with \mathcal{K}' takes α to an element of \mathcal{K}' that does not lie in \mathcal{O}' . Such a map corresponds to a twist of $\pi_{(s')^m}$ by an integral unramified character.

Let Π be an irreducible quotient of the representation $\text{c-Ind}_{K_m}^{G_m} \tilde{\kappa}_m \otimes \text{St}_{(s')^m}$ on which $H(G_m, K_m, \tilde{\kappa}_m \otimes \text{St}_{(s')^m})$ acts via ϕ . Then Π is integral, as the cuspidal support of Π is an integral twist of $\pi_{(s')^m}$. In particular there is an ℓ -adically separated G_m -stable lattice L_Π in Π , and this lattice is preserved by $H(G_m, K_m, \tilde{\kappa}_m \otimes L)$. This contradicts the fact that α acts on Π by a scalar that is not a unit. \square

Moreover, $H(G_m, K_m, \tilde{\kappa}_m \otimes L)_{K_m z K_m}$ is free of rank one over $W(k)$. There is thus a polynomial in the $\Theta_{i,m}$ (with coefficients in $W(k)$) that is an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$ whose image in $H(G_m, K_m, \tilde{\kappa}_m \otimes L)_{K_m z K_m}$ generates the latter over $W(k)$. It follows that the map:

$$R_{K_m z K_m} \rightarrow H(G_m, K_m, \tilde{\kappa}_m \otimes L)_{K_m z K_m}$$

is surjective. This map factors through $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$. In particular (as E_z is local with maximal ideal \mathfrak{m}_z), the image of $R_{K_m z K_m}$ in $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$ cannot be contained in $\mathfrak{m}_z H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$, and must thus be all of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)_{K_m z K_m}$ when z is not a power of $z_{m,m}$.

We have thus shown that the subalgebra of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$ consisting of the images of strongly central elements contains the image of E_{K_m, τ_m} , completing the proof of Theorem 7.13. \square

The proof of Theorem 7.13 yields the following further observation:

Proposition 7.20. *Let y be a strongly central element of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$. There exist elements x_1, x_2 of E_{K_m, τ_m} , and a nonnegative integer a , such that:*

- x_2 is supported on double cosets of the form $K_m z_{m,m}^r K_m$, and
- we have $f_m(x_2) = \ell^a (y - f_m(x_1))$.

Proof. We may consider y as an element of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$, since the proof of Theorem 7.13 identifies the strongly central elements of $E_{K_{m'}, \tau_{m'}}^{\otimes j}$ with a subalgebra of $H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}'_m)$, in a manner compatible with the map f_m . Let y_1 be the part of y supported away from double cosets of the form $K_m z_{m,m}^r K_m$, and let $y_2 = y - y_1$. Proposition 7.9 shows that $y_1 = f_m(x_1)$ for some x_1 .

On the other hand, y_2 is supported on double cosets of the form $K_m z_{m,m} K_m$, and we have a support-preserving isomorphism:

$$H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}_m) / I^{\text{cusp}} \otimes \mathcal{K} \cong H(G_m, K_m, \tilde{\kappa}_m \otimes \mathcal{P}') \otimes \mathcal{K}.$$

There is thus an x_2 such that, for some a , $f_m(x_2) = \ell^a y_2$, and the result follows. \square

We conclude this section by giving a complete description of E_{K_m, τ_m} for small m ; that is, for $m < \ell$. There are two cases to consider; in the first, $m = 1$, and in the second $m = e_{q^f} > 1$.

When $m = 1$, the element $\Theta_{1,1}^r$ of E_{K_1, τ_1} generates $(E_{K_1, \tau_1})_{K_1 z_{1,1}^r K_1}$ as an E_{σ_1} -module for all r , and hence we have

$$E_{K_1, \tau_1} = E_{\sigma_1}[\Theta_{1,1}^{\pm 1}].$$

This case was already studied by Dat in [D3]; in particular Dat shows that E_{σ_1} is a universal deformation ring of σ_1 , and that, after completing at any maximal ideal of characteristic ℓ , E_{K_1, τ_1} becomes the universal deformation ring of the corresponding supercuspidal representation.

When $m > 1$ but $m < \ell$, then any $s \neq (s'_1)^m$ with $s^{\text{reg}} = (s'_1)^m$ is irreducible. We have an ideal I_0 of E_{σ_m} that is the kernel of the action of E_{σ_m} on $\text{St}_{(s'_1)^m}$.

Proposition 7.21. *When $1 < m < \ell$, we have an isomorphism:*

$$E_{K_m, \tau_m} \cong E_{\sigma_m}[\Theta_{1,m}, \dots, \Theta_{m,m}^{\pm 1}] / \langle \Theta_{1,m}, \dots, \Theta_{m-1,m} \rangle \cdot I^{\text{cusp}}.$$

Proof. As E_{σ} and the $\Theta_{i,m}$ are contained in $E_{K, \tau}$, we have an inclusion:

$$E_{\sigma}[\Theta_{1,m}, \dots, \Theta_{m,m}^{\pm 1}] \hookrightarrow E_{K, \tau}.$$

It is easy to see that $\langle \Theta_{1,m}, \dots, \Theta_{m-1,m} \rangle$ map to zero in $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ for $s \neq (s'_1)^m$, so that $\langle \Theta_{1,m}, \dots, \Theta_{m-1,m} \rangle \cdot I^{\text{cusp}}$ is in the kernel of the map to $E_{K, \tau}$. That this is precisely the kernel, and that the resulting map is surjective, follows by considering the elements supported on KzK for each z in $Z_{0,m}$. \square

Remark 7.22. When $\ell > n$, and σ is not supercuspidal, Paige gives an explicit description of E_σ as a $W(k)$ -algebra in his forthcoming thesis [Pg]. The above proposition thus gives a complete description of $E_{K,\tau}$ in this case.

8. FINITENESS RESULTS

Our next goal is to establish fundamental finiteness results for $\mathcal{P}_{K,\tau}$. In order to do so it will be necessary to work integrally with lattices inside a generic pseudo-type $(K, \tilde{\kappa} \otimes \text{St}_s)$. Choose a finite extension \mathcal{K}' of \mathcal{K} such that St_s is defined over \mathcal{K}' , and let \mathcal{O}' be the ring of integers in \mathcal{K}' . We can then consider \mathcal{O}' -lattices L_s inside St_s , and consider the “integral generic pseudo-type” $(K, \tilde{\kappa} \otimes L_s)$, and try to determine the structure of $\text{c-Ind}_K^G \tilde{\kappa} \otimes L$ as a module over $H(G, K, \tilde{\kappa} \otimes L_s)$.

There will be two lattices in St_s of particular interest to us. We construct the first of these as follows: denote by \overline{M} the Levi subgroup \overline{M}_s of \overline{G} . The representation $\text{St}_{\overline{M},s}$ is irreducible, cuspidal and defined over \mathcal{K}' , and remains irreducible when reduced mod ℓ . There is thus a \overline{M} -stable \mathcal{O}' -lattice $L_{\overline{M}}$ in $\text{St}_{\overline{M},s}$, and such an $L_{\overline{M}}$ is unique up to homothety. Then $i_{\overline{P}}^{\overline{G}} L_{\overline{M},s}$ is an \mathcal{O}' -lattice in I_s . Let L_s be the image of this lattice in St_s .

The second lattice we will make use of will be denoted L'_s , and is defined as follows: the representation St_s is a direct summand of $\mathcal{P}_\sigma \otimes_{W(k)} \mathcal{K}'$. Let L'_s be the image of $\mathcal{P}_\sigma \otimes_{W(k)} \mathcal{O}'$ under the projection to St_s ; this defines L'_s up to homothety. The lattice L'_s is the one that is of interest to us in applications, but is more complicated; we will study it via its relationship with L_s . Note that there exist a, b such that $\ell^a L_s \subset L'_s \subset \ell^b L_s$. Let τ_{L_s} and $\tau_{L'_s}$ denote the representations $\tilde{\kappa} \otimes L_s$, and $\tilde{\kappa} \otimes L'_s$.

The pair (K, τ_{L_s}) is not difficult to understand; indeed, the arguments of section 6 apply. In particular, consider the pair $(K_M, \tilde{\kappa}_M \otimes L_{\overline{M}})$, where M, K_M , and $\tilde{\kappa}_M$ are as in section 6. It follows from [V2], IV.2.5 and IV.2.6 that the pair $(K'', \tilde{\kappa}'' \otimes L_{\overline{M}})$ is a G -cover (as an \mathcal{O}' -module) of $(K_M, \tilde{\kappa}_M \otimes L_{\overline{M}})$, and that the center of $H(G, K'', \tilde{\kappa}'' \otimes L_{\overline{M}})$ is isomorphic to $\mathcal{O}'[Z_s]^{W_{\overline{M}}(s)}$. (Indeed, a choice of a compatible family of cuspidals for the tower of types that contains (K, τ) gives rise to explicit isomorphisms:

$$\begin{aligned} H(M, K_M, \tilde{\kappa}_M \otimes L_{\overline{M}}) &\cong \mathcal{O}'[Z_s] \\ Z(H(G, K'', \tilde{\kappa}'' \otimes L_{\overline{M}})) &\cong \mathcal{O}'[Z_s]^{W_{\overline{M}}(s)}. \end{aligned}$$

Henceforth we fix such a choice.) The intertwining calculations of section 6 give rise to a support preserving isomorphism $H(G, K'', \tilde{\kappa}'' \otimes L_{\overline{M}})$ with $H(G, K_{\overline{P}}, \tilde{\kappa}_{\overline{P}} \otimes L_{\overline{M}})$, and an isomorphism of the latter with $H(G, K, \tilde{\kappa} \otimes i_{\overline{P}}^{\overline{G}} L_{\overline{M}})$.

We now observe:

Lemma 8.1. *Let x be a central element of $H(G, K, \tilde{\kappa} \otimes i_{\overline{P}}^{\overline{G}} L_{\overline{M}})$. Then x descends to an endomorphism of $\text{c-Ind}_K^G \tilde{\kappa} \otimes L_s$ via the surjection of $i_{\overline{P}}^{\overline{G}} L_{\overline{M}}$ onto L_s .*

Proof. The results of section 6 show that this holds after inverting ℓ . We have a surjection:

$$\text{c-Ind}_K^G \tilde{\kappa} \otimes (i_{\overline{P}}^{\overline{G}} L_{\overline{M}}) \rightarrow \text{c-Ind}_K^G \tilde{\kappa} \otimes L_s,$$

and it suffices to show that x preserves the kernel of this surjection. But as this holds after inverting ℓ , and both the left-hand and right-hand sides are ℓ -torsion free, the result follows. \square

We next turn to questions of admissibility:

Lemma 8.2. *The module $\mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa} \otimes L_M$ is an admissible $H(M, K_M, \tilde{\kappa}_M \otimes L_M)$ -module.*

Proof. Let π be an $\mathcal{O}'[M]$ -module such that $\pi \otimes_{\mathcal{O}'} K'$ is absolutely irreducible and such that the restriction of π to K_M admits a nonzero map from $\tilde{\kappa} \otimes L_M$. Then $\mathrm{Hom}_{\mathcal{O}'[K_M]}(\tilde{\kappa} \otimes L_M, \pi)$ is a free \mathcal{O}' -module of rank one. Consider the $\mathcal{O}'[M]$ -module $\pi \otimes_{\mathcal{O}'} \mathcal{O}'[M/M_0]$, on which M acts on $\mathcal{O}'[M/M_0]$ via the natural character $M \rightarrow \mathcal{O}'[M/M_0]$. We have

$$\mathrm{Hom}_{\mathcal{O}'[K_M]}(\tilde{\kappa} \otimes L_M, \pi) \cong \mathcal{O}'[M/M_0],$$

and $H(M, K_M, \tau) = \mathcal{O}'[Z]$ acts on the right hand side via the inclusion of $\mathcal{O}'[Z]$ in $\mathcal{O}'[M/M_0]$. This yields an isomorphism:

$$(\mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa} \otimes L_M) \otimes_{\mathcal{O}'[Z_s]} \mathcal{O}'[M/M_0] \cong \pi \otimes_{\mathcal{O}'} \mathcal{O}'[M/M_0].$$

In particular the left hand side is admissible over $\mathcal{O}'[M/M_0]$, and so $\mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa} \otimes L_M$ is admissible over $\mathcal{O}'[Z_s]$. \square

Lemma 8.3. *Let R be commutative $W(k)$ -algebra, let $P = MU$ be a parabolic subgroup of G , and let π be an admissible $R[M]$ -module such that for any parabolic subgroup $P' = M'U'$ of M , $r_M^{P'}\pi$ is admissible as an $R[M']$ -module. Then $i_P^G M$ is an admissible $R[G]$ -module, and, for any parabolic subgroup $P'' = M''U''$ of G , $r_G^{P''} i_P^G \pi$ is an admissible $R[M'']$ -module.*

Proof. This is an immediate consequence of Bernstein-Zelevinski's filtration of the composite functor $r_G^{P''} i_P^G$ ([BZ], 2.12) together with the fact that parabolic induction takes admissible representations to admissible representations. \square

With these results in hand, we can show:

Proposition 8.4. *The module $\mathrm{c}\text{-Ind}_K^G \tau_{L_s}$ is admissible over $H(G, K, \tau_{L_s})[G]$. More generally, for any parabolic $P' = M'U'$ in G , the module $r_G^{P'} \mathrm{c}\text{-Ind}_K^G \tau_{L_s}$ is admissible over $H(G, K, \tau_{L_s})[M']$.*

Proof. The module $\mathrm{c}\text{-Ind}_K^G \tau_{L_s}$ is a quotient of $\mathrm{c}\text{-Ind}_K^G \tilde{\kappa} \otimes i_P^{\overline{G}} L_{\overline{M}}$, and the latter is isomorphic to $\mathrm{c}\text{-Ind}_{K''}^G \tilde{\kappa}'' \otimes L_{\overline{M}}$. It thus suffices to show that $r_G^{P'} \mathrm{c}\text{-Ind}_{K''}^G \tilde{\kappa}'' \otimes L_{\overline{M}}$ is admissible over the center $\mathcal{O}'[Z]^{W_{\overline{M}}(s)}$ of $H(G, K'', \tilde{\kappa}'' \otimes L_{\overline{M}})$.

As $\tilde{\kappa}_M \otimes L_{\overline{M}}$ is a lattice in a maximal distinguished cuspidal M -type, the $W(k)[M]$ -module $\mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa}_M \otimes L_{\overline{M}}$ is admissible and cuspidal over the Hecke algebra $H(M, K_M, \tilde{\kappa}_M \otimes L_{\overline{M}})$, and the latter is isomorphic to $\mathcal{O}'[Z]$. It follows that $i_P^G \mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa}_M \otimes L_{\overline{M}}$ is an admissible $\mathcal{O}'[Z]$ -module, as is $r_G^{P'} i_P^G \mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa}_M \otimes L_{\overline{M}}$ for any P' .

On the other hand, the relationship between G -covers and parabolic induction yields an isomorphism:

$$(\mathrm{c}\text{-Ind}_K^G \tilde{\kappa}'' \otimes L_{\overline{M}}) \otimes_{\mathcal{O}'[Z]^{W_{\overline{M}}(s)}} \mathcal{O}'[Z] \cong i_P^G \mathrm{c}\text{-Ind}_{K_M}^M \tilde{\kappa}_M \otimes L_{\overline{M}}.$$

As $\mathcal{O}'[Z_s]$ is a finitely generated, free $\mathcal{O}'[Z_s]^{W_{\overline{M}}(s)}$ -module the result is now immediate. \square

Proposition 8.5. *Every irreducible representation in the block $\text{Rep}_{\overline{K}}(G)_{M,\pi}$ has mod ℓ inertial supercuspidal support equal to that of a cuspidal representation of type (K, τ) .*

Proof. It follows directly from the definition of mod ℓ inertial supercuspidal support that every irreducible representation in the block $\text{Rep}_{\overline{K}}(G)_{M,\pi}$ has the same mod ℓ inertial supercuspidal support. Choose an integral irreducible representation π' in this block. Then we have a map $\text{c-Ind}_K^G \tau_{L_s} \rightarrow \pi'$ whose image is an \mathcal{O}' -lattice in π' . Tensoring with k , we find that every simple $W(k)$ -subquotient of π killed by ℓ is also a subquotient of $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$, where \overline{L}_s is $L_s \otimes_{\mathcal{O}'} k$. In particular it suffices to show that every irreducible subquotient of $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$ has supercuspidal support given by (K, τ) . Note that, as τ is equal to $\kappa \otimes \sigma$, and σ is a subquotient of \overline{L}_s , $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$ has a subquotient isomorphic to $\text{c-Ind}_K^G \tau$.

On the other hand, Vigneras has shown ([V2], IV.6.2) that the subcategory of $\text{Rep}_k(G)$ consisting of all representations with supercuspidal support given by (K, τ) is a block of $\text{Rep}_k(G)$. It thus suffices to show that $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$ is contained in a single block of $\text{Rep}_k(G)$. But the endomorphism ring of $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$ is simply $H(G, K, \tau_{L_s}) \otimes_{\mathcal{O}'} k$, and is therefore equal to $k[Z]^{W_{\overline{M}}(s)}$. In particular this endomorphism ring is a domain, so $\text{c-Ind}_K^G \kappa \otimes \overline{L}_s$ is a domain and the result follows. \square

We now compare $\text{c-Ind}_K^G \tau_{L_s}$ and $\text{c-Ind}_K^G \tau_{L'_s}$. The inclusions $\ell^a L_s \subset L'_s \subset \ell^b L$ give rise to inclusions: $\ell^a \text{c-Ind}_K^G \tau_{L_s} \subset \text{c-Ind}_K^G \tau_{L'_s} \subset \ell^b \text{c-Ind}_K^G \tau_{L_s}$.

The endomorphism ring $E_{K,\tau}$ of $\mathcal{P}_{K,\tau}$ preserves the factor $\text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$ of $\mathcal{P}_{K,\tau} \otimes \overline{K}$, and hence preserves the image of $\mathcal{P}_{K,\tau} \otimes \mathcal{O}'$ in $\text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$. This image is equal to $\text{c-Ind}_K^G \tau_{L'_s}$. In particular we obtain a map of $E_{K,\tau} \otimes \mathcal{O}'$ into $H(G, K, \tau_{L'_s})$.

For some m , we have $(K, \tau) = (K_m, \tau_m)$, where K_m and τ_m are as in section 7. Our choice of compatible family of cuspidals in section 7 identifies $H(G, K, \tilde{\kappa} \otimes \text{St}_s)$ with $\overline{K}[Z_s]^{W_{\overline{M}}(s)}$. Under this identification $H(G, K, \tau_{L_s})$ is the subalgebra $\mathcal{O}'[Z_s]^{W_{\overline{M}}(s)}$, and $H(G, K, \tau_{L'_s})$ is another, yet-to-be-determined \mathcal{O}' -subalgebra that contains $E_{K,\tau} \otimes \mathcal{O}'$.

We observe that as a subalgebra of $\mathcal{O}'[Z_s]$, $H(G, K, \tau_{L'_s})$ contains the images of the elements $\Theta_{i,m}$ of $C_{K,\tau}$, as well as the image of $\Theta_{m,m}^{-1}$. These map to elements $\theta_{i,s}$ of $W(k)[Z_s]^{W_{\overline{M}}(s)}$. We observe:

Proposition 8.6. *The algebra $W(k)[Z_s]^{W_{\overline{M}}(s)}$ is a finitely generated module over $W(k)[\theta_{1,s}, \dots, \theta_{m,s}, \theta_{m,s}^{-1}]$.*

Proof. Let s_1, \dots, s_r be the irreducible constituents of s , and let z_1, \dots, z_r be the elements of Z_s such that, when considered as an element of $\text{GL}_{\frac{n}{ef}}(E)$, z_i is scalar with entries ϖ_E on the block of Z_s corresponding to s_i , and the identity on all other blocks. Let d_i be the degree of s_i over \mathbb{F}_{q^f} , and let d be the degree of s' over \mathbb{F}_{q^f} , where $s^{\text{reg}} = (s')^m$. Then, by definition, we have:

$$\theta_{i,s} = \sum_S \prod_{j \in S} z_j,$$

where S runs over those subsets of $1, \dots, r$ such that

$$\sum_{j \in S} d_j = di.$$

Now consider the polynomial

$$P(t) = \prod_{i=1}^r (t^{\frac{d_i}{d}} + z_i).$$

For $1 \leq i \leq r$, the coefficient of t^{r-i} in $P(t)$ is $\theta_{i,s}$. It follows that the elements $(-z_i)^{\frac{d}{d_i}}$ are integral over $W(k)[\theta_{1,s}, \dots, \theta_{m,s}]$, and so the elements z_i themselves are. As $W(k)[Z_s]$ is generated by the z_i , together with $\theta_{m,s}^{-1}$, it follows that $W(k)[Z_s]$ is integral, and hence finitely generated as a module, over $W(k)[\theta_{1,s}, \dots, \theta_{m,s}, \theta_{m,s}^{-1}]$, and the result is immediate. \square

We now show:

Proposition 8.7. *The module $\text{c-Ind}_K^G \tau_{L'_s}$ is an admissible $E_{K,\tau}[G]$ -module. Moreover, for any $P' = M'U'$ in G , $r_G^{P'} \text{c-Ind}_K^G \tau_{L'}$ is admissible as a $E_{K,\tau}[M']$ -module.*

Proof. The module $r_G^{P'} \text{c-Ind}_K^G \tau_{L'_s}$ is admissible over $H(G, K, \tau_{L'_s})$, which we have identified with $W(k)[Z_s]^{W_{\overline{M}(s)}}$. It is thus also admissible over $\mathcal{O}'[\theta_{1,s}, \dots, \theta_{m,s}, \theta_{m,s}^{-1}]$, by Proposition 8.6. The $\theta_{i,s}$ preserve both $\text{c-Ind}_K^G \tau_{L'_s}$ and $\text{c-Ind}_K^G \tau_{L'_s}$, and any embedding of the latter in the former is equivariant for the $\theta_{i,s}$. Fix such an embedding; this yields an embedding of $r_G^{P'} \text{c-Ind}_K^G \tau_{L'_s}$ in $r_G^{P'} \text{c-Ind}_K^G \tau_{L'_s}$ that is compatible with the action of the elements $\theta_{i,s}$. As the former is admissible over $\mathcal{O}'[\theta_{1,s}, \dots, \theta_{m,s}, \theta_{m,s}^{-1}]$, the latter must be as well, and the result follows. \square

We now return to the study of $\mathcal{P}_{K,\tau}$. We choose \mathcal{K}' (and by extension \mathcal{O}' sufficiently large that every map from E_σ to $\overline{\mathcal{K}}$ has image contained in \mathcal{O}' . Then St_s is defined over \mathcal{K}' for every s . Moreover, the embeddings:

$$\mathcal{P}_{K,\tau} \otimes \overline{\mathcal{K}} \hookrightarrow \bigoplus_s \text{c-Ind}_K^G \tilde{\kappa} \otimes \text{St}_s$$

$$E_{K,\tau} \otimes \overline{\mathcal{K}} \hookrightarrow \prod_s H(G, K, \tilde{\kappa} \otimes \text{St}_s)$$

factor through embeddings:

$$\mathcal{P}_{K,\tau} \otimes \mathcal{O}' \hookrightarrow \bigoplus_s \text{c-Ind}_K^G \tau_{L'_s}$$

$$E_{K,\tau} \otimes \mathcal{O}' \hookrightarrow \prod_s H(G, K, \tau_{L'_s}).$$

We thus have:

Theorem 8.8. *For any parabolic subgroup $P' = M'U'$ of G , $r_G^{P'} \mathcal{P}_{K,\tau}$ is an admissible $C_{K,\tau}[M']$ -module (and hence also an admissible $E_{K,\tau}[M']$ -module.)*

Proof. For each s , $r_G^{P'} \text{c-Ind}_K^G \tau_{L'_s}$ is admissible over $C_{K,\tau} \otimes \mathcal{O}'$, and hence also over $C_{K,\tau}$. The result is thus immediate from the above decomposition. \square

Corollary 8.9. *The $W(k)$ -algebra $E_{K,\tau}$ is a finitely generated $C_{K,\tau}$ -module.*

9. STRUCTURE OF $\mathcal{P}_{(M,\pi)}$

We now turn to the study of the $W(k)[G]$ -module $\mathcal{P}_{(M,\pi)}$ for an inertial equivalence class (M, π) of irreducible cuspidal representations of M over k . Write π as a tensor product of irreducible cuspidal representations π_i of general linear groups $\mathrm{GL}_{n_i}(F)$, and for each i , let (K_i, τ_i) be a maximal distinguished cuspidal k -type contained in π_i .

Theorem 9.1. *Suppose $\ell > n$, and let R be the tensor product, over $W(k)$, of the rings E_{K_i, τ_i} for all i . Then $\mathcal{P}_{(M,\pi)}$ is an admissible $R[G]$ -module, and for any parabolic subgroup $P' = M'U'$ of G , $r_G^{P'} \mathcal{P}_{(M,\pi)}$ is an admissible $R[M']$ -module.*

Proof. This is an immediate consequence of Theorem 8.8 and Lemma 8.3. \square

Let (M', π') be the supercuspidal support of (M, π) ; then (M', π') is determined up to inertial equivalence.

Proposition 9.2. *Let Π be an irreducible supercuspidal $\overline{\mathcal{K}}$ -representation of a Levi subgroup M'' of G , and suppose that the mod ℓ supercuspidal support of the inertial equivalence class (M'', Π) is not equal to (M', π') . Then the idempotent $e_{M'', \Pi, \overline{\mathcal{K}}}$ of the Bernstein center of $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)$ annihilates $\mathcal{P}_{(M,\pi)} \otimes \overline{\mathcal{K}}$. Equivalently, every simple subquotient of $\mathcal{P}_{(M,\pi)} \otimes \overline{\mathcal{K}}$ has mod ℓ inertial supercuspidal support (M', π') .*

Proof. When the sequence $\{(K_i, \tau_i)\}$ consists of a single type (K, τ) , this follows immediately from Corollary 6.7 and Proposition 8.5. The general case follows by the additivity of supercuspidal support under parabolic induction. \square

We now give more precise results about the block decomposition of $\mathcal{P}_{(M,\pi)} \otimes \overline{\mathcal{K}}$. We have $\mathcal{P}_{M,\pi} = i_P^G[\mathcal{P}_{K_1, \tau_1} \otimes \cdots \otimes \mathcal{P}_{K_r, \tau_r}]$. Let $\tau_i = \kappa_i \otimes \sigma_i$ and fix an s_i such that σ_i is the cuspidal representation corresponding to the ℓ -regular part s'_i of s_i . We then have a map $\phi_i : E_{\sigma_i} \rightarrow \overline{\mathcal{K}}$, giving the action of E_{σ_i} on St_{s_i} . Let M_{i, s_i} be the Levi subgroup corresponding to this choice of s_i . Choose, for each (K_i, τ_i) a corresponding compatible system of cuspiduals in the sense of the discussion preceding Theorem 7.3, in such a way that if (K_i, τ_i) is equivalent to (K_j, τ_j) , then we use the same compatible system for both. This yields in particular a cuspidal representation π_{i, s_i} of M_{i, s_i} for each pair i, s_i .

Let $\vec{s} = \{s_1, \dots, s_r\}$ be a collection in which each, for each i , the ℓ -regular part of s_i is equal to s'_i . Let $M_{\vec{s}}$ be the product of the M_{s_i} ; it is a Levi subgroup of G . Let $\pi_{\vec{s}}$ be the tensor product of the π_{s_i} . Finally, consider the $\overline{\mathcal{K}}[G]$ -modules

$$(\mathcal{P}_{(M,\pi)})_{\vec{s}} = i_P^G \left(\bigotimes_i \mathrm{c}\text{-Ind}_{K_i}^{G_i} \tilde{\kappa}_i \otimes \mathrm{St}_{s_i} \right),$$

where $P_{\vec{s}}$ is a parabolic of G with Levi component $M_{\vec{s}}$. Then $(\mathcal{P}_{(M,\pi)})_{\vec{s}}$ lies in $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)_{M_{\vec{s}}, \pi_{\vec{s}}}$. As parabolic induction commutes with tensor products, we have a direct sum decomposition:

$$\mathcal{P}_{(M,\pi)} \otimes \overline{\mathcal{K}} = \bigoplus_{\vec{s}} i_P^G \mathrm{c}\text{-Ind}_{K_i}^{G_i} \tilde{\kappa}_i \otimes \mathrm{St}_{s_i}.$$

For each i we have maps:

$$E_{K_i, \tau_i} \rightarrow H(G, K, \tilde{\kappa}_i \otimes \mathrm{St}_{s_i}) \cong \overline{\mathcal{K}}[Z_{i, s_i}]^{W_{M_{i, s_i}}(s_i)} \hookrightarrow \overline{\mathcal{K}}[Z_{i, s_i}]$$

where the isomorphism is the one determined by the pair (M_{s_i}, π_{s_i}) . Taking the tensor product of all of these maps yields a map

$$\phi_{\bar{s}} : \bigotimes_i E_{K_i, \tau_i} \rightarrow \overline{\mathcal{K}}[Z_{\bar{s}}],$$

where $Z_{\bar{s}}$ is the subgroup of $M_{\bar{s}}$ given by the product of the Z_{i, s_i} .

The pair $(M_{\bar{s}}, \pi_{\bar{s}})$ induces an isomorphism of $A_{M_{\bar{s}}, \pi_{\bar{s}}}$ with $\overline{\mathcal{K}}[Z_{\bar{s}}]^{W_{M_{\bar{s}}(\pi_{\bar{s}})}}$. We thus obtain a map from $A_{M_{\bar{s}}, \pi_{\bar{s}}}$ to the tensor product of the E_{K_i, τ_i} . As an immediate consequence of Proposition 3.6, we then have:

Proposition 9.3. *The center $A_{M_{\bar{s}}, \pi_{\bar{s}}}$ of $\text{Rep}_{\overline{\mathcal{K}}}(G)_{M_{\bar{s}}, \pi_{\bar{s}}}$ acts on $(\mathcal{P}_{(M, \pi)})_{\bar{s}}$ via the map*

$$A_{M_{\bar{s}}, \pi_{\bar{s}}} \rightarrow \bigotimes_i E_{K_i, \tau_i}$$

defined above. In particular each $(\mathcal{P}_{(M, \pi)})_{\bar{s}}$ is a faithful module over $A_{M_{\bar{s}}, \pi_{\bar{s}}}$.

10. THE BERNSTEIN DECOMPOSITION AND BERNSTEIN'S SECOND ADJOINTNESS

Our next goal is to apply our results on the structure of $\mathcal{P}_{K, \tau}$ to establish a Bernstein decomposition for the category of smooth $W(k)[G]$ -modules. With the results of the previous section in hand, this is an easy consequence of Bernstein's second adjointness for the category of smooth $W(k)[G]$ -modules, which is due, in this generality, to Dat in [D2]. However, at this point it is not too much work to give an alternative proof of Bernstein's second adjointness which is quite different in spirit from Dat's approach. We thus detour for a moment to show how Bernstein's second adjointness follows from the results so far.

For technical reasons (namely, the fact that parabolic induction is naturally a right adjoint, and therefore takes injectives to injectives), it will be useful for us to work with injective objects rather than the projectives $\mathcal{P}_{K, \tau}$. To obtain a suitable supply of injectives, we define:

Definition 10.1. Let Π be a smooth $W(k)[G]$ -module. We denote by Π^\vee the $W(k)[G]$ -submodule of smooth vectors in $\text{Hom}_{W(k)}(\Pi, \mathcal{K}/W(k))$.

As $\mathcal{K}/W(k)$ is an injective $W(k)$ -module, and the functor that takes a $W(k)[G]$ -module to the submodule consisting of its smooth vectors is exact, the functor $\Pi \mapsto \Pi^\vee$ is exact as well. Moreover, we have:

Lemma 10.2. *Let Π and Π' be smooth $W(k)[G]$ -modules. Then there is a natural isomorphism:*

$$\text{Hom}_{W(k)[G]}(\Pi, (\Pi')^\vee) \rightarrow \text{Hom}_{W(k)[G]}(\Pi', \Pi^\vee).$$

Proof. Both $\text{Hom}_{W(k)[G]}(\Pi, (\Pi')^\vee)$ and $\text{Hom}_{W(k)[G]}(\Pi', \Pi^\vee)$ are in bijection with the set of G -equivariant pairings $\Pi \times \Pi' \rightarrow \mathcal{K}/W(k)$ that are smooth with respect to the action of G . \square

Corollary 10.3. *If Π is a projective $W(k)[G]$ -module, then Π^\vee is injective.*

Proof. The functors $\text{Hom}_{W(k)[G]}(-, \Pi^\vee)$ and $\text{Hom}_{W(k)[G]}(\Pi, (-)^\vee)$ are naturally equivalent, and the latter is exact. \square

This duality is well-behaved with respect to normalized parabolic induction:

Lemma 10.4. *Let $P = MU$ be a Levi subgroup of G , and let π be a smooth $W(k)[M]$ -module. Then there is a natural isomorphism:*

$$(i_P^G \pi)^\vee \rightarrow i_P^G \pi^\vee.$$

In particular, if Π is a simple $W(k)[G]$ -module with cuspidal (resp. supercuspidal) support (M, π) , then Π^\vee has cuspidal (resp. supercuspidal) support (M, π^\vee) .

Proof. Integration over G/P defines a G -equivariant bilinear map:

$$[i_P^G \pi] \times [i_P^G \pi^\vee] \rightarrow \mathcal{K}/W(k),$$

and hence a G -equivariant map:

$$i_P^G \pi^\vee \rightarrow [i_P^G \pi]^\vee.$$

This map is easily seen to be an isomorphism by passing to U -invariants for a cofinal family of sufficiently small compact open subgroups U of G . \square

Note that if Π is an *admissible* $W(k)[G]$ -module, (or, alternatively, an admissible $\mathcal{K}[G]$ -module), then $(\Pi^\vee)^\vee$ is naturally isomorphic to Π . In particular this is true if Π is simple.

If (K, τ) is a maximal distinguished cuspidal k -type, we define $I_{K, \tau}$ to be the injective $W(k)[G]$ -module $\mathcal{P}_{K, \tau}^\vee$. Similarly, if (M, π) is a pair consisting of a Levi subgroup of G and an irreducible cuspidal representation π of M over k , we set $I_{(M, \pi)}$ to be the $W(k)[G]$ -module $\mathcal{P}_{(M, \pi)}^\vee$.

Proposition 10.5. *The $W(k)[G]$ -module $I_{(M, \pi)}$ is injective. Moreover, every simple $W(k)[G]$ -module Π with mod ℓ inertial cuspidal support (M, π) embeds in $I_{(M, \pi)}$.*

Proof. For a suitable parabolic subgroup P , and suitable maximal distinguished cuspidal types (K_i, τ_i) , we have:

$$\begin{aligned} \mathcal{P}_{(M, \pi)}^\vee &= [i_P^G \mathcal{P}_{K_1, \tau_1}^\vee \otimes \cdots \otimes \mathcal{P}_{K_r, \tau_r}^\vee]^\vee \\ &= i_P^G I_{K_1, \tau_1} \otimes I_{K_r, \tau_r}. \end{aligned}$$

As i_P^G is a right adjoint of an exact functor (by Frobenius reciprocity), the latter module is clearly injective.

Now given Π , Π^\vee has mod ℓ inertial cuspidal support (M, π^\vee) ; by Proposition 4.16 we have a surjection $\mathcal{P}_{(M, \pi^\vee)} \rightarrow \Pi^\vee$. Dualizing, and using the fact that $(\Pi^\vee)^\vee = \Pi$, we obtain our desired result. \square

Proposition 10.6. *Let (M, π) and (M', π') be two pairs consisting of a Levi subgroup of G and an irreducible cuspidal representation of that Levi subgroup over k . Let $I = I_{(M, \pi)}$; $I' = I_{(M', \pi')}$. If $\text{Hom}_{W(k)[G]}(I, I')$ is nonzero, then the mod ℓ inertial supercuspidal supports of (M, π) and (M', π') coincide.*

Proof. Set $\mathcal{P} = \mathcal{P}_{(M, \pi)}^\vee$; $\mathcal{P}' = \mathcal{P}_{(M', \pi')}^\vee$, so that $I = \mathcal{P}^\vee$ and $I' = (\mathcal{P}')^\vee$. We have an injection of $\text{Hom}_{W(k)[G]}(I, I')$ into $\text{Hom}_{W(k)[G]}((I')^\vee, I^\vee)$; the latter is equal to $\text{Hom}_{W(k)[G]}(\mathcal{P}^{\vee\vee}, (\mathcal{P}')^{\vee\vee})$. We also have an embedding:

$$\text{Hom}_{W(k)[G]}(\mathcal{P}^{\vee\vee}, (\mathcal{P}')^{\vee\vee}) \rightarrow \text{Hom}_{W(k)[G]}(\mathcal{P}^{\vee\vee} \otimes \bar{\mathcal{K}}, (\mathcal{P}')^{\vee\vee} \otimes \bar{\mathcal{K}}),$$

so it suffices to show that the latter is zero unless the mod ℓ inertial supercuspidal supports of (M, π) and (M', π') coincide.

We have an embedding of $\mathcal{P}^{\vee\vee} \otimes \bar{\mathcal{K}}$ into $(\mathcal{P} \otimes \bar{\mathcal{K}})^{\vee\vee}$.

If (L, π') is a pair consisting of a Levi subgroup of G and an irreducible supercuspidal representation of L over $\overline{\mathcal{K}}$, and π' has mod ℓ inertial supercuspidal support different from (M, π) , then the projection $(\mathcal{P} \otimes \overline{\mathcal{K}})_{L, \pi}$ of $\mathcal{P} \otimes \overline{\mathcal{K}}$ to $\text{Rep}_{\overline{\mathcal{K}}}(G)_{L, \pi}$ vanishes, and therefore so does $(\mathcal{P} \otimes \overline{\mathcal{K}})_{L, \pi}^{\vee\vee}$. It follows that $(\mathcal{P} \otimes \overline{\mathcal{K}})^{\vee\vee}$ has a direct sum decomposition in which each summand lies in some block of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ corresponding to an inertial supercuspidal support whose mod ℓ reduction is (M, π) . Similarly, $(\mathcal{P}' \otimes \overline{\mathcal{K}})^{\vee\vee}$ has a direct summand decomposition in which each summand lies in some block of $\text{Rep}_{\overline{\mathcal{K}}}(G)$ corresponding to an inertial supercuspidal support whose mod ℓ reduction is (M', π') .

Thus, if the mod ℓ inertial supercuspidal supports of (M, π) and (M', π') differ, then no summand of $(\mathcal{P} \otimes \overline{\mathcal{K}})^{\vee\vee}$ lies in the same block as any summand of $(\mathcal{P}' \otimes \overline{\mathcal{K}})^{\vee\vee}$, and the result follows. \square

Corollary 10.7. *Every simple subquotient of $I_{(M, \pi)}$ has mod ℓ inertial supercuspidal support equal to that of (M, π) .*

Proof. Let Π be a simple subquotient of $I_{(M, \pi)}$, with mod ℓ inertial supercuspidal support (M', π') . Then Π embeds in $I_{(M', \pi')}$; as the latter is injective we obtain a nonzero map $I_{(M, \pi)} \rightarrow I_{(M', \pi')}$. The preceding proposition now implies that (M', π') has the same mod ℓ inertial supercuspidal support as (M, π) . \square

An immediate corollary is the ‘‘Bernstein decomposition’’ for $\text{Rep}_{W(k)}(G)$. Let M be a Levi subgroup of G , and let π be an irreducible supercuspidal representation of M over k . If Π is a simple smooth $W(k)[G]$ -module with mod ℓ inertial supercuspidal support given by (M, π) , then the mod ℓ cuspidal support of Π falls into one of finitely many possible mod ℓ inertial equivalence classes. Choose representatives (M_j, π_j) for these inertial equivalence classes, and let $I_{[M, \pi]} = I_{(M_1, \pi_1)} \oplus \cdots \oplus I_{(M_r, \pi_r)}$. Then every simple subquotient of $I_{[M, \pi]}$ has mod ℓ inertial supercuspidal support (M, π) . On the other hand, any simple smooth $W(k)[G]$ -module π with mod ℓ inertial supercuspidal support (M, π) has mod ℓ inertial cuspidal support (M_j, π_j) for some j , and hence embeds in $I_{(M_j, \pi_j)}$ (and thus also in $I_{[M, \pi]}$).

On the other hand, if Π is a simple smooth $W(k)[G]$ -module whose mod ℓ inertial supercuspidal support is not in the inertial equivalence class (M, π) , then there is an infinite collection of possible inertial equivalence classes into which the mod ℓ inertial cuspidal support of Π could fall. If we let $I_{\overline{[M, \pi]}}$ denote the direct sum of $I_{(M', \pi')}$ as (M', π') runs over a set of representatives for the inertial equivalence classes of pairs (M', π') over k whose supercuspidal support is *not* in the inertial equivalence class (M, π) , then no subquotient of $I_{\overline{[M, \pi]}}$ has mod ℓ inertial supercuspidal support equal to (M, π) , and every simple object of $\text{Rep}_{W(k)}(G)$ whose mod ℓ inertial supercuspidal support is not equivalent to (M, π) is a subobject of $I_{\overline{[M, \pi]}}$.

Theorem 10.8. *The full subcategory $\text{Rep}_{W(k)}(G)_{[M, \pi]}$ of $\text{Rep}_{W(k)}(G)$ consisting of smooth $W(k)[G]$ -modules Π such that every simple subquotient of Π has mod ℓ inertial supercuspidal support given by (M, π) is a block of $\text{Rep}_{W(k)}(G)$. Moreover, every element of $\text{Rep}_{W(k)}(G)_{[M, \pi]}$ has a resolution by direct sums of copies of $I_{[M, \pi]}$.*

Proof. This is immediate from the above discussion and Proposition 2.4. \square

Our first application of this Bernstein decomposition will be to establish Bernstein’s second adjointness for smooth $W(k)[G]$ -modules. This will allow us, at last,

to conclude that the modules $\mathcal{P}_{(M,\pi)}$ are projective. We follow the argument in the lecture notes by Bernstein-Rumelhart [BR], adapting it as necessary so that it will work in $\text{Rep}_{W(k)}(G)$.

Definition 10.9. Let $P = MU$ be a parabolic subgroup of G , let K be a compact open subgroup of G that is decomposed with respect to P , and let λ be a totally positive central element of M . Let T_λ be the element of $H(G, K, 1)$ given by $T^+(1_{K_M \lambda K_M})$. A smooth $W(k)[G]$ -module Π is K, P -stable, with constant $c_{K,P,\lambda}$, if there exists a positive integer $c_{K,P,\lambda}$ such that Π^K splits as a direct sum:

$$\Pi^K = \Pi^K [T_\lambda^{c_{K,P,\lambda}}] \oplus \Pi_{T_\lambda\text{-invert}}^K,$$

where $\Pi^K [T_\lambda^{c_{K,P,\lambda}}]$ is the $W(k)$ -submodule of Π^K consisting of elements killed by $T_\lambda^{c_{K,P,\lambda}}$, and $\Pi_{T_\lambda\text{-invert}}^K$ is the maximal $W(k)$ -submodule of Π^K on which $[\lambda]$ is invertible.

The key to establishing Bernstein's second adjointness will be proving that for every pair K, P , and every supercuspidal inertial equivalence class (L, π) , all objects of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ are K, P -stable. We first make a few observations:

Lemma 10.10. *Let Π be a smooth K, P -stable $W(k)[G]$ -module. Then Π^\vee is also K, P -stable.*

Proof. We have

$$(\Pi^\vee)^K \cong (\Pi^K)^\vee \cong \Pi^K [T_\lambda^{c_{K,P,\lambda}}]^\vee \oplus [\Pi_{T_\lambda\text{-invert}}^K]^\vee;$$

the result follows immediately. \square

Lemma 10.11. *Finite direct sums of K, P -stable modules are K, P -stable. Infinite direct sums of modules which are K, P -stable with a uniform constant $c_{K,P,\lambda}$ are K -stable. Kernels and cokernels of maps of K, P -stable modules are K, P -stable.*

Lemma 10.12. *Let Π be a smooth $W(k)[G]$ -module. Then the natural projection:*

$$\Pi^K \rightarrow (\Pi_U)^{K_M}$$

identifies $(\Pi_U)^{K_M}$ with $\Pi^K \otimes_{W(k)[T_\lambda]} W(k)[T_\lambda, T_\lambda^{-1}]$. In particular, if Π is K, P -stable, then the map $\Pi^K \rightarrow (\Pi_U)^{K_M}$ is surjective, and one has a direct sum decomposition:

$$\Pi^K = \Pi^K [T_\lambda^{c_{K,P,\lambda}}] \oplus (\Pi_U)^{K_M}.$$

This decomposition is independent of λ .

Proof. We make $(\Pi_U)^{K_M}$ into a $W(k)[T_\lambda]$ -module by letting T_λ act on $(\Pi_U)^{K_M}$ via λ . It is then clear that the map

$$\Pi^K \rightarrow (\Pi_U)^{K_M}$$

is T_λ -equivariant. It thus suffices to show that every element of the kernel of this map is killed by a power of T_λ , and that, for every element x of $(\Pi_U)^{K_M}$, $\lambda^m x$ is in the image of this map for some sufficiently large m .

For the first claim, let e_{K^+} be the idempotent projector onto the K^+ invariants of a K -module. As $K = K^- K_M K^+$, we have $e_K = e_{K^+} e_{K_M} e_{K^-}$. For each m , we have

$$\begin{aligned} e_K \lambda^m e_K &= e_{K^+} e_{K_M} e_{K^-} \lambda^m e_K \\ &= e_{K^+} \lambda^m e_{K_M} e_{\lambda^{-m} K^- \lambda^m} e_K = \lambda^m e_{\lambda^{-m} K^+ \lambda^m} e_K. \end{aligned}$$

(Here we have used that λ is positive.) Now if \tilde{x} is an element of Π^K that maps to zero in Π_U , then $e_K \tilde{x} = \tilde{x}$, and there exists a compact open subgroup U_1 of U such that $e_{U_1} \tilde{x} = 0$. But as λ is strictly positive, there exists an m such that $\lambda^{-m} K^+ \lambda^m$ contains U_1 . Thus \tilde{x} is killed by $e_{\lambda^{-m} K^+ \lambda^m}$ and fixed by e_K . Then \tilde{x} is killed by $e_K \lambda^m e_K$, as required.

As for the second claim, let \tilde{x} be a lift of x to Π^{K_M} . There is a compact open subgroup K' of G that fixes \tilde{x} ; then \tilde{x} is in particular invariant under $K' \cap U^\circ$. As λ is strictly positive, there exists an m such that $\lambda^m \tilde{x}$ is invariant under K^- . Thus

$$e_{K^+} \lambda^m \tilde{x} = e_{K^+} e_{K_M} e_{K^-} \lambda^m \tilde{x} = e_K \lambda^m \tilde{x},$$

so $e_{K^+} \lambda^m \tilde{x}$ lies in Π^K . As e_{K^+} acts trivially on Π_U (because K^+ is contained in U), $e_{K^+} \lambda^m \tilde{x}$ maps to $\lambda^m \tilde{x}$ under the map

$$\Pi^K \rightarrow (\Pi_U)^{K_M},$$

as required.

Finally, if Π is K, P -stable, then Π^K surjects onto $\Pi^K \otimes_{W(k)[T_\lambda]} W(k)[T_\lambda, T_\lambda^{-1}]$; this surjection identifies $(\Pi_U)^{K_M}$ with the maximal T_λ -divisible submodule of Π^K and thus yields the asserted direct sum decomposition. To see that this decomposition is independent of λ , choose another strictly positive element λ' . Then the maximal $T_{\lambda\lambda'}$ -divisible submodule of Π^K is contained in both the maximal T_λ -divisible submodule and the maximal $T_{\lambda'}$ -divisible submodule; since projection onto $(\Pi_U)^{K_M}$ is an isomorphism on each of these submodules they must all coincide. \square

Lemma 10.13. *Let R be a commutative Noetherian $W(k)$ -algebra, and let Π be an admissible $R[G]$ -module. Suppose that $r_G^P \Pi$ is also admissible. Then Π is K, P -stable.*

Proof. As $r_G^P \Pi$ is a twist of Π_U , the hypotheses imply that Π^K and $(\Pi_U)^{K_M}$ are finitely generated R -modules. On the other hand, we have an isomorphism

$$(\Pi_U)^{K_M} \cong \Pi^K \otimes_{R[T_\lambda]} R[T_\lambda, T_\lambda^{-1}].$$

The result is now an immediate consequence of [BR], Lemma 33. \square

Corollary 10.14. *For any Levi subgroup L of G , and any irreducible cuspidal representation π of L over k , the modules $\mathcal{P}_{(M,\pi)}$ and $I_{(M,\pi)}$ are K, P -stable.*

Proof. For $\mathcal{P}_{(M,\pi)}$, the result follows from the previous lemma, together with Theorem 9.1. We also have the result for $\mathcal{P}_{[M,\pi]}$, as this is a finite direct sum of modules $\mathcal{P}_{[M',\pi']}$. As $I_{(M,\pi)} = \mathcal{P}_{(M,\pi^\vee)}^\vee$, we also have the result for $I_{(M,\pi)}$, and hence also for $I_{[M,\pi]}$. \square

Proposition 10.15. *For any Levi subgroup L of G , and any irreducible supercuspidal k -representation π of L , every object of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ is K, P -stable.*

Proof. Any object of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ has a resolution by direct sums of $I_{[L,\pi]}$. These are K, P -stable, so the result follows from the fact that kernels of maps of K, P -stable modules are K, P -stable. \square

It follows that for any object Π of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$, the map $\Pi^K \rightarrow (\Pi_U)^{K_M}$ is surjective.

Proposition 10.16. *Let Π be an object of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$. There is a canonical isomorphism:*

$$r_G^P \Pi^\vee \cong (r_G^{P^\circ} \Pi)^\vee.$$

Proof. We follow the proof of [B], Theorem 2. First note that $r_G^P \Pi^\vee$ is a twist of $(\Pi^\vee)_U$ by our (fixed) square root of the modulus character of P , and $r_G^P \Pi$ is a twist of Π_U by a square root of the modulus character of P° . As these two modulus characters are inverses of each other, it suffices to construct an isomorphism $(\Pi_{U^\circ})^\vee \cong (\Pi^\vee)_U$ of $W(k)[M]$ -modules.

For each K that is decomposed with respect to P , we have an isomorphism:

$$(\Pi^K)^\vee \rightarrow (\Pi^\vee)^K$$

coming from a perfect pairing $\Pi^K \times (\Pi^\vee)^K \rightarrow \mathcal{K}/W(k)$. Under this pairing, the adjoint of T_λ is $T_{\lambda^{-1}}$. If we take λ to be strictly positive with respect to P , then λ^{-1} is strictly positive with respect to P° . Moreover, Π is both (K, P) -stable and (K, P°) -stable. In particular, $(\Pi_{U^\circ})^{K_M}$ is isomorphic to the maximal $T_{\lambda^{-1}}$ -divisible submodule of Π^K , and $((\Pi^\vee)_U)^{K_M}$ is isomorphic to the maximal T_λ -divisible submodule of $(\Pi^\vee)^K$. Moreover, under these identifications $(\Pi_{U^\circ})^{K_M}$ and $((\Pi^\vee)_U)^{K_M}$ are direct summands of Π^K and $(\Pi^\vee)^K$, respectively. The pairing on the latter thus descends to a perfect pairing:

$$(\Pi_{U^\circ})^{K_M} \times ((\Pi^\vee)_U)^{K_M} \rightarrow \mathcal{K}/W(k).$$

By [B], Lemma 1, we can find a K decomposed with respect to P inside any compact open subgroup of G . Taking the limit over a cofinal system of such K gives the desired perfect pairing

$$\Pi_{U^\circ} \times (\Pi^\vee)_U \rightarrow \mathcal{K}/W(k),$$

and hence the desired identification of $(\Pi^\vee)_U$ with $(\Pi_{U^\circ})^\vee$. \square

Theorem 10.17 (Bernstein's second adjointness). *Let Π_1 and Π_2 be objects of $\text{Rep}_{W(k)}(M)$ and $\text{Rep}_{W(k)}(G)_{[L,\pi]}$, respectively. Then there is a canonical isomorphism:*

$$\text{Hom}_{W(k)[G]}(i_P^G \Pi_1, \Pi_2) \cong \text{Hom}_{W(k)[M]}(\Pi_1, r_G^{P^\circ} \Pi_2).$$

Proof. We follow the argument of the "claim" after Theorem 20 of [BR]. We first establish the case in which $\Pi_2 = (\Pi'_2)^\vee$ for some Π'_2 in $\text{Rep}_{W(k)}(G)_{[L,\pi^\vee]}$. We then have a sequence of functorial isomorphisms:

$$\begin{aligned} \text{Hom}_{W(k)[M]}(\Pi_1, r_G^{P^\circ} \Pi_2) &\cong \text{Hom}_{W(k)[M]}(\Pi_1, (r_G^P \Pi'_2)^\vee) \\ &\cong \text{Hom}_{W(k)[M]}(r_G^P \Pi'_2, \Pi_1^\vee) \\ &\cong \text{Hom}_{W(k)[G]}(\Pi'_2, i_P^G \Pi_1^\vee) \\ &\cong \text{Hom}_{W(k)[G]}(\Pi'_2, (i_P^G \Pi_1)^\vee) \\ &\cong \text{Hom}_{W(k)[G]}(i_P^G \Pi_1, \Pi_2) \end{aligned}$$

In particular the result holds for $\Pi_2 = I_{[L,\pi]}$. If Π_1 is finitely generated, then the functors $\text{Hom}_{W(k)[G]}(i_P^G \Pi_1, -)$ and $\text{Hom}_{W(k)[M]}(\Pi_1, r_G^{P^\circ} -)$ commute with arbitrary direct sums, so the result holds for Π_1 finitely generated and Π_2 an arbitrary direct sum of copies of $I_{[L,\pi]}$. As any Π_1 is the limit of its finitely generated submodules, the result holds for an arbitrary Π_1 , when Π_2 is an arbitrary direct sum of copies of $I_{[L,\pi]}$. Finally, we can resolve an arbitrary Π_2 by direct sums of copies of $I_{[L,\pi]}$, and the result then follows for all Π_1, Π_2 . \square

Corollary 10.18. *The representations $\mathcal{P}_{[L,\pi]}$ are projective and small.*

Proof. It suffices to show that each $\mathcal{P}_{(M,\pi')}$, with π' irreducible and cuspidal over K is projective and small, as $\mathcal{P}_{[L,\pi]}$ is a finite direct sum of representations of this form. For a suitable sequence of types (K_i, τ_i) , we have

$$\mathcal{P}_{(M,\pi')} = i_P^G \mathcal{P}_{K_1, \tau_1} \otimes \cdots \otimes \mathcal{P}_{K_r, \tau_r}.$$

For each i the representation $\mathcal{P}_{K_i, \tau_i}$ is projective and small (as $\mathcal{P}_{K_i, \tau_i}$ is the compact induction of a finite-length $W(k)$ -module). Thus the tensor product Π of the $\mathcal{P}_{K_i, \tau_i}$ is projective and small when considered as a $W(k)[M]$ -module. We have shown that $\mathcal{P}_{(M,\pi')}$ lies in $\text{Rep}_{W(k)}(G)_{[L,\pi]}$.

Fix a smooth representation Π' of G , and let $\Pi'_{[L,\pi]}$ be the direct summand of Π' that lies in $\text{Rep}_{W(k)}(G)_{[L,\pi]}$. Then

$$\begin{aligned} \text{Hom}_{W(k)[G]}(\mathcal{P}_{(M,\pi')}, \Pi') &\cong \text{Hom}_{W(k)[G]}(\mathcal{P}_{(M,\pi')}, \Pi'_{[L,\pi]}) \\ &\cong \text{Hom}_{W(k)[M]}(\Pi, r_G^{P^\circ} \Pi'_{[L,\pi]}) \end{aligned}$$

As $r_G^{P^\circ}$ commutes with direct sums it is easy to see this implies $\mathcal{P}_{(M,\pi')}$ is small; projectivity of $\mathcal{P}_{(M,\pi')}$ follows from exactness of $r_G^{P^\circ}$. \square

Corollary 10.19. *The Bernstein center $A_{[L,\pi]}$ of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ is isomorphic to the center of $\text{End}_{W(k)[G]}(\mathcal{P}_{[L,\pi]})$.*

Proof. Every simple object of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ is a quotient of $\mathcal{P}_{(M,\pi')}$ for some pair (M, π') with inertial supercuspidal support (L, π) , and hence such an object is a quotient of $\mathcal{P}_{[L,\pi]}$. It follows that $\mathcal{P}_{[L,\pi]}$ is faithfully projective in $\text{Rep}_{W(k)}(G)_{[L,\pi]}$, and the result follows immediately. \square

11. COMPUTATION OF THE BERNSTEIN CENTER

In this section we compute the center $A_{[L,\pi]}$ of $\text{End}_{W(k)[G]}(\mathcal{P}_{[L,\pi]})$.

Proposition 11.1. *There is a natural isomorphism:*

$$A_{[L,\pi]} \otimes \bar{\mathcal{K}} \cong \prod_{(M, \tilde{\pi})} A_{M, \tilde{\pi}},$$

where $(M, \tilde{\pi})$ runs over inertial equivalence classes of pairs in which M is a Levi subgroup of G and $\tilde{\pi}$ is a cuspidal representation of M over $\bar{\mathcal{K}}$ whose mod ℓ inertial supercuspidal support equals (L, π) . This isomorphism is uniquely characterised by the property that for any Π in $\text{Rep}_{\bar{\mathcal{K}}}(G)$, and any x in $A_{[L,\pi]}$, the action of x on Π coincides with that of its image in $\prod_{(M, \tilde{\pi})} A_{M, \tilde{\pi}}$.

Proof. The module \mathcal{P} defined by $\mathcal{P} = \text{c-Ind}_{\{e\}}^G W(k)$, where $\{e\}$ is the trivial subgroup of G , is a faithfully projective module in $\text{Rep}_{W(k)}(G)$. We have $\mathcal{P} \otimes \bar{\mathcal{K}} = \text{c-Ind}_{\{e\}}^G \bar{\mathcal{K}}$; in particular $\mathcal{P} \otimes \bar{\mathcal{K}}$ is faithfully projective in $\text{Rep}_{\bar{\mathcal{K}}}(G)$. As \mathcal{P} is ℓ -torsion free, we have an injection:

$$\text{End}_{W(k)[G]}(\mathcal{P}) \rightarrow \text{End}_{W(k)[G]}(\mathcal{P}) \otimes \bar{\mathcal{K}} \cong \text{End}_{\bar{\mathcal{K}}[G]}(\mathcal{P} \otimes \bar{\mathcal{K}}).$$

In particular we have an isomorphism:

$$Z(\text{End}_{W(k)[G]}(\mathcal{P})) \otimes \bar{\mathcal{K}} \cong Z(\text{End}_{\bar{\mathcal{K}}[G]}(\mathcal{P} \otimes \bar{\mathcal{K}})).$$

Multiplying both sides by the central idempotent $e_{[L,\pi]}$ gives us the desired isomorphism. This isomorphism is equivariant for the actions of both sides on $\mathcal{P} \otimes \overline{\mathcal{K}}$, and hence for all objects of $\text{Rep}_{\overline{\mathcal{K}}}(G)$. \square

This isomorphism identifies $A_{[L,\pi]}$ with the $W(k)$ -subalgebra of $\prod_{(M,\bar{\pi})} A_{M,\bar{\pi}}$ consisting of tuples $(x_{M,\bar{\pi}})$ such that the action of $(x_{M,\bar{\pi}})$ on $\mathcal{P}_{[L,\pi]} \otimes \overline{\mathcal{K}}$ preserves $\mathcal{P}_{[L,\pi]}$. As we have a direct sum decomposition:

$$\mathcal{P}_{[L,\pi]} = \bigoplus_{M',\pi'} \mathcal{P}_{(M',\pi')}$$

where (M',π') runs over the inertial equivalence classes of pairs in which M' is a Levi subgroup of G , π' is an irreducible cuspidal representation of M' over k , and the supercuspidal support of (M',π') is (L,π) . It is clear that any central endomorphism of $\mathcal{P}_{[L,\pi]}$ preserves this direct sum decomposition. We thus have:

Lemma 11.2. *A tuple $(x_{M,\bar{\pi}})$ preserves $\mathcal{P}_{[L,\pi]}$ inside $\mathcal{P}_{[L,\pi]} \otimes \overline{\mathcal{K}}$ if, and only if, it preserves $\mathcal{P}_{(M',\pi')}$ inside $\mathcal{P}_{(M',\pi')} \otimes \overline{\mathcal{K}}$ for all (M',π') with supercuspidal support (L,π) .*

For a fixed (M',π') , let $B_{(M',\pi')}$ be the $W(k)$ -subalgebra of $\prod_{(M,\bar{\pi})} A_{M,\bar{\pi}}$ consisting of tuples $(x_{M,\bar{\pi}})$ that preserve $\mathcal{P}_{(M',\pi')}$ in $\mathcal{P}_{(M',\pi')} \otimes \overline{\mathcal{K}}$.

Lemma 11.3. *Let P be a parabolic subgroup of G , with Levi subgroup M , and let Π be a smooth $W(k)[M]$ -module that is ℓ -torsion free. Then the natural map:*

$$\text{End}_{W(k)[M]}(\Pi) \rightarrow \text{End}_{W(k)[M]}(i_P^G \Pi)$$

is injective. Moreover, if f is an element of $\text{End}_{W(k)[G]}(i_P^G \Pi)$ such that $\ell^a f$ is in the image of this map for some a , then f is also in the image of this map.

Proof. Let g be an element of $\text{End}_{W(k)[M]}(\Pi)$. Then g fits in a commutative diagram:

$$\begin{array}{ccc} r_G^P i_P^G \Pi & \rightarrow & \Pi \\ \downarrow & & \downarrow \\ r_G^P i_P^G \Pi & \rightarrow & \Pi \end{array}$$

in which the horizontal arrows are the natural maps (and are therefore surjective), and the vertical arrows are $r_G^P i_P^G g$ and g , respectively. The surjectivity of the horizontal maps shows we can recover g from $r_G^P i_P^G g$, proving the first claim. As for the second, suppose that $\ell^a f = i_P^G g$ for some g . Then the image of $r_G^P i_P^G g$ is contained in $\ell^a r_G^P i_P^G \Pi$ and the above diagram shows that the image of g is contained in $\ell^a \Pi$. As Π is ℓ -torsion free it follows that g is (uniquely) divisible by ℓ^a in $\text{End}_{W(k)[M]}(\Pi)$; that is, $g = \ell^a g'$. Then we have $f = i_P^G g'$. \square

Recall that by definition, $\mathcal{P}_{(M',\pi')} = i_P^G \otimes_i \mathcal{P}_{(K_i,\tau_i)}$ for a suitable sequence of maximal distinguished cuspidal types (K_i,τ_i) .

Lemma 11.4. *Let $(x_{M,\bar{\pi}})$ be an element of $B_{(M',\pi')}$. Let f be the induced endomorphism of $\mathcal{P}_{M',\pi'}$. Then f arises by applying i_P^G to a (unique) endomorphism of $\otimes_i \mathcal{P}_{(K_i,\tau_i)}$.*

Proof. This is immediate from Lemma 11.3 and Proposition 9.3. \square

We denote by $E_{(M',\pi')}$ the tensor product $\bigotimes_i E_{K_i,\tau_i}$; the lemma above realizes $B_{(M',\pi')}$ as a subalgebra of $E_{(M',\pi')}$.

Proposition 11.5. *The image of $B_{(M', \pi')}$ in $\text{End}_{W(k)[G]}(\mathcal{P}_{(M', \pi')})$ is isomorphic to the subalgebra of $\bigotimes_i E_{K_i, \tau_i}$ consisting of endomorphisms f such that:*

- (1) *For any sequence $\vec{s} = \{s_1, \dots, s_r\}$, such that for each i , σ_i is the cuspidal representation attached to the ℓ -regular part s'_i of s_i , the action of f on $(\mathcal{P}_{(M', \pi')})_{\vec{s}}$ is given by an element $x_{\vec{s}}$ of $A_{M_{\vec{s}}, \pi_{\vec{s}}}$.*
- (2) *For any pair \vec{s}_1, \vec{s}_2 , such that $(M_{\vec{s}_1}, \pi_{\vec{s}_1})$ is inertially equivalent to $(M_{\vec{s}_2}, \pi_{\vec{s}_2})$, we have $x_{\vec{s}_1} = x_{\vec{s}_2}$.*

Proof. Fix $(x_{M, \tilde{\pi}})$ in $B_{(M', \pi')}$; we have shown that any such element arises by parabolic induction from a unique element f of $\bigotimes_i E_{K_i, \tau_i}$. The action of $i_P^G f$ on $\mathcal{P}_{(M, \tilde{\pi})}$ coincides with that of $(x_{M, \tilde{\pi}})$; in particular the action of f on $(\mathcal{P}_{(M, \tilde{\pi})})_{\vec{s}}$ coincides with $x_{M_{\vec{s}}, \pi_{\vec{s}}}$. It is thus clear that (1) and (2) hold for f .

Conversely, given an f satisfying (1) and (2), define an element $(x_{M, \tilde{\pi}})$ of $B_{(M', \pi')}$ by setting $x_{M, \tilde{\pi}} = x_{\vec{s}}$ for any \vec{s} such that $(M_{\vec{s}}, \pi_{\vec{s}})$ is inertially equivalent to $(M, \tilde{\pi})$. Such an endomorphism of $\mathcal{P}_{M, \tilde{\pi}} \otimes \overline{\mathcal{K}}$ clearly preserves $\mathcal{P}_{M, \tilde{\pi}}$, as its action on $\mathcal{P}_{M, \tilde{\pi}} \overline{\mathcal{K}}$ coincides with that of $i_P^G f$. \square

In principle this proposition gives a precise description of $A_{[L, \pi]}$, although one that is too cumbersome for practical applications. In order to produce a more useful description of $A_{[L, \pi]}$, we must be more explicit about the maps from $B_{(M', \pi')}$ to $A_{M_{\vec{s}}, \pi_{\vec{s}}}$ for various \vec{s} . We will normalize these maps by making a choice of a compatible system of cuspidals as in the discussion preceding Theorem 7.3 for each tower of maximal distinguished cuspidal k -types attached to a maximal distinguished cuspidal type as in section 7.

We can make these choices systematically for all i as follows: for each j , and each inertial equivalence class of supercuspidal representations of $\text{GL}_j(F)$ over k , we choose:

- (1) a representative Π in this inertial equivalence class,
- (2) a maximal distinguished cuspidal k -type $(K_{1, \Pi}, \tau_{1, \Pi})$ contained in Π , of the form $\kappa_{1, \Pi} \otimes \sigma_{1, \Pi}$
- (3) an ℓ -regular element $s'_{1, \Pi}$ of a suitable $\overline{G}_{1, \Pi}$ that gives rise to the supercuspidal representation $\sigma_{1, \Pi}$
- (4) a lift $\pi_{1, \Pi}$ of Π to an absolutely irreducible supercuspidal representation of $\text{GL}_j(F)$ over \mathcal{K} , containing the type $(K_{1, \Pi}, \tilde{\kappa}_{1, \Pi} \otimes \text{St}_{s'_{1, \Pi}})$.

Attached to these choices we have the sequence of cuspidal k -types $(K_{m, \Pi}, \tau_{m, \Pi})$, and a compatible system of cuspidals $\pi_{s, \Pi}$ for all m and all s such that $s^{\text{reg}} = (s'_{1, \Pi})^m$. We may assume that for each i , the k -type (K_i, τ_i) is equal to the type $(K_{m_i, \Pi}, \tau_{m_i, \Pi})$ for some m_i and Π_i . For any \vec{s} , we take $\pi_{\vec{s}}$ to be the tensor product of the representations π_{s_i, Π_i} . By construction, $\pi_{\vec{s}}$ is invariant under the action of $W_{M_{\vec{s}}}(\pi_{\vec{s}})$.

Proposition 9.3 (together with the construction preceding it) then gives, for each \vec{s} , maps:

$$\phi_{\vec{s}} : \bigotimes_i E_{K_i, \tau_i} \rightarrow \overline{\mathcal{K}}[Z_{\vec{s}}]$$

$$A_{M_{\vec{s}}, \pi_{\vec{s}}} \cong \overline{\mathcal{K}}[Z_{\vec{s}}]^{W_{M_{\vec{s}}}(\pi_{\vec{s}})}.$$

These maps have the property that an element f in $\bigotimes_i E_{K_i, \tau_i}$ acts on $(\mathcal{P}_{(M, \tilde{\pi})})_{\vec{s}}$ by an element x of the Bernstein center if, and only if, the image of f and x in $\overline{\mathcal{K}}[Z_{\vec{s}}]$ agree.

From this point of view, an f in $\otimes_i E_{K_i, \tau_i}$ satisfies condition (1) of Proposition 11.5 if, and only if, for all sequences \vec{s} , $\phi_{\vec{s}}(f)$ lies in $\overline{\mathcal{K}}[Z_{\vec{s}}]^{W_{\overline{M}_{\vec{s}}}}(\pi_{\vec{s}})$.

Condition (2) of Proposition 11.5 can also be reformulated in this way, though it is more awkward to do so. We first observe that if we have \vec{s}_1 and \vec{s}_2 such that the pairs $(M_{\vec{s}_1}, \pi_{\vec{s}_1})$ and $(M_{\vec{s}_2}, \pi_{\vec{s}_2})$ are inertially equivalent, then exists a $w \in W(G)$ such that $wM_{\vec{s}_1}w^{-1} = M_{\vec{s}_2}$, and $\pi_{\vec{s}_1}^w = \pi_{\vec{s}_2}$.

Suppose we have \vec{s}_1 , \vec{s}_2 , and w as in the lemma. Then $A_{M_{\vec{s}_1}, \pi_{\vec{s}_1}}$ is canonically isomorphic to $A_{M_{\vec{s}_2}, \pi_{\vec{s}_2}}$. Then conjugation by w takes $Z_{\vec{s}_1}$ to $Z_{\vec{s}_2}$ and $W_{\overline{M}_{\vec{s}_1}}(\pi_{\vec{s}_1})$ to $W_{\overline{M}_{\vec{s}_2}}(\pi_{\vec{s}_2})$. We then have a commutative diagram:

$$\begin{array}{ccc} A_{M_{\vec{s}_1}, \pi_{\vec{s}_1}} & \rightarrow & \overline{\mathcal{K}}[Z_{\vec{s}_1}]^{W_{\overline{M}_{\vec{s}_1}}(\pi_{\vec{s}_1})} \\ \downarrow & & \downarrow \\ A_{M_{\vec{s}_2}, \pi_{\vec{s}_2}} & \rightarrow & \overline{\mathcal{K}}[Z_{\vec{s}_2}]^{W_{\overline{M}_{\vec{s}_2}}(\pi_{\vec{s}_2})} \end{array}$$

where the left-hand vertical map is the natural isomorphism and the right-hand vertical map, which we denote by ψ_w , is induced by conjugation by w .

An f in $\otimes_i E_{K_i, \tau_i}$ satisfies conditions (1) and (2) of Proposition 11.5 if, and only if, for all triples \vec{s}_1, \vec{s}_2, w as in the lemma, we have $\phi_{\vec{s}_1}(f) = \psi_w(\phi_{\vec{s}_2}(f))$. (Condition (1) follows from looking at those triples with $\vec{s}_1 = \vec{s}_2$, for example.)

This description allows us to obtain a natural tensor factorization of $B_{M', \pi'}$. For each representative Π of a given inertial equivalence class of supercuspidal representations we have chosen, let $M'(\Pi)$ be the product of the blocks of M' corresponding to those i for which the type $(K_i, \tau_i) = (K_{m_i, \Pi_i}, \tau_{m_i, \Pi_i})$ satisfies $\Pi_i = \Pi$. Let $\pi'(\Pi)$ be the tensor product of those tensor factors of π' corresponding to blocks of $M'(\Pi)$. Then we have

$$\begin{aligned} M' &= \prod_{\Pi} M'(\Pi), \\ \pi' &= \bigotimes_{\Pi} \pi'(\Pi) \\ E_{M', \pi'} &= \bigotimes_{\Pi} E_{M'(\Pi), \pi'(\Pi)}. \end{aligned}$$

Moreover, if for each \vec{s} , we let $\vec{s}(\Pi)$ be the subsequence consisting of those s_i for which (K_i, τ_i) has underlying supercuspidal representation Π , then we obtain factorizations:

$$\begin{aligned} M_{\vec{s}} &= \prod_{\Pi} M_{\vec{s}(\Pi)}, \\ \pi_{\vec{s}} &= \bigotimes_{\Pi} \pi_{\vec{s}(\Pi)}, \end{aligned}$$

and the map: $\phi_{\vec{s}}$ of $E_{M', \pi'}$ into $\overline{\mathcal{K}}[Z_{\vec{s}}]$ then factors as a tensor product:

$$\phi_{\vec{s}} = \bigotimes_{\Pi} \phi_{\vec{s}(\Pi)} : E_{M'(\Pi), \pi'(\Pi)} \rightarrow \overline{\mathcal{K}}[Z_{\vec{s}(\Pi)}].$$

For each tuple $(\vec{s}_1, \vec{s}_2, w)$, the map ψ_w takes the tensor factor $\overline{\mathcal{K}}[Z_{\vec{s}_1(\Pi)}]$ of $\overline{\mathcal{K}}[Z_{\vec{s}_1}]$ to the factor $\overline{\mathcal{K}}[Z_{\vec{s}_2(\Pi)}]$ of $\overline{\mathcal{K}}[Z_{\vec{s}_2}]$. It is then immediate that we have:

Proposition 11.6. *The natural isomorphism:*

$$E_{M', \pi'} \cong \bigotimes_{\Pi} E_{M'(\Pi), \pi'(\Pi)}$$

restricts to an isomorphism:

$$B_{M', \pi'} \cong \bigotimes_{\Pi} B_{M'(\Pi), \pi'(\Pi)}.$$

Note that for each (K_i, τ_i) , we have a distinguished subalgebra C_{K_i, τ_i} of E_{K_i, τ_i} . The tensor product of these subalgebras is a subalgebra of $E_{M', \pi'}$. Let $C_{M', \pi'}$ be the intersection in $E_{M', \pi'}$ of $B_{M', \pi'}$ with $\bigotimes_i C_{K_i, \tau_i}$. It is clear that each $C_{M', \pi'}$ factors as the tensor product over Π of $C_{M'(\Pi), \pi'(\Pi)}$.

The algebra $C_{M'(\Pi), \pi'(\Pi)}$ can be made very explicit. Write

$$E_{M'(\Pi), \pi'(\Pi)} = \bigotimes_i E_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}.$$

The subalgebra $C_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}$ of $E_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}$ is generated by $\Theta_{1, m_i}, \dots, \Theta_{m_i, m_i}$, together with Θ_{m_i, m_i}^{-1} . Let m be the sum of the m_i , and define elements $\Theta_1, \dots, \Theta_m$ of $\bigotimes_i E_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}$ by setting

$$\Theta_r = \sum_{\vec{j}} \bigotimes_i \Theta_{j_i, m_i},$$

where \vec{j} runs over sequences $\{j_i\}$ such that $1 \leq j_i \leq m_i$ for all i and the sum of the j_i is equal to m .

Proposition 11.7. *The algebra $C_{M'(\Pi), \pi'(\Pi)}$ is generated by $\Theta_1, \dots, \Theta_m$, together with Θ_m^{-1} .*

Proof. We first verify that the elements $\Theta_1, \dots, \Theta_m$ lie in $B_{M'(\Pi), \pi'(\Pi)}$. For each \vec{s} , define

$$\theta_{r, \vec{s}} \in \bigotimes_i \overline{\mathcal{K}}[Z_{s_i}]^{W_{\overline{M}_{s_i}}(s_i)}$$

by the formula:

$$\theta_{r, \vec{s}} = \sum_{\vec{j}} \bigotimes_i \theta_{j_i, s_i},$$

where \vec{j} runs over sequences $\{j_i\}$ such that $1 \leq j_i \leq m_i$ for all i , and the sum of the $d_i j_i$ is equal to m , where d_i is the degree of $\mathbb{F}_q(s_i)$ over $\mathbb{F}_q(s'_{1, \Pi})$. It is clear from Theorem 7.3 that the map

$$\phi_{\vec{s}(\Pi)} : \bigotimes_i E_{K_{m_i, \Pi}, \tau_{m_i, \Pi}} \rightarrow \overline{\mathcal{K}}[Z_{\vec{s}}]$$

takes Θ_r to $\theta_{r, \vec{s}}$. Furthermore, $\theta_{r, \vec{s}}$ is invariant under $W_{\overline{M}_{\vec{s}(\Pi)}(\pi_{\vec{s}(\Pi)})}$. Finally, observe that if w is an element of $W(G)$ that conjugates $(M_{\vec{s}_1(\Pi)}, \pi_{\vec{s}_1(\Pi)})$ to $(M_{\vec{s}_2(\Pi)}, \pi_{\vec{s}_2(\Pi)})$, then $\theta_{r, \vec{s}_1} = w \theta_{r, \vec{s}_2} w^{-1}$. It follows that the Θ_r lie in $C_{M'(\Pi), \pi'(\Pi)}$ as claimed.

Conversely, note that if we define \vec{s} by setting $s_i = (s'_{1, \Pi})^{m_i}$, then the tensor product of the $C_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}$ injects into $\overline{\mathcal{K}}[Z_{\vec{s}}]$, and the $W_{\overline{M}_{\vec{s}}(\pi_{\vec{s}})}$ -invariant subalgebra of $\overline{\mathcal{K}}[Z_{\vec{s}}]$ is generated by the images of the Θ_i and Θ_m^{-1} . In particular, the only elements of the tensor product of the $C_{K_{m_i, \Pi}, \tau_{m_i, \Pi}}$ that can lie in $B_{M'(\Pi), \pi'(\Pi)}$ are those in the subalgebra generated by the Θ_i and Θ_m^{-1} . \square

We record the following by-product of the proof of the above proposition:

Proposition 11.8. *Let \vec{s} be the sequence obtained by setting $s_i = (s'_{1,\Pi})^{m_i}$. Then The map*

$$\psi_{\vec{s}} : B_{M',\pi'} \rightarrow \overline{\mathcal{K}}[Z_{\vec{s}}]^{W_{\overline{M}_{\vec{s}}(\vec{s})}}$$

identifies $C_{M',\pi'}$ with $W(k)[Z_{\vec{s}}]^{W_{\overline{M}_{\vec{s}}(\vec{s})}}$.

Theorem 11.9. *The algebra $B_{M',\pi'}$ is a finitely generated $C_{M',\pi'}$ -module.*

Proof. It suffices to show this for each $B_{M'(\Pi),\pi'(\Pi)}$ over $C_{M'(\Pi),\pi'(\Pi)}$. Moreover, it is clear that the tensor product $\otimes_i C_{K_{m_i,\Pi},\tau_{m_i,\Pi}}$ is finitely generated over $C_{M'(\Pi),\pi'(\Pi)}$, and that $\otimes_i E_{K_{m_i,\Pi},\tau_{m_i,\Pi}}$ is finitely generated over this tensor product. It follows that $\otimes_i E_{K_{m_i,\Pi},\tau_{m_i,\Pi}}$ is finitely generated over $C_{M'(\Pi),\pi'(\Pi)}$; as $B_{M'(\Pi),\pi'(\Pi)}$ is a $C_{M'(\Pi),\pi'(\Pi)}$ -submodule of $\otimes_i E_{K_{m_i,\Pi},\tau_{m_i,\Pi}}$, the result follows. \square

To go further, we must introduce a partial order on the inertial equivalence classes of pairs (M', π') of irreducible cuspidal representations π' of M' over k with inertial supercuspidal support (L, π) . We do this by extending the partial order defined in section 7. Attached to (M', π') we have a collection of types $(K_{m_i,\Pi_i}, \tau_{m_i,\Pi_i})$. Define an “elementary operation” on such a collection to be the act of replacing a single type $(K_{m_i,\Pi_i}, \tau_{m_i,\Pi_i})$ with $j = \frac{m'_i}{m_i}$ copies of the type $(K_{m'_i,\Pi_i}, \tau_{m'_i,\Pi_i})$, where m'_i immediately precedes m_i . This new collection gives rise to a new pair (M'', π'') , with M'' contained (up to conjugacy) in M' . We say that $(M'', \pi'') \preceq (M', \pi')$, if we can obtain (M'', π'') from (M', π') by a sequence of such “elementary operations.” Note that if $(M'', \pi'') \preceq (M', \pi')$, then π' is inertially equivalent to the unique cuspidal Jordan-Hölder constituent of the parabolic induction of π'' to M' , and conversely; this gives an alternative characterization of the partial order.

There is thus, up to inertial equivalence, a unique maximal pair (M'_{\max}, π'_{\max}) among the pairs (M', π') with supercuspidal support (L, π) . Morally, (M'_{\max}, π'_{\max}) is the inertial equivalence class that is “farthest from supercuspidal.”

Proposition 11.10. *Let $(x_{\vec{s}})$ be an element of $\prod_{\vec{s}} A_{M_{\vec{s}},\pi_{\vec{s}}}$ that preserves \mathcal{P}_{M_1,π'_1} in $\mathcal{P}_{M_1,\pi'_1} \otimes \overline{\mathcal{K}}$. Then $(x_{M,\vec{s}})$ preserves \mathcal{P}_{M_2,π'_2} in $\mathcal{P}_{M_2,\pi'_2} \otimes \overline{\mathcal{K}}$ for all $(M_2, \pi'_2) \preceq (M_1, \pi'_1)$.*

Proof. It suffices to establish this when one can obtain (M'_2, π'_2) from (M'_1, π'_1) by a single “elementary operation” of the kind described above. Our tensor factorization allows us to reduce to the case where the type attached to (M'_1, π'_1) is a tensor product of $(K_{m_i,\Pi}, \tau_{m_i,\Pi})$ for a single fixed Π , and $i = 1, \dots, r$, and the type attached to (M'_2, π'_2) is obtained by replacing $(K_{m_r,\Pi}, \tau_{m_r,\Pi})$ with a tensor product of j copies of $(K_{m',\Pi}, \tau_{m',\Pi})$, where m' immediately precedes m_r .

Theorem 7.13 then gives us a map:

$$f_m : E_{K_{m_r,\Pi},\tau_{m_r,\Pi}} \rightarrow E_{K_{m',\Pi},\tau_{m',\Pi}}^{\otimes j}$$

and we extend this by taking tensor products to a map:

$$E_{M_1,\pi'_1} \rightarrow E_{M_2,\pi'_2}.$$

The compatibility properties of f_m , together with our description of B_{M_1,π'_1} , shows that this map takes B_{M_1,π'_1} to B_{M_2,π'_2} , and the result follows. \square

Corollary 11.11. *The map $A_{[L,\pi]} \rightarrow B_{M_{\max},\pi_{\max}}$ giving the action of $A_{[L,\pi]}$ on $\mathcal{P}_{M_{\max},\pi_{\max}}$ is an isomorphism.*

In particular the embedding of $C_{M_{\max}, \pi_{\max}}$ in $B_{(M_{\max}, \pi_{\max})}$ yields an embedding of $C_{M_{\max}, \pi_{\max}}$ in $A_{[L, \pi]}$. When embedded in this way we write $C_{[L, \pi]}$ for the image of $C_{M_{\max}, \pi_{\max}}$ in $A_{[L, \pi]}$. Note that $C_{[L, \pi]}$ is a subalgebra of $A_{[L, \pi]}$ isomorphic to a polynomial ring over $W(k)$, and that $A_{[L, \pi]}$ is a finitely generated $C_{[L, \pi]}$ -module.

We make the following observations about the subalgebra $C_{[L, \pi]}$ of $A_{[L, \pi]}$:

Proposition 11.12. *The action of $C_{[L, \pi]}$ on $\mathcal{P}_{M', \pi'}$ identifies $C_{[L, \pi]}$ with $C_{M', \pi'}$ for all pairs (M', π') with supercuspidal support (L, π) .*

Proof. The map f_m of Theorem 7.13 takes the subalgebra C_{K_m, τ_m} of E_{K_m, τ_m} to a subalgebra of $E_{K_{m'}, \tau_{m'}}$. Thus, the map $B_{M_{\max}, \pi_{\max}} \rightarrow B_{M', \pi'}$ from Proposition 11.10 maps $C_{[L, \pi]}$ to $C_{M', \pi'}$. The resulting map from $C_{[L, \pi]}$ to $C_{M', \pi'}$ can be seen to be an isomorphism by invoking Proposition 11.8. \square

Proposition 11.8, together with our tensor factorization of $C_{[L, \pi]}$, gives an isomorphism of $C_{[L, \pi]}$ with $W(k)[Z_{\vec{s}}]^{W_{\overline{M}_{\vec{s}}(\vec{s})}}$, where \vec{s} is obtained by setting $s_i = (s'_{1, \Pi_i})^{m_i}$.

For each \vec{s}' , we then have a map:

$$C_{[L, \pi]} \hookrightarrow A_{[L, \pi]} \rightarrow A_{M_{\vec{s}'}, \pi_{\vec{s}'}} \cong W(k)[Z_{\vec{s}'}]^{W_{\overline{M}_{\vec{s}'}(\vec{s}')}}.$$

We can describe the composed map explicitly on generators as follows: the algebra $C_{[L, \pi]}$ is generated by elements of the form $\Theta_{r_1, \Pi_1} \otimes \cdots \otimes \Theta_{r_u, \Pi_u}$. The group $Z_{\vec{s}'}$ decomposes as a product of the groups $Z_{\vec{s}'(\Pi_i)}$ as i varies.

Proposition 11.13. *The map*

$$C_{[L, \pi]} \rightarrow W(k)[Z_{\vec{s}'}]^{W_{\overline{M}_{\vec{s}'}(\vec{s}')}}$$

takes the element $\Theta_{r_1, \Pi_1} \otimes \cdots \otimes \Theta_{r_u, \Pi_u}$ to the element: $\theta_{r_1, \vec{s}'(\Pi_1)} \otimes \cdots \otimes \theta_{r_u, \vec{s}'(\Pi_u)}$ of $W(k)[Z_{\vec{s}'}]$, where the latter is considered as the tensor product of the rings $W(k)[Z_{\vec{s}'(\Pi_i)}]$.

Proof. This is immediate from the tensor factorization, together with the calculation in the proof of Proposition 11.7. \square

The action of $C_{[L, \pi]}$ on representations over k has a description similar to the description of action of the Bernstein center in characteristic zero. As in the characteristic zero theory, let $\Psi(L)$ be the group of unramified characters of L , considered as an algebraic group over k , and let H be the subgroup of $\Psi(L)$ consisting of characters χ such that $\pi \otimes \chi$ is isomorphic to π . Let $W(L, \pi)$ be the subgroup of $w \in W(L)$ such that π^w is inertially equivalent to π .

Note that an unramified character χ of L over k corresponds to a map $f_\chi : k[L/L_0] \rightarrow k$, and for any two such characters χ, χ' , we have $\pi \otimes \chi$ inertially equivalent to $\pi \otimes \chi'$ if, and only if the restrictions of f_χ and $f_{\chi'}$ to $(k[L/L_0]^H)^{W_L(\pi)}$ agree.

Theorem 11.14. *There is a natural isomorphism:*

$$C_{L, \pi} \otimes_{W(k)} k \cong (k[L/L_0]^H)^{W_L(\pi)},$$

such that for any Π over k with supercuspidal support $(L, \pi \otimes \chi)$, $C_{L, \pi}$ acts on Π via the map $f_\chi : k[L/L_0] \rightarrow k$. (In particular, one can recover the supercuspidal support of Π from the action of $C_{L, \pi}$.)

Proof. Define \bar{s}' by $s'_i = (s'_{\Pi_i})^{m_i}$ for all i , where $(K_{m_i, \Pi_i}, \tau_{m_i, \Pi_i})$ are the types giving rise to (M_{\max}, π_{\max}) . Then $M_{\bar{s}'}$ is conjugate to L , and $\pi_{\bar{s}'}$ is conjugate to an unramified twist of a lift $\tilde{\pi}$ of π . In particular this conjugation takes $W_{M_{\bar{s}'}}(\pi_{\bar{s}'})$ to $W_L(\pi)$, and so we have an isomorphism of $A_{M_{\bar{s}'}, \pi_{\bar{s}'}}$ with $(\bar{\mathcal{K}}[L/L_0]^H)^{W_L(\pi)}$. Proposition 11.8 then shows that the composed map:

$$A_{L, \pi} \rightarrow A_{M_{\bar{s}'}, \pi_{\bar{s}'}} \cong (\bar{\mathcal{K}}[L/L_0]^H)^{W_L(\pi)}$$

identifies $C_{L, \pi}$ with $(W(k)[L/L_0]^H)^{W_L(\pi)}$. We normalize the second isomorphism via the pair $(L, \tilde{\pi})$, so that, under this isomorphism, the action of $C_{L, \pi}$ on a representation $\tilde{\Pi} = \tilde{\pi} \otimes \tilde{\chi}$ in the block corresponding to $(M_{\bar{s}'}, \pi_{\bar{s}'})$ is via $f_{\tilde{\chi}} : W(k)[L/L_0] \rightarrow \bar{\mathcal{K}}$.

Let $\tilde{\Pi}$ be a lift of Π to a representation over $\bar{\mathcal{K}}$ in the inertial equivalence class corresponding to $(M_{\bar{s}'}, \pi_{\bar{s}'})$; then there exists an unramified character $\tilde{\chi}$ such that $\tilde{\Pi} = \tilde{\pi} \otimes \tilde{\chi}$. Then $C_{L, \pi}$ acts on $\tilde{\Pi}$ via $f_{\tilde{\chi}}$, so it acts on Π via f_{χ} . \square

12. COROLLARIES

Let L be a Levi subgroup of G and fix an irreducible supercuspidal representation π of L over k . We summarize several implications of section 11 for $A_{[L, \pi]}$ below:

Theorem 12.1. *The ring $A_{[L, \pi]}$ is a finitely generated, reduced, ℓ -torsion free $W(k)$ -algebra.*

Proof. As $A_{[L, \pi]}$ is a finitely generated $C_{[L, \pi]}$ -module, and $C_{[L, \pi]}$ is a polynomial ring over $W(k)$, it is immediate that $A_{[L, \pi]}$ is a finitely generated $W(k)$ -algebra. The fact that $A_{[L, \pi]}$ is ℓ -torsion free follows from $A_{[L, \pi]} \subset \text{End}_{W(k)[G]}(\mathcal{P}_{[L, \pi]})$ and the fact that $\mathcal{P}_{[L, \pi]}$ is ℓ -torsion free. Reducedness follows from the fact that $A_{[L, \pi]}$ embeds in $A_{[L, \pi]} \otimes_{W(k)} \bar{\mathcal{K}}$, and the latter is reduced by Proposition 11.1. \square

Theorem 12.2. *If Π and Π' are irreducible representations of G over k that lie in $\text{Rep}_{W(k)}(G)_{[L, \pi]}$, and $f_{\Pi}, f_{\Pi'}$ are the maps $A_{[L, \pi]} \rightarrow k$ giving the action of $A_{L, \pi}$ on Π and Π' respectively, then $f_{\Pi} = f_{\Pi'}$ if, and only if, Π and Π' have the same supercuspidal supports.*

Proof. Suppose first that Π and Π' both have supercuspidal support (L, π') for some π' inertially equivalent to π . Let $\tilde{\pi}'$ be a lift of π' to $\bar{\mathcal{K}}$. Then both Π and Π' are subquotients of $i_P^G \tilde{\pi}'$. On the other hand the action of $A_{[L, \pi]}$ on $i_P^G \tilde{\pi}'$ factors through $A_{[L, \pi]} \otimes \bar{\mathcal{K}}$, and the latter acts on $i_P^G \tilde{\pi}'$ by scalars. It follows that any element of $A_{[L, \pi]}$ acts on Π and Π' by the same scalar. (Note that this direction uses nothing of our explicit computations from the previous section.)

The converse is an immediate consequence of Theorem 11.14 of the previous section. \square

Proposition 12.3. *The faithfully projective module $\mathcal{P}_{L, \pi}$ is an admissible $A_{L, \pi}[G]$ -module.*

Proof. The module $\mathcal{P}_{L, \pi}$ is a direct sum of modules $\mathcal{P}_{M', \pi'}$; each of the latter is an admissible $E_{M', \pi'}[G]$ -module by Theorem 9.1. As $E_{M', \pi'}$ is a finitely generated $C_{M', \pi'}$ -module, $\mathcal{P}_{M', \pi'}$ is also admissible over $C_{M', \pi'}$. Inside $A_{L, \pi}$ we have $C_{[L, \pi]}$, and the map $A_{[L, \pi]} \rightarrow \text{End}_{W(k)[G]}(\mathcal{P}_{M', \pi'})$ identifies $C_{[L, \pi]}$ with $C_{M', \pi'}$. Thus each $\mathcal{P}_{M', \pi'}$ is admissible over $A_{[L, \pi]}$, and we are done. \square

Corollary 12.4. *Let Π be an object of $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ that is finitely generated as a $W(k)[G]$ -module. Then Π is an admissible $A_{L,\pi}[G]$ -module.*

Proof. As $\mathcal{P}_{[L,\pi]}$ is faithfully projective, there is a surjection of a direct sum of (possibly infinitely many) copies of $\mathcal{P}_{[L,\pi]}$ onto Π . Any element x of Π is in the image of this surjection, and any element \tilde{x} that maps to x is nonzero only in finitely many copies of $\mathcal{P}_{[L,\pi]}$. Thus if Π is generated by finitely many elements, then there is a finite direct sum of copies of $\mathcal{P}_{[L,\pi]}$ whose image in Π contains all the generators, and this direct sum surjects onto Π . \square

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