

Towards a geometric Jacquet-Langlands correspondence for unitary Shimura varieties

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Let G be a unitary group over a totally real field, and X a Shimura variety associated to G . For certain primes p of good reduction for X , we construct cycles $X_{\tau_0, i}$ on the characteristic p fiber of X . These cycles are defined as the loci on which the Verschiebung map has small rank on particular pieces of the Lie algebra of the universal abelian variety on X .

The geometry of these cycles turns out to be closely related to Shimura varieties for a *different* unitary group G' , which is isomorphic to G at all finite places but not isomorphic to G at archimedean places. More precisely, each cycle $X_{\tau_0, i}$ has a natural desingularization $\tilde{X}_{\tau_0, i}$, which is “almost” isomorphic to a scheme parametrizing certain subbundles of the Lie algebra of the universal abelian variety over a Shimura variety X' associated to G' .

We exploit this relationship to construct an injection of the étale cohomology of X' into that of X . This yields a geometric construction of “Jacquet-Langlands transfers” of automorphic representations of G' to automorphic representations of G .

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1 Introduction

Suppose G and G' are two algebraic groups over \mathbb{Q} , isomorphic at all finite places of \mathbb{Q} but not necessarily isomorphic at infinity. The Jacquet-Langlands correspondence predicts, in many cases, that there is a natural bijection between the automorphic representations of G and those for G' , such that if π' is the representation of G' corresponding to a representation π of G , then π_v is isomorphic to π'_v for all finite places v .

This correspondence is proven in many cases by comparing the trace formulas for G and G' . In this way one can conclude that there is an isomorphism between suitable spaces of automorphic forms for G and G' as abstract representations, but not in any canonical fashion. One might therefore hope for a more natural way of understanding this correspondence.

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For GL_2 , Serre [Se] was the first to suggest an alternative approach to Jacquet-Langlands, in the context of modular forms mod p . By considering modular forms as sections of a line bundle on a modular curve, and restricting these sections to the supersingular locus of this curve in characteristic p , Serre relates the space of modular forms mod p to a space of functions on this supersingular locus; the latter can be interpreted as a space of “algebraic modular forms” for the quaternion algebra B ramified at p and infinity. Ghitza has since adapted this approach for symplectic groups [Gh1], [Gh2].

In contrast to the traditional approach to Jacquet-Langlands, the approach of Serre and Ghitza yields canonical isomorphisms between spaces that arise naturally from the geometry of Shimura varieties attached to the groups under consideration (rather than simply a bijection of isomorphism classes of representations.)

Another approach can be found in the work of Ribet ([Ri2], [Ri1]). Ribet finds a relationship between the reductions at various primes of two Shimura curves associated to two *different* quaternion algebras over \mathbb{Q} . He uses this to obtain an explicit isomorphism between certain Hecke modules for the two quaternion algebras, and thereby proves the Jacquet-Langlands correspondence in that setting. This sharpening of the Jacquet-Langlands correspondence is a key ingredient in his proof of level-lowering for classical modular forms.

More recently, work of the author in [He] adapts Ribet’s techniques to the case of a unitary group G isomorphic (up to a factor of \mathbb{R}^\times) to a product of $U(1, 1)$ ’s at infinity. As with Ribet’s approach, the key is to understand the reduction of a Shimura variety X attached to G that has an analogue of $\Gamma_0(p)$ level structure at p . We obtain an explicit description of the global structure of the special fiber in this setting: the irreducible components each are (nearly) isomorphic to products of projective bundles over Shimura varieties X' attached to unitary groups G' that are isomorphic to G at all finite places but not in general isomorphic to G at infinity. Via the theory of vanishing cycles, one can then relate the étale cohomology of X to the étale cohomology of the various X' that arise; the upshot is that the highest weight quotient of the étale cohomology of X can be interpreted in terms of a space of algebraic modular forms (over \mathbb{Q}_ℓ) for a unitary group G' that is compact at infinity.

In this paper we present a different approach, that works for arbitrary unitary groups, but proceeds by considering Shimura varieties at primes of *good* reduction. Given a Shimura variety X arising from a unitary group G , and a suitable prime p of good reduction, we consider cycles $X_{\tau_0, i}$ on the characteristic p fiber of X . (Here i is a positive integer and τ_0 determines a map $p_{\tau_0} : \mathcal{O}_F \rightarrow \overline{\mathbb{F}}_p$, where F is the CM -field arising in the definition of G .) Loosely speaking, $X_{\tau_0, i}$ is the locus of abelian varieties A (with \mathcal{O}_F -action) such that the space $\text{Hom}(\alpha_p, A[p])_{p_{\tau_0}}$ of maps on which \mathcal{O}_F acts via p_{τ_0} has dimension at least i larger than the “expected dimension”. Alternatively, $X_{\tau_0, i}$ can be thought of as the locus of abelian varieties A such that

$$\text{Ver} : \text{Lie}(A^{(p)})_{p_{\tau_0}} \rightarrow \text{Lie}(A)_{p_{\tau_0}}$$

has rank at least i less than the “expected rank”. Such cycles are closed strata

in the so-called “a-number” stratification of X , and their local structure has been studied extensively.

Our approach requires an understanding of the *global* structure of these strata in addition to the local structure. Questions of this nature are not nearly as well-understood; fortunately in the cases we are interested in they can be attacked by fairly standard techniques. We construct a natural desingularization of each cycles $\tilde{X}_{\tau_0, i}$. The geometry of this desingularization turns out to be closely related to the geometry of a Shimura variety X' arising from a different unitary group G' . As in [He], G' is isomorphic to G at finite places but not at infinity. In particular, we construct a scheme $(X')^{\tau_0, i}$, defined naturally in terms of the universal abelian variety over X' , such that there exists a scheme $\hat{X}_{\tau_0, i}$, together with maps:

$$\begin{aligned} \hat{X}_{\tau_0, i} &\rightarrow \tilde{X}_{\tau_0, i} \\ \hat{X}_{\tau_0, i} &\rightarrow (X')^{\tau_0, i} \end{aligned}$$

that are bijections on points and isomorphisms on étale cohomology. Loosely speaking, this says that $\tilde{X}_{\tau_0, i}$ and $(X')^{\tau_0, i}$ are “nearly” isomorphic. The fibers of $(X')^{\tau_0, i}$ over X' are Grassmannians of various dimensions. (This generalizes results of [He] for the case of $U(2)$ Shimura varieties).

Rather attempting to establish a geometric Jacquet-Langlands correspondence as in [He], by way of a suitable “Deligne-Rapoport model”, we use the above construction to give an explicit injection of the étale cohomology of X' into that of X . (Theorem 5.3.) The existence of this map follows from a general construction in étale cohomology; its injectivity is more difficult to prove. The key ingredients are the Leray spectral sequence for $(X')^{\tau_0, i}$ and the Thom-Porteus formula, which allows us to compute the self-intersection of $X_{\tau_0, i}$. This argument is the main focus of sections 5 and 6.

We obtain cases of Jacquet-Langlands transfer as an easy corollary of the existence of this injection. (Theorem 7.2) In particular we show that for every cohomological automorphic representation π' of G' , there is an automorphic representation π of G such that π and π' agree at all finite places. Our approach suffers from the limitation that it is only possible to transfer such representations from one Shimura variety to another Shimura variety of higher dimension; going in the other direction requires some way of controlling the *image* of the map constructed in Theorem 5.3, which we do not yet have at our disposal.

This approach appears to cover some new cases of Jacquet-Langlands transfer that have not yet appeared in the literature. In particular the traditional trace formula approach to Jacquet-Langlands runs into difficulty with unitary groups that are not compact at infinity. On the other hand, Harris and Labesse ([HL], particularly Theorems 2.1.2, 3.1.6, and Proposition 3.1.7) have used base change techniques to establish rather general Jacquet-Langlands results for unitary groups, but need the representation under consideration to be supercuspidal at certain places.

It should also be noted that whereas traditional approaches to Jacquet-Langlands yield a bijection of isomorphism classes of automorphic representations, our approach yields information about a natural map between spaces

that arise naturally in geometry, and have considerable arithmetic interest. One might therefore hope for arithmetic applications of this approach, in analogy with the application of Ribet's results on character groups to level-lowering.

2 Basic definitions and properties

We begin with the definition and basic properties of unitary Shimura varieties.

Fix a totally real field F^+ , of degree d over \mathbb{Q} . Let E be an imaginary quadratic extension of \mathbb{Q} , of discriminant D , and let x be a square root of D in E . Let F be the field EF^+ .

Also fix a square root \sqrt{D} of D in \mathbb{C} . Then any embedding $\tau : F^+ \rightarrow \mathbb{R}$ induces two embeddings $p_\tau, q_\tau : F \rightarrow \mathbb{C}$, via

$$\begin{aligned} p_\tau(a + bx) &= \tau(a) + \tau(b)\sqrt{D} \\ q_\tau(a + bx) &= \tau(a) - \tau(b)\sqrt{D}. \end{aligned}$$

Fix an integer n , and an n -dimensional F -vector space \mathcal{V} , equipped with an alternating, nondegenerate pairing $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{Q}$. We require that

$$\langle \alpha x, y \rangle = \langle x, \bar{\alpha} y \rangle$$

for all α in F .

Each embedding $\tau : F^+ \rightarrow \mathbb{R}$ gives us a complex vector space $\mathcal{V}_\tau = \mathcal{V} \otimes_{F^+, \tau} \mathbb{R}$. The pairing $\langle \cdot, \cdot \rangle$ on \mathcal{V} is the ‘‘imaginary part’’ of a unique Hermitian pairing $[\cdot, \cdot]_\tau$ on \mathcal{V}_τ ; we denote the number of 1's in the signature of $[\cdot, \cdot]_\tau$ by $r_\tau(\mathcal{V})$, and the number of -1 's by $s_\tau(\mathcal{V})$. If \mathcal{V} is obvious from the context, we will often omit it, and denote $r_\tau(\mathcal{V})$ and $s_\tau(\mathcal{V})$ by r_τ and s_τ . We fix a $\hat{\mathcal{O}}_F$ -lattice T inside $\mathcal{V}(\mathbb{A}_\mathbb{Q}^f)$, such that λ induces a map $T \rightarrow \text{Hom}(T, \hat{\mathbb{Z}})$.

Let G be the algebraic group over \mathbb{Q} such that for any \mathbb{Q} -algebra R , $G(R)$ is the subgroup of $\text{Aut}_F(\mathcal{V} \otimes_{\mathbb{Q}} R)$ consisting of all g such that there exists an r in R^\times with $\langle gx, gy \rangle = r \langle x, y \rangle$ for all x and y in $\mathcal{V} \otimes_{\mathbb{Q}} R$. The discussion in the previous paragraph shows that $G(\mathbb{R})$ is the subgroup of

$$\prod_{\tau: F^+ \rightarrow \mathbb{R}} \text{GU}(r_\tau, s_\tau)$$

consisting of those tuples $(g_\tau)_{\tau: F^+ \rightarrow \mathbb{R}}$ such that the ‘‘similitude ratio’’ of g_τ is the same for all τ .

Now fix a compact open subgroup U of $G(\mathbb{A}_\mathbb{Q}^f)$, preserving T , and consider the Shimura variety associated to G and U . If U is sufficiently small, this variety can be thought of as a fine moduli space for abelian varieties with PEL structures. We now describe such a model over a suitable ring of Witt vectors.

Fix a prime p split in E , such that the cokernel of the map $T \rightarrow \text{Hom}(T, \hat{\mathbb{Z}})$ is supported away from p and such that U_p is equal to the subgroup of all elements of $G(\mathbb{Q}_p)$ preserving $T(\mathbb{Q}_p)$. Also fix a finite field k_0 of characteristic p large enough to contain subfields isomorphic to each of the residue fields of \mathcal{O}_F/p ,

and an identification of the Witt vectors $W(k_0)$ with a subring of \mathbb{C} . This choice of identification induces a bijection of the set of archimedean places of F with the set of algebra maps $\mathcal{O}_F \rightarrow W(k_0)$. In an abuse of notation we will use the symbols p_τ and q_τ to represent both the embeddings $F \rightarrow \mathbb{C}$ defined above, and the maps $\mathcal{O}_F \rightarrow W(k_0)$ defined above.

Note that if S is a $W(k_0)$ -scheme, and M is a $W(k_0)$ -module with an action of \mathcal{O}_F , then M splits as a direct sum

$$M = \bigoplus_{\tau} M_{p_\tau} \oplus M_{q_\tau},$$

where \mathcal{O}_F acts of M_{p_τ} (resp. M_{q_τ}) by $p_\tau : \mathcal{O}_F \rightarrow W(k_0)$ (resp. $q_\tau : \mathcal{O}_F \rightarrow W(k_0)$.)

Consider the functor that associates to each $W(k_0)$ -scheme S the set of isomorphism classes of triples (A, λ, ρ) where:

1. A is an abelian scheme over S of dimension nd , with an action of \mathcal{O}_F
2. λ is a polarization of A , of degree prime to p , such that the Rosati involution associated to λ induces complex conjugation on $\mathcal{O}_F \subset \text{End}(A)$.
3. ρ is a U -orbit of isomorphisms $T^{(p)} \rightarrow T_{\hat{\mathbb{Z}}^{(p)}} A$, sending the Weil pairing on $T_{\hat{\mathbb{Z}}^{(p)}} A$ to a scalar multiple of the pairing \langle, \rangle on $T^{(p)}$. (Here $T_{\hat{\mathbb{Z}}^{(p)}} A$ denotes the product over all $l \neq p$ of the l -adic Tate modules of A , and the superscript (p) denotes the non-pro- p part of T or $\hat{\mathbb{Z}}$.)
4. When considered as an endomorphism of $\text{Lie}(A/S)$, an element α of \mathcal{O}_F has characteristic polynomial

$$\prod_{\tau: F^+ \rightarrow \mathbb{R}} (x - p_\tau(\alpha))^{r_\tau(\mathcal{V})} (x - q_\tau(\alpha))^{s_\tau(\mathcal{V})}.$$

Since $\text{Lie}(A/S)$ is an $\mathcal{O}_F \otimes S$ -module, we can rephrase the characteristic polynomial condition as follows: For each $\tau : F^+ \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \text{rank}_S \text{Lie}(A/S)_{p_\tau} &= r_\tau(\mathcal{V}) \\ \text{rank}_S \text{Lie}(A/S)_{q_\tau} &= s_\tau(\mathcal{V}). \end{aligned}$$

If U is sufficiently small, this functor is represented by a smooth $W(k_0)$ -scheme, which we denote $X_U(\mathcal{V})$. It is a model for the Shimura variety associated to G and U , over $W(k_0)$. Henceforth we will refer to such an object as a 'unitary Shimura variety'. Its dimension is given by the formula

$$\dim X_U(\mathcal{V}) = \sum_{\tau: F^+ \rightarrow \mathbb{R}} r_\tau s_\tau.$$

Remark 2.1 The scheme $X_U(\mathcal{V})$ depends not only on U and \mathcal{V} but on all of the choices we have made in this section. To avoid clutter, we have chosen to suppress most of these choices in our notation.

3 Dieudonné theory and points on $X_U(\mathcal{V})$

Let k be a perfect field containing k_0 , and let (A, λ, ρ) be a k -valued point of $X_U(\mathcal{V})$. We begin by studying the (contravariant) Dieudonné modules of $A[p]$ and $A[p^\infty]$.

Let \mathcal{D}_A denote the contravariant Dieudonné module of $A[p^\infty]$. It is a free $W(k)$ -module of rank $2nd$, equipped with endomorphisms F and V , that satisfy $FV = VF = p$. These endomorphisms do not commute with the action of $W(k)$, but instead satisfy:

$$\begin{aligned} F\alpha &= \alpha^\sigma F \\ V\alpha^\sigma &= \alpha V, \end{aligned}$$

where $\alpha \in W(k)$, and the superscript σ denotes the Witt vector Frobenius.

The \mathcal{O}_F -action on A induces an \mathcal{O}_F action on \mathcal{D}_A ; we therefore have a direct sum decomposition:

$$\mathcal{D}_A = \bigoplus_{\tau: F^+ \rightarrow \mathbb{R}} (\mathcal{D}_A)_{p_\tau} \oplus (\mathcal{D}_A)_{q_\tau}.$$

For $\tau: F^+ \rightarrow \mathbb{R}$, let $p_{\sigma\tau}$ denote the map $\mathcal{O}_F \rightarrow W(k_0)$ obtained by taking the map $\mathcal{O}_F \rightarrow W(k_0)$ corresponding to p_τ and composing it with the Witt vector Frobenius. Define $q_{\sigma\tau}$ similarly. Then the σ -linearity properties of F and V mean that they induce maps:

$$\begin{aligned} F: (\mathcal{D}_A)_{p_\tau} &\rightarrow (\mathcal{D}_A)_{p_{\sigma\tau}} \\ V: (\mathcal{D}_A)_{p_{\sigma\tau}} &\rightarrow (\mathcal{D}_A)_{p_\tau}, \end{aligned}$$

and similarly for the q_τ . Since $FV = VF = p$, we find that $(\mathcal{D}_A)_{p_\tau}$ and $(\mathcal{D}_A)_{p_{\sigma\tau}}$ have the same rank for all τ , as do $(\mathcal{D}_A)_{q_\tau}$ and $(\mathcal{D}_A)_{q_{\sigma\tau}}$.

If we fix a prime \mathfrak{p} of \mathcal{O}_F over p , then the Dieudonné module of $A[\mathfrak{p}^\infty]$ is the direct sum of $(\mathcal{D}_A)_{p_\tau}$ for those p_τ (or possibly q_τ) for which the preimage of the ideal (p) of $W(k_0)$ under the corresponding map $\mathcal{O}_F \rightarrow W(k_0)$ is \mathfrak{p} . These form a single orbit under the action of σ described above, so they all have the same rank. But since the height of $A[\mathfrak{p}^\infty]$ is n times the residue class degree of \mathfrak{p} over p , it follows that $(\mathcal{D}_A)_{p_\tau}$ and $(\mathcal{D}_A)_{q_\tau}$ are free $W(k)$ -modules of rank n for all τ .

Now consider the quotient $\overline{\mathcal{D}}_A = \mathcal{D}_A/p\mathcal{D}_A$. It is canonically isomorphic to the Dieudonné module of $A[p]$. The above discussion shows that for each τ , $(\overline{\mathcal{D}}_A)_{p_\tau}$ and $(\overline{\mathcal{D}}_A)_{q_\tau}$ are n -dimensional k -vector spaces. Moreover, Oda [Od] has shown that there is a natural isomorphism $H_{\text{DR}}^1(A/k) \cong \overline{\mathcal{D}}_A$, and that this isomorphism identifies the Hodge flag $\text{Lie}(A/k)^* \subset H_{\text{DR}}^1(A/k)$ with the subspace $V\overline{\mathcal{D}}_A$ of $\overline{\mathcal{D}}_A$.

In particular, we have $\dim V((\overline{\mathcal{D}}_A)_{p_{\sigma\tau}}) = \dim \text{Lie}(A/k)_{p_\tau}^* = r_\tau$. Since the image of V is equal to the kernel of F on $\overline{\mathcal{D}}_A$, we also have $\dim F((\overline{\mathcal{D}}_A)_{p_\tau}) = s_\tau$.

Thus, for each τ , $(\overline{\mathcal{D}}_A)_{p_\tau}$ is an n -dimensional k -vector space with two distinguished subspaces, $F_\tau = F((\overline{\mathcal{D}}_A)_{p_{\sigma^{-1}\tau}})$ (of dimension $s_{\sigma^{-1}\tau}$), and $V_\tau = V((\overline{\mathcal{D}}_A)_{p_{\sigma\tau}})$ (of dimension r_τ .)

Fix a particular τ_0 , and assume, for the rest of the paper, that $r_{\sigma^{-1}\tau_0} \leq r_{\tau_0}$. (If this does not hold, then $s_{\sigma^{-1}\tau_0} \leq s_{\tau_0}$, and everything that follows will still be true once one reverses the roles of p_τ and q_τ .) In this case, if F_{τ_0} and V_{τ_0} are in general position with respect to each other, then their sum will span all of $(\overline{\mathcal{D}}_A)_{\tau_0}$. Of course, F_{τ_0} and V_{τ_0} need not be in general position with respect to one another, which motivates the following definition:

Definition 3.1 *Let i be an integer between 0 and $\min(r_{\sigma^{-1}\tau}, s_\tau)$, inclusive. A point (A, λ, ρ) is (τ_0, i) -special if $\dim F_{\tau_0} + V_{\tau_0} \leq n - i$. A subspace H of $(\overline{\mathcal{D}}_A)_{p_{\tau_0}}$ is (τ_0, i) -special if it has dimension $n - i$ and contains both F_{τ_0} and V_{τ_0} .*

Note that (A, λ, ρ) admits an H that is (τ_0, i) -special if and only if (A, λ, ρ) itself is (τ_0, i) -special, and that such an H will be unique if and only if (A, λ, ρ) is (τ_0, i) -special but not $(\tau_0, i + 1)$ -special.

Suppose we have (A, λ, ρ) , along with a (τ_0, i) -special H for this abelian variety. Define a submodule M_H of $\overline{\mathcal{D}}_A$ as follows:

1. $(M_H)_{p_{\tau_0}} = H$
2. $(M_H)_{p_\tau} = (\overline{\mathcal{D}}_A)_{p_\tau}$ for $\tau \neq \tau_0$
3. $(M_H)_{q_\tau} = (M_H)_{p_\tau}^\perp$, where \perp denotes orthogonal complement under the perfect pairing $(\overline{\mathcal{D}}_A)_{p_\tau} \times (\overline{\mathcal{D}}_A)_{q_\tau} \rightarrow k$ induced by the polarization λ .

It is clear that M_H is stable under $W(k)$, \mathcal{O}_F , F , and V . In particular, it is a Dieudonné submodule of $\overline{\mathcal{D}}_A$. We thus obtain an exact sequence:

$$0 \rightarrow M_H \rightarrow \overline{\mathcal{D}}_A \rightarrow \overline{\mathcal{D}}_K \rightarrow 0$$

where $\overline{\mathcal{D}}_K$ is the Dieudonné module of a group scheme K over k . The surjection $\overline{\mathcal{D}}_A \rightarrow \overline{\mathcal{D}}_K$ corresponds to an inclusion of K in $A[p]$; henceforth we identify K with its image in $A[p]$. Since M_H is a maximal isotropic subspace of $\overline{\mathcal{D}}_A$ under the pairing induced by λ , K is a maximal isotropic subgroup of $A[p]$ (under the Weil pairing induced by λ).

Let $B = A/K$, and let $f : A \rightarrow B$ denote the quotient map. Since $K \subset A[p]$, multiplication by p (considered as an endomorphism of A) factors through f . In this way we obtain a map f' such that $f'f = f'f = p$. Note that $f'(B[p])$ is equal to K .

Consider the polarization $(f')^\vee \lambda f'$ of B . For any α, β in $B[p]$, we have

$$\langle \alpha, (f')^\vee \lambda f' \beta \rangle_B = \langle f' \alpha, \lambda f' \beta \rangle_A.$$

The right-hand side vanishes identically since K is isotropic and $f'(B[p]) = K$. Thus $B[p]$ lies in the kernel of $(f')^\vee \lambda f'$, and so there is a unique polarization λ' of B such that $p\lambda' = (f')^\vee \lambda f'$. (Note that λ' can also be characterized as the unique polarization of B such that $p\lambda = f^\vee \lambda' f$.) The degree of λ is easily seen to be prime to p .

Proposition 3.2 *Suppose that $\sigma\tau_0 \neq \tau_0$. Then*

1. $\dim \operatorname{Lie}(B/k)_{p_{\tau_0}} = r_{\tau_0} + i$.
2. $\dim \operatorname{Lie}(B/k)_{p_{\sigma^{-1}\tau_0}} = r_{\sigma^{-1}\tau_0} - i$.
3. $\dim \operatorname{Lie}(B/k)_{p_\tau} = r_\tau$ for τ not equal to τ_0 or $\sigma^{-1}\tau_0$.
4. $\dim \operatorname{Lie}(B/k)_{q_\tau} = n - \dim \operatorname{Lie}(B/k)_{p_\tau}$ for all τ .

Proof. The quotient map $f : A \rightarrow B$ induces a map $\overline{\mathcal{D}}_B \rightarrow \overline{\mathcal{D}}_A$, where $\overline{\mathcal{D}}_B$ is the Dieudonné module of $B[p]$. The image of this map is precisely M_H . On the level of p -divisible groups, therefore, f induces an inclusion of \mathcal{D}_B into \mathcal{D}_A , that identifies \mathcal{D}_B with the submodule of \mathcal{D}_A consisting of those elements whose images in $\overline{\mathcal{D}}_A$ lie in M_V . We identify \mathcal{D}_B with this submodule for the remainder of the argument.

By the isomorphism between Dieudonné modules and DeRham cohomology,

$$\dim \operatorname{Lie}(B/k)_{p_\tau} = \dim V((\mathcal{D}_B)_{p_{\sigma\tau}})/p(\mathcal{D}_B)_{p_\tau}.$$

On the other hand, we have:

1. $(\mathcal{D}_B)_{p_\tau} = (\mathcal{D}_A)_{p_\tau}$ for $\tau \neq \tau_0$.
2. $(\mathcal{D}_A)_{p_{\tau_0}}/(\mathcal{D}_B)_{p_{\tau_0}}$ has dimension i .
3. $V((\mathcal{D}_A)_{p_{\sigma\tau}})/p(\mathcal{D}_A)_{p_\tau}$ has dimension r_τ for all τ .

Statements (1), (2), and (3) of the proposition follow immediately from the above paragraph. Statement (4) follows from the existence of the prime-to- p polarization λ' on B . \square

Note that if $\sigma\tau_0 = \tau_0$, then the result above fails. (In particular, the proof of the result shows in this case that $\dim \operatorname{Lie}(B/K)_{p_\tau} = r_\tau$ for all τ .) Since the above proposition is crucial to our argument, we assume, for the remainder of the paper, that $\sigma\tau_0 \neq \tau_0$.

The upshot of the above proposition is that (B, λ') is “nearly” a k -valued point a unitary Shimura variety. It lacks only a level structure. We cannot define such a level structure in terms of \mathcal{V} , however, as $r_{\tau_0}(\mathcal{V}) = r_{\tau_0}$ but $\dim \operatorname{Lie}(B/k)_{p_{\tau_0}} = r_{\tau_0} + i$. We thus invoke the following lemma, proven in the appendix of [He]:

Lemma 3.3 *There exists an n -dimensional F -vector space \mathcal{V}' , together with a pairing \langle, \rangle' satisfying the conditions of section 2, such that:*

1. $r_{\tau_0}(\mathcal{V}') = r_{\tau_0} + i$.
2. $r_{\sigma^{-1}\tau_0}(\mathcal{V}') = r_{\sigma^{-1}\tau_0} - i$.
3. $r_\tau(\mathcal{V}') = r_\tau$ for τ not equal to τ_0 or $\sigma^{-1}\tau_0$.
4. *There exists an isomorphism ϕ of $\mathcal{V}(\mathbb{A}_{\mathbb{Q}}^f)$ with $\mathcal{V}'(\mathbb{A}_{\mathbb{Q}}^f)$ that takes the pairing \langle, \rangle to a scalar multiple of \langle, \rangle' .*

We fix, once and for all, a \mathcal{V}' , \langle, \rangle' and ϕ as in the lemma. Let $T' = \phi(T)$, and let G' be the algebraic group such that for each \mathbb{Q} -algebra R , $G'(R)$ is the subset of $\text{Aut}_F(\mathcal{V}' \otimes_{\mathbb{Q}} R)$ consisting of those automorphisms that send \langle, \rangle' to a scalar multiple of itself. Then ϕ induces an isomorphism $G(\mathbb{A}_{\mathbb{Q}}^f) \cong G'(\mathbb{A}_{\mathbb{Q}}^f)$, and this identifies U with a subgroup U' of G' . If ρ is a U -level structure on (A, λ) , then it follows from this construction that $f \circ \rho \circ \phi^{-1}$ is a U' -level structure on (B, λ') . In particular, $(B, \lambda', f \circ \rho \circ \phi^{-1})$ is a k -valued point of the unitary Shimura variety $X_{U'}$ associated to the subgroup U' of G' .

The map that associates to each (A, λ, ρ, V) the point $(B, \lambda', f \circ \rho \circ \phi^{-1})$ is not in general a bijection. We will now proceed to remedy this, by describing the extra information needed to recover (A, λ, ρ, V) from $(B, \lambda', f \circ \rho \circ \phi^{-1})$.

Definition 3.4 *Let (B, λ', ρ') be a point on $X_{U'}(k)$. A subspace W of $(\overline{\mathcal{D}}_B)_{p_{\tau_0}}$ is (τ_0, i) -constrained if it has dimension i and is contained in both $V((\overline{\mathcal{D}}_B)_{p_{\sigma\tau_0}})$ and $F((\overline{\mathcal{D}}_B)_{p_{\sigma^{-1}\tau_0}})$.*

Lemma 3.5 *Let (A, λ, ρ, V) be a point on $X_U(k)$ together with a (τ_0, i) -special V , and let $(B, \lambda', f \circ \rho \circ \phi^{-1})$ be the corresponding point of $X_{U'}(k)$. Let*

$$W = \ker f : (\overline{\mathcal{D}}_B)_{p_{\tau_0}} \rightarrow (\overline{\mathcal{D}}_A)_{p_{\tau_0}}.$$

Then W is (τ_0, i) -constrained.

Proof. Note that since $f : (\overline{\mathcal{D}}_B)_{p_{\tau}} \rightarrow (\overline{\mathcal{D}}_A)_{p_{\tau}}$ is an isomorphism for $\tau \neq \tau_0$, we have that

$$W = \ker f : \bigoplus_{\tau} (\overline{\mathcal{D}}_B)_{p_{\tau}} \rightarrow \bigoplus_{\tau} (\overline{\mathcal{D}}_A)_{p_{\tau}}.$$

In particular W is stable under F and V ; but since F and V send W to $(\overline{\mathcal{D}}_{p_{\sigma\tau_0}})$ and $(\overline{\mathcal{D}}_{p_{\sigma^{-1}\tau_0}})$, and neither of these contain any nonzero element of W , we have that W is killed by both F and V . The result follows immediately. \square

We have thus associated to each tuple (A, λ, ρ, V) a tuple $(B, \lambda', f \circ \rho \circ \phi^{-1}, W)$. We now describe an inverse construction.

Let (B, λ', ρ') be a point in $X_{U'}(k)$, and let W be a (τ_0, i) -constrained subspace of $(\overline{\mathcal{D}}_B)_{p_{\tau_0}}$. Define a submodule N_W of $\overline{\mathcal{D}}_B$ by:

1. $(N_W)_{p_{\tau_0}} = W$
2. $(N_W)_{\tau} = 0$ for $\tau \neq \tau_0$
3. $(N_W)_{q_{\tau}} = (N_W)_{p_{\tau}}^{\perp}$ for all τ .

It is clear that N_W is stable under F and V , and is a maximal isotropic submodule of $\overline{\mathcal{D}}_B$. The inclusion of N_W in $\overline{\mathcal{D}}_B$ fits into an exact sequence

$$0 \rightarrow N_W \rightarrow \overline{\mathcal{D}}_B \rightarrow \overline{\mathcal{D}}(K') \rightarrow 0,$$

where $\overline{\mathcal{D}}(K')$ is the Dieudonné module of a subgroup K' of B .

Let $A = B/K'$, and let $f' : B \rightarrow A$ be the natural quotient map. Then, just as before, there is a natural polarization λ on A such that $p\lambda = (f')^{\vee} \lambda' f'$.

Lemma 3.6 *The dimension of $\text{Lie}(A/k)_{p_\tau}$ (resp. $\text{Lie}(A/k)_{q_\tau}$) is r_τ (resp. s_τ) for all τ .*

Proof. The proof of this lemma is identical to the proof of Lemma 3.3, and we omit it. \square

It follows that the triple $(A, \lambda, \frac{1}{p}f' \circ \rho' \circ \phi)$ is a k -valued point of X_U . Moreover, define H by

$$H = \ker f' : (\overline{\mathcal{D}}_A)_{p_{\tau_0}} \rightarrow (\overline{\mathcal{D}}_B)_{p_{\tau_0}}.$$

Then we have:

Lemma 3.7 *The space H is (τ_0, i) -special.*

Proof. Since the image of $f' : \overline{\mathcal{D}}_A \rightarrow \overline{\mathcal{D}}_B$ is N_W , and $(N_W)_{p_{\tau_0}}$ has dimension i , V has dimension $n - i$. The submodule $M_H = \ker f' : \overline{\mathcal{D}}_A \rightarrow \overline{\mathcal{D}}_B$ is stable under F and V , so in particular $F((M_H)_{p_{\sigma^{-1}\tau_0}})$ is contained in $(M_H)_{p_{\tau_0}}$. But the former is all of $F((\overline{\mathcal{D}}_A)_{p_{\sigma^{-1}\tau_0}})$, whereas the latter is just H . In particular H contains $F((\overline{\mathcal{D}}_A)_{p_{\sigma^{-1}\tau_0}})$. Similarly H contains $V((\overline{\mathcal{D}}_A)_{p_{\sigma\tau_0}})$, so H is (τ_0, i) -special. \square

Theorem 3.8 *The constructions above that associate to each (A, λ, ρ, H) the corresponding (B, λ', ρ', W) (and vice versa) are inverse to each other. In particular there is a natural bijection between the space of tuples (A, λ, ρ, H) where $(A, \lambda, \rho) \in X_U(k)$ and H is (τ_0, i) -special, and the space of tuples (B, λ', ρ', W) where $(B, \lambda', \rho') \in X_{U'}(k)$ and W is (τ_0, i) -constrained.*

Proof. Fix a particular (A, λ, ρ, H) , and let (B, λ', ρ', W) be the point associated to it by the first construction above. Let $(A'', \lambda'', \rho'', H'')$ be the point associated to (B, λ', ρ', W) by the second construction above.

We need to show that the tuples (A, λ, ρ, H) and $(A'', \lambda'', \rho'', H'')$ are isomorphic. Let $f : A \rightarrow B$ be the map used in the construction of B from A , and $f' : B \rightarrow A''$ be the map used in the construction of A'' from B . The composition $f'f$ induces the zero map $\overline{\mathcal{D}}_{A''} \rightarrow \overline{\mathcal{D}}_A$, and hence its kernel contains $A[p]$. Degree considerations then show that the kernel is exactly $A[p]$, so that $\frac{1}{p}f'f$ is an isomorphism of A with A'' . It is easy to check that this isomorphism carries λ to λ'' and ρ to ρ'' . We henceforth identify A with A'' via this isomorphism.

Note now that by construction, we have

$$H'' = \ker f' : (\overline{\mathcal{D}}_A)_{p_{\tau_0}} \rightarrow (\overline{\mathcal{D}}_B)_{p_{\tau_0}}.$$

By our definition of f , we have that $H = f((\overline{\mathcal{D}}_B)_{p_{\tau_0}})$. Since

$$f(\overline{\mathcal{D}}_B) = \ker f' : \overline{\mathcal{D}}_A \rightarrow \overline{\mathcal{D}}_B,$$

it follows that $H = H''$. Thus the second construction is a left inverse to the first.

The proof that the second construction is a right inverse to the first is similar, and will be omitted. \square

4 Geometrizing the Construction

We now make our calculations with points in the previous section into a geometric relationship between $X_{U'}$ and X_U , by realizing the bijection above as arising from a map of varieties. We also study the relationship of these varieties to X_U and $X_{U'}$. We do so by systematically replacing the Dieudonné modules appearing in the previous section with DeRham cohomology modules.

Definition 4.1 *Let S be a k_0 -scheme, and (A, λ, ρ) a point of $X_U(S)$. A subbundle H of $H_{DR}^1(A/S)_{p_{\tau_0}}$ is (τ_0, i) -special if H has rank $n-i$, and contains both $\text{Lie}(A/S)_{p_{\tau_0}}^*$ and $\text{Fr}(H_{DR}^1(A^{(p)}/S)_{p_{\tau_0}})$, where Fr denotes the relative Frobenius $A \rightarrow A^{(p)}$.*

This generalizes our previous notion for the case when $S = \text{Spec } k$, k perfect.

Lemma 4.2 *Let (A, λ, ρ) be a point of $X_U(S)$. Then (A, λ, ρ) admits a (τ_0, i) -special H if and only if the rank of*

$$\text{Ver} : \text{Lie}(A^{(p)}/S)_{p_{\tau_0}} \rightarrow \text{Lie}(A/S)_{p_{\tau_0}}$$

is less than or equal to $r_{\sigma^{-1}\tau_0} - i$.

Proof. The kernel of

$$\text{Ver} : H_{DR}^1(A/S) \rightarrow H_{DR}^1(A^{(p)}/S)$$

is equal to the image of

$$\text{Fr} : H_{DR}^1(A^{(p)}/S) \rightarrow H_{DR}^1(A/S).$$

Since the dual of the map $\text{Ver} : \text{Lie}(A^{(p)}/S) \rightarrow \text{Lie}(A/S)$ is the restriction of the map $\text{Ver} : H_{DR}^1(A^{(p)}/S) \rightarrow H_{DR}^1(A/S)$ to the submodule $\text{Lie}(A^{(p)}/S)^*$ of $H_{DR}^1(A^{(p)}/S)$, the rank of the map

$$\text{Ver} : \text{Lie}(A^{(p)}/S)_{p_{\tau_0}} \rightarrow \text{Lie}(A/S)_{p_{\tau_0}}$$

is less than or equal to $r_{\sigma^{-1}\tau_0} - i$ if and only if the rank of the intersection of the subsheaves $\text{Lie}(A/S)_{p_{\tau_0}}^*$ and $\text{Fr}(H_{DR}^1(A^{(p)}/S)_{p_{\tau_0}})$ of $H_{DR}^1(A^{(p)})$ has rank at least $r_{\tau_0} - r_{\sigma^{-1}\tau_0} + i$. This is true if and only if their sum has rank at most $n - i$, which in turn is true if and only if there exists a subbundle H of rank $n - i$ containing both of them. \square

Let \mathcal{A} denote the universal abelian variety on X_U . We let $(X_U)_{\tau_0, i}$ denote the subscheme of X_U on which the map $\text{Ver} : \text{Lie}(\mathcal{A}^{(p)}/X_U)_{p_{\tau_0}} \rightarrow \text{Lie}(\mathcal{A}/X_U)_{p_{\tau_0}}$ has rank less than or equal to $r_{\sigma^{-1}\tau_0} - i$. The closed points on $(X_U)_{\tau_0, i}$ are precisely the (τ_0, i) -special points in the language of the preceding section. (In particular, the results of the previous section show that $(X_U)_{\tau_0, i}$ is nonempty.)

Let $\tilde{X}_{U, \tau_0, i}$ denote the k_0 -scheme parametrizing tuples (A, λ, ρ, H) , where $(A, \lambda, \rho) \in X_U(S)$ and H is a (τ_0, i) -special subspace of $H_{DR}^1(A/S)_{p_{\tau_0}}$. There is a natural map $\tilde{X}_{U, \tau_0, i} \rightarrow X_U$, whose image is contained in $(X_U)_{\tau_0, i}$.

Our first goal is to understand the map $\tilde{X}_{U,\tau_0,i} \rightarrow X_U$. We will do so by constructing a local model for this map.

For $\tau \neq \tau_0$, let $\mathcal{M}_\tau = G(r_\tau, n)_{\mathbb{F}_p}$ be the Grassmannian parametrizing r_τ -planes in \mathbb{F}_p^n . Define \mathcal{M}_{τ_0} to be the Schubert cycle in $G(r_{\tau_0}, n)$ parametrizing r_{τ_0} -planes in \mathbb{F}_p^n that intersect the span of the first $n - r_{\sigma^{-1}\tau}$ basis vectors in \mathbb{F}_p^n in a subspace of dimension at least $r_{\tau_0} - r_{\sigma^{-1}\tau_0} + i$.

Finally, define $\tilde{\mathcal{M}}_{\tau_0}$ to be the moduli space parametrizing pairs (V, H) , where V is a subspace of \mathbb{F}_p^n of dimension r_{τ_0} , and H is a subspace of \mathbb{F}_p^n of dimension $n - i$ containing both V and the span of the first $n - r_{\sigma^{-1}\tau_0}$ basis vectors in \mathbb{F}_p^n . There is a natural map $\tilde{\mathcal{M}}_{\tau_0} \rightarrow \mathcal{M}_{\tau_0}$ that forgets H ; this map is generically one-to-one.

On the other hand, we have a natural map $\tilde{\mathcal{M}}_{\tau_0} \rightarrow G(n - i, n)_{\mathbb{F}_p}$ that forgets V . The fibers of this map over a given H are simply $G(r_{\tau_0}, H)$. It follows that $\tilde{\mathcal{M}}_{\tau_0}$ is smooth; it is a natural desingularization of \mathcal{M}_{τ_0} .

For $\tau \neq \tau_0$, set $\tilde{\mathcal{M}}_\tau = \mathcal{M}_\tau$. Let \mathcal{M} be the product (over \mathbb{F}_p) of the \mathcal{M}_τ for all τ , and let $\tilde{\mathcal{M}}$ be the product of the $\tilde{\mathcal{M}}_\tau$ for all τ . We have a natural map $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$.

Theorem 4.3 *The map $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is a local model for the map $\tilde{X}_{U,\tau_0,i} \rightarrow (X_U)_{\tau_0,i}$, in the sense that for any field k , and every $x \in (X_U)_{\tau_0,i}(k)$, there is a point p of $\mathcal{M}(k)$ and étale neighborhoods U_x of x and U_p of p such that the base change of $\tilde{X}_U \rightarrow (X_U)_{\tau_0,i}$ to U_x is isomorphic to the base change of $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ to U_p .*

To prove this, we first introduce two new schemes $(X_U)_{\tau_0,i}^+$ and $\tilde{X}_{U,\tau_0,i}^+$. The former parametrizes tuples $(A, \lambda, \rho, \{e_{i,\tau}\})$, where i runs from 1 to n for each $\tau : F^+ \rightarrow \mathbb{R}$, and the set $\{e_{1,\tau}, \dots, e_{n,\tau}\}$ is a basis for $H_{\text{DR}}^1(A)_{p_\tau}$ for all τ , such that the subset $\{e_{1,\tau_0}, \dots, e_{n-r_{\sigma^{-1}\tau_0}, \tau_0}\}$ of $\{e_{1,\tau_0}, \dots, e_{n,\tau_0}\}$ is a basis for the subbundle $\text{Fr}(H_{\text{DR}}^1(A^{(p)}))_{p_{\tau_0}}$ of $H_{\text{DR}}^1(A)_{p_{\tau_0}}$. The latter parametrizes the same data, plus a (τ_0, i) -special subbundle H of $H_{\text{DR}}^1(A)_{p_{\tau_0}}$.

Clearly $(X_U)_{\tau_0,i}^+$ and $\tilde{X}_{U,\tau_0,i}^+$ possess natural maps to $(X_U)_{\tau_0,i}$ and $\tilde{X}_{U,\tau_0,i}$, respectively, by forgetting the $e_{i,\tau}$. They also possess natural maps to \mathcal{M} and $\tilde{\mathcal{M}}$, which we will now construct.

Given an S -valued point $(A, \lambda, \rho, \{e_{i,\tau}\})$ of $(X_U)_{\tau_0,i}$, the basis $e_{i,\tau}$ allows us to identify $H_{\text{DR}}^1(A)_{p_\tau}$ with \mathcal{O}_S^n . Then the subbundle $\text{Lie}(A/S)_{p_\tau}^*$ of $H_{\text{DR}}^1(A)_{p_\tau}$ gives us a corresponding subbundle V of \mathcal{O}_S^n , and hence a point of \mathcal{M}_τ . We thus obtain a map $(X_U)_{\tau_0,i}^+ \rightarrow \mathcal{M}$. If in addition we have a (τ_0, i) -special subbundle H of $H_{\text{DR}}^1(A)_{p_{\tau_0}}$, then the pair $(\text{Lie}(A)^*, H)$ corresponds to a point of $\tilde{\mathcal{M}}_{\tau_0}$. We therefore obtain a map $\tilde{X}_{U,\tau_0,i}^+ \rightarrow \tilde{\mathcal{M}}$. These fit into a commutative diagram:

$$\begin{array}{ccccc} \tilde{X}_{U,\tau_0,i} & \leftarrow & \tilde{X}_{U,\tau_0,i}^+ & \rightarrow & \tilde{\mathcal{M}} \\ \downarrow & & \downarrow & & \downarrow \\ (X_U)_{\tau_0,i} & \leftarrow & (X_U)_{\tau_0,i}^+ & \rightarrow & \mathcal{M}. \end{array}$$

The left-hand horizontal maps are clearly smooth. We will show the right-hand horizontal maps are also smooth.

The right-hand square in the above diagram is cartesian, so it suffices to show that the map $(X_U)_{\tau_0, i}^+ \rightarrow \mathcal{M}$ is smooth. There is a standard way to do this using the crystalline deformation theory of abelian varieties. We first summarize the necessary facts:

Let S be a scheme, and S' a thickening of S equipped with divided powers. Let $\mathbb{C}_{S'}$ denote the category of abelian varieties over S' , and \mathbb{C}_S denote the category of abelian varieties over S . For A an object of $\mathbb{C}_{S'}$, let \bar{A} denote its base change to \mathbb{C}_S .

Fix an A in $\mathbb{C}_{S'}$, and consider the module $H_{\text{cris}}^1(\bar{A}/S)_{S'}$. This is a locally free $\mathcal{O}_{S'}$ -module, and we have a canonical isomorphism:

$$H_{\text{cris}}^1(\bar{A}/S)_{S'} \cong H_{\text{DR}}^1(A/S').$$

Moreover, we have a natural submodule

$$\text{Lie}(A/S')^* \subset H_{\text{DR}}^1(A/S').$$

The preceding isomorphism thus gives us a subbundle of $H_{\text{cris}}^1(\bar{A}/S)_{S'}$ that lifts the subbundle $\text{Lie}(\bar{A}/S)^*$ of $H_{\text{DR}}^1(\bar{A}/S)$.

Knowing this lift allows us to recover A from \bar{A} . More precisely, let \mathbb{C}_S^+ denote the category of pairs (\bar{A}, ω) , where \bar{A} is an object of \mathbb{C}_S and ω is a subbundle of $H_{\text{cris}}^1(\bar{A}/S)_{S'}$ that lifts $\text{Lie}(\bar{A}/S)^*$. Then the construction outlined above gives us a functor from $\mathbb{C}_{S'}$ to \mathbb{C}_S^+ .

Theorem 4.4 (Grothendieck) *The functor $\mathbb{C}_{S'} \rightarrow \mathbb{C}_S^+$ defined above is an equivalence of categories.*

Proof. The proof is sketched in [Gr], pp. 116-118. A complete proof can be found in [MM]. \square

Proposition 4.5 *The map $(X_U)_{\tau_0, i}^+ \rightarrow \mathcal{M}$ is smooth.*

Proof. Let R' be a ring, and I an ideal of R such that $I^2 = \{0\}$. Let R be the ring R'/I . It suffices to show for any diagram

$$\begin{array}{ccc} \text{Spec } R & \rightarrow & (X_U)_{\tau_0, i}^+ \\ \downarrow & & \downarrow \\ \text{Spec } R' & \rightarrow & \mathcal{M} \end{array}$$

there is a map $\text{Spec } R' \rightarrow (X_U)_{\tau_0, i}^+$.

In terms of the moduli, such a diagram consists of the following data:

1. an R -valued point (A, λ, ρ) of $(X_U)_{\tau_0, i}^+$,
2. for each τ , bases $e_{i, \tau}$ of $H_{\text{DR}}^1(A/R)_{p_\tau}$, such that the set $e_{1, \tau_0}, \dots, e_{s_{\sigma-1}, \tau_0}$ is a basis for the submodule $\text{Fr}(H_{\text{DR}}^1(A^{(p)}/R)_{p_{\tau_0}})$ of $H_{\text{DR}}^1(A/R)_{p_{\tau_0}}$,

3. For each τ , a rank r_τ subbundle V_τ of $(R')^n$ whose reduction modulo I is the subbundle of R^n that corresponds to the subbundle $\text{Lie}(A/R)_{p_\tau}^*$ of $H_{\text{DR}}^1(A/R)_{p_\tau}$ under the identification of the latter with R^n induced by the $e_{i,\tau}$. The bundle V_{τ_0} has the additional property that its intersection with the span of the first $s_{\sigma^{-1}\tau}$ standard basis vectors of $(R')^n$ has rank at least $r_{\tau_0} - r_{\sigma^{-1}\tau} + i$.

For each τ and i , let $\tilde{e}_{\tau,i}$ be a lift of $e_{\tau,i}$ to $(H_{\text{cris}}^1(A/R)_{R'})_{p_\tau}$. (If $\tau = \tau_0$ and $i \leq s_{\sigma^{-1}\tau_0}$, then we require that $\tilde{e}_{\tau,i}$ lies in the subbundle $\text{Fr}(H_{\text{cris}}^1(A^{(p)}/R)_{R'})_{p_\tau}$ of $(H_{\text{cris}}^1(A/R)_{R'})_{p_\tau}$.)

Under this choice of basis, each V_τ corresponds to a subbundle ω_{p_τ} of $(H_{\text{cris}}^1(A/R)_{R'})_{p_\tau}$ that lifts the subbundle $\text{Lie}(A/R)_{p_\tau}^*$ of $H_{\text{DR}}^1(A/R)_{p_\tau}$. Define $\omega_{q_\tau} = \omega_{p_\tau}^\perp$ for all τ , where \perp denotes orthogonal complement with respect to the pairing

$$(H_{\text{cris}}^1(A/R)_{R'})_{p_\tau} \times (H_{\text{cris}}^1(A/R)_{R'})_{q_\tau} \rightarrow R'$$

induced by λ .

By crystalline deformation theory, this defines a lift of A to an abelian scheme over $\text{Spec } R'$. The relation $\omega_{q_\tau} = \omega_{p_\tau}^\perp$ implies that λ lifts to a prime-to- p polarization of this lift as well. We thus obtain a point $(\tilde{A}, \tilde{\lambda}, \tilde{\rho})$ of $X_U(R')$. Moreover, since the rank of the intersection of V_{τ_0} with the span of the first $s_{\sigma^{-1}\tau_0}$ basis vectors of $(R')^n$ has rank at least $r_{\tau_0} - r_{\sigma^{-1}\tau_0} + i$, the same can be said for the intersection of $\omega_{p_{\tau_0}}$ with $\text{Fr}(H_{\text{cris}}^1(A/R)_{R'})_{p_{\tau_0}}$, and hence also for the intersection of $\text{Lie}(\tilde{A}/R')_{p_{\tau_0}}^*$ with $\text{Fr}(H_{\text{DR}}^1(A/R)_{p_{\tau_0}})$. Thus $(\tilde{A}, \tilde{\lambda}, \tilde{\rho})$ lies in $(X_U)_{\tau_0,i}^+$. Finally, the basis $\tilde{e}_{\tau,i}$ corresponds to a basis of $H_{\text{DR}}^1(\tilde{A}/R')_{p_\tau}$ for each τ , and these bases, together with the point $(\tilde{A}, \tilde{\lambda}, \tilde{\rho})$ define the required point of $(X_U)_{\tau_0,i}^+$. \square

It is easy to see (for instance, by computing the dimension of the tangent space to a fiber) that the smooth maps $(X_U)_{\tau_0,i}^+ \rightarrow (X_U)_{\tau_0,i}$ and $(X_U)_{\tau_0,i}^+ \rightarrow \mathcal{M}$ have the same relative dimension. Thus if x is a point of $(X_U)_{\tau_0,i}$, x^+ is a lift of x to $(X_U)_{\tau_0,i}^+$, and p is the image of x^+ in \mathcal{M} , the complete local ring $\hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x^+}$ is simultaneously a power series ring over $\hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x}$ and a power series ring over $\hat{\mathcal{O}}_{\mathcal{M},p}$, in the same number of variables.

Corollary 4.6 of [dJ] then implies that $\hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x}$ and $\hat{\mathcal{O}}_{\mathcal{M},p}$ are isomorphic. More precisely, the proof of this corollary shows that there is a map $\hat{\mathcal{O}}_{\mathcal{M},p} \rightarrow \hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x^+}$ whose composition with the map $\hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x^+} \rightarrow \hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x}$ is an isomorphism, and whose composition with the map $\hat{\mathcal{O}}_{(X_U)_{\tau_0,i},x^+} \rightarrow \hat{\mathcal{O}}_{\mathcal{M},p}$ is the identity on $\hat{\mathcal{O}}_{\mathcal{M},p}$.

It follows by Artin approximation ([Ar], especially Corollary 2.5) that there are étale neighborhoods U_x , U_{x^+} , and U_p of x , x^+ , and p respectively, a diagram

$$\begin{array}{ccccc} U_x & \leftarrow & U_{x^+} & \rightarrow & U_p \\ \downarrow & & \downarrow & & \downarrow \\ (X_U)_{\tau_0,i} & \leftarrow & (X_U)_{\tau_0,i}^+ & \rightarrow & \mathcal{M} \end{array}$$

in which both squares are Cartesian, and a section $U_p \rightarrow U_{x^+}$ whose composition with the map $U_{x^+} \rightarrow U_x$ is an isomorphism, and whose composition with the map $U_{x^+} \rightarrow U_p$ is the identity on U_p .

Define

$$\begin{aligned}\tilde{U}_x &= \tilde{X}_{U,\tau_0,i} \times_{(X_U)_{\tau_0,i}} U_x, \\ \tilde{U}_{x^+} &= \tilde{X}_{U,\tau_0,i}^+ \times_{(X_U)_{\tau_0,i}^+} U_{x^+}, \\ \tilde{U}_p &= \tilde{\mathcal{M}} \times_{\mathcal{M}} U_p.\end{aligned}$$

We obtain from the section $U_p \rightarrow U_{x^+}$ a map $\tilde{U}_p \rightarrow \tilde{U}_{x^+}$ whose composition with the natural map $\tilde{U}_{x^+} \rightarrow \tilde{U}_x$ is an isomorphism. This yields a commutative square

$$\begin{array}{ccc}\tilde{U}_x & \cong & \tilde{U}_p \\ \downarrow & & \downarrow \\ U_x & \cong & U_p,\end{array}$$

and thus establishes Theorem 4.3.

Theorem 4.3 implies that the singularities of $(X_U)_{\tau_0,i}$ look (étale locally) like products of an affine space with a singularity of the Schubert cycle \mathcal{M}_{τ_0} . Moreover, $\tilde{X}_{U,\tau_0,i}$ is a natural desingularization of $(X_U)_{\tau_0,i}$. For $j \geq 0$, the fiber of the map $\tilde{X}_{U,\tau_0,i} \rightarrow (X_U)_{\tau_0,i}$ over a point of $(X_U)_{\tau_0,i+j} \setminus (X_U)_{\tau_0,i+j+1}$ is a Grassmannian parametrizing j -planes in an $i+j$ -dimensional space.

The points of $\tilde{X}_{U,\tau_0,i}$ over a perfect field k correspond to tuples (A, λ, ρ, H) , where (A, λ, ρ) is a k -valued point of X_U , and H is a (τ_0, i) -special subspace of $\mathcal{D}(A[p])_{p_{\tau_0}}$. In order to geometrize the construction in the previous section, we would like to have a map from $\tilde{X}_{U,\tau_0,i}$ to $X_{U'}$. Unfortunately, $\tilde{X}_{U,\tau_0,i}$ does not admit such a map. We must therefore introduce another moduli problem:

Definition 4.6 *Let S be a k_0 -scheme, (A, λ, ρ) a point of $X_U(S)$, and (B, λ', ρ') a point of $X_{U'}(S)$. A (τ_0, i) -special isogeny $f : (A, \lambda, \rho) \rightarrow (B, \lambda', \rho')$ is an \mathcal{O}_F -isogeny $f : A \rightarrow B$, of degree p^{nd} , such that:*

1. $p\lambda = f^\vee \lambda' f$,
2. the U' -level structure ρ' on B corresponds to $f \circ \rho$ under the identification of T with T' fixed in the previous section,
3. for each $\tau \neq \tau_0$, the map f induces an isomorphism of $H_{\text{DR}}^1(B/S)_{p_\tau}$ with $H_{\text{DR}}^1(A/S)_{p_\tau}$, and
4. the image of $H_{\text{DR}}^1(B/S)_{p_{\tau_0}}$ in $H_{\text{DR}}^1(A/S)_{p_{\tau_0}}$ under f has rank $n - i$. (It is necessarily a subbundle of $H_{\text{DR}}^1(A/S)_{p_{\tau_0}}$.)

We denote by $\hat{X}_{U,\tau_0,i}$ the scheme parametrizing tuples $(A, \lambda, \rho, B, \lambda', \rho', f)$, where (A, λ, ρ) is a point of (X_U) , (B, λ', ρ') is a point of $X_{U'}$, and f is a (τ_0, i) -special isogeny from (A, λ, ρ) to (B, λ', ρ') .

If $(A, \lambda, \rho, B, \lambda', \rho', F)$ is a point of $\hat{X}_{U, \tau_0, i}(S)$, then $f(H_{\text{DR}}^1(B/S)_{p_{\tau_0}})$ is a (τ_0, i) -special subbundle of $H_{\text{DR}}^1(A/S)_{p_{\tau_0}}$. Indeed, we know that the kernel of

$$f : H_{\text{DR}}^1(B/S)_{p_{\tau_0}} \rightarrow H_{\text{DR}}^1(A/S)_{p_{\tau_0}}$$

has rank i . The subbundle $\text{Lie}(B/S)_{p_{\tau_0}}^*$ of $H_{\text{DR}}^1(B/S)_{p_{\tau_0}}$ has rank $r_{\tau_0} + i$, and $\text{Lie}(A/S)_{p_{\tau_0}}$ has rank r_{τ_0} . As f maps the former to the latter, $f(H_{\text{DR}}^1(B/S)_{p_{\tau_0}})$ must contain $\text{Lie}(A/S)_{p_{\tau_0}}^*$. An identical argument shows that $f(H_{\text{DR}}^1(B/S)_{p_{\tau_0}})$ contains $\text{Fr}(H_{\text{DR}}^1(A^{(p)}/S)_{p_{\tau_0}})$. The morphism of functors that associates the tuple $(A, \lambda, \rho, f(H_{\text{DR}}^1(B/S)_{p_{\tau_0}}))$ to the tuple $(A, \lambda, \rho, B, \lambda', \rho', f)$ therefore induces a map $\hat{X}_{U, \tau_0, i} \rightarrow \tilde{X}_{U, \tau_0, i}$.

Proposition 4.7 *The map $\hat{X}_{U, \tau_0, i} \rightarrow \tilde{X}_{U, \tau_0, i}$ is a bijection on k -valued points for any perfect field k .*

Proof. The construction in the previous section associates to every (A, λ, ρ) in $X_U(k)$, and every (τ_0, i) -special subspace H of $\mathcal{D}(A[p])_{p_{\tau_0}}$ (or equivalently of $H_{\text{DR}}^1(A/k)_{p_{\tau_0}}$) a (B, λ', ρ') and a (τ_0, i) -special isogeny $f : (A, \lambda, \rho) \rightarrow (B, \lambda', \rho')$. This construction is inverse to the map

$$\hat{X}_{U, \tau_0, i}(k) \rightarrow \tilde{X}_{U, \tau_0, i}(k)$$

constructed above. □

This has strong consequences for the geometry of the map $\hat{X}_{U, \tau_0, i} \rightarrow \tilde{X}_{U, \tau_0, i}$. In particular we have the following result:

Proposition 4.8 *Let Y and Z be schemes of finite type over a perfect field k of characteristic p , such that Z is normal and Y is reduced. Let $f : Y \rightarrow Z$ be a proper map that is a bijection on points. Then there is a map $f' : Z_{p^r} \rightarrow Y$ such that*

$$f f' : Z_{p^r} \rightarrow Z$$

is the r th power of the Frobenius. (In particular f is an isomorphism on étale cohomology.)

This is proven in [He], Proposition 4.8.

Remark 4.9 One might wonder if the map $\hat{X}_{U, \tau_0, i} \rightarrow \tilde{X}_{U, \tau_0, i}$ is actually an isomorphism, but in fact a straightforward calculation, using Theorem 4.4, shows that it often fails to be an isomorphism on tangent spaces.

The scheme $\hat{X}_{U, \tau_0, i}$ admits an obvious map to $X_{U'}$. In fact, as one expects from the previous section, it admits a map to a scheme $(X_{U'})^{\tau_0, i}$ parametrizing (τ_0, i) -constrained subspaces. More precisely:

Definition 4.10 *Let S be a k_0 -scheme, and (B, λ', ρ') a point of $X_{U'}(S)$. A subbundle W of $H_{\text{DR}}^1(B/S)_{p_{\tau_0}}$ is (τ_0, i) -constrained if it has rank i and is contained in both $\text{Lie}(B/S)_{p_{\tau_0}}^*$ and $\text{Fr}(H_{\text{DR}}^1(B^{(p)}/S)_{p_{\tau_0}})$. We denote by $(X_{U'})^{\tau_0, i}$ the scheme parametrizing tuples (B, λ', ρ', W) , where (B, λ', ρ') is a point of $X_{U'}$ and W is a (τ_0, i) -constrained subbundle of $H_{\text{DR}}^1(B/S)_{p_{\tau_0}}$.*

Proposition 4.11 *Let $(A, \lambda, \rho, B, \lambda', \rho', f)$ be an element of $\hat{X}_{U, \tau_0, i}(S)$, and let W be the kernel of the map*

$$f : H_{\text{DR}}^1(B/S)_{p_{\tau_0}} \rightarrow H_{\text{DR}}^1(A/S)_{p_{\tau_0}}.$$

Then W is a (τ_0, i) -constrained subbundle of $H_{\text{DR}}^1(B/S)_{p_{\tau_0}}$.

Proof. The rank of W is clearly i , so it suffices to show that W is contained in $\text{Lie}(B/S)_{p_{\tau_0}}^*$ and $\text{Fr}(H_{\text{DR}}^1(B^{(p)}/S)_{p_{\tau_0}})$. The former has rank $r_{\tau_0} + i$, whereas $\text{Lie}(A/S)_{p_{\tau_0}}^*$ has rank r_{τ_0} . Thus the kernel of the map

$$f : \text{Lie}(B/S)_{p_{\tau_0}}^* \rightarrow \text{Lie}(A/S)_{p_{\tau_0}}^*$$

has dimension at least i . Since this kernel is contained in W , it must be equal to W , and hence W is contained in $\text{Lie}(B/S)_{p_{\tau_0}}^*$. The proof of containment in $\text{Fr}(H_{\text{DR}}^1(B/S)_{p_{\tau_0}})$ is similar. \square

We thus have a map $\hat{X}_{U, \tau_0, i} \rightarrow (X_{U'})^{\tau_0, i}$ that takes $(A, \lambda, \rho, B, \lambda', \rho', f)$ to (B, λ', ρ', W) , with W as above. For any perfect field k of characteristic p , composing the map

$$\hat{X}_{U, \tau_0, i}(k) \rightarrow (X_{U'})^{\tau_0, i}(k)$$

with the bijection

$$\tilde{X}_{U, \tau_0, i}(k) \rightarrow \hat{X}_{U, \tau_0, i}(k)$$

yields the bijection

$$\tilde{X}_{U, \tau_0, i}(k) \rightarrow (X_{U'})^{\tau_0, i}(k)$$

constructed in the previous section. In particular the map $\hat{X}_{U, \tau_0, i} \rightarrow (X_{U'})^{\tau_0, i}$ is a bijection on points.

Lemma 4.12 *The scheme $(X_{U'})^{\tau_0, i}$ is smooth over k_0 .*

Proof. The dimension of $(X_{U'})^{\tau_0, i}$ is equal to that of $\tilde{X}_{U, \tau_0, i}$, and hence to that of \mathcal{M} . Thus $(X_{U'})^{\tau_0, i}$ has dimension equal to

$$\left(\sum_{\tau} r_{\tau} s_{\tau} \right) - i(i + r_{\tau_0} - r_{\sigma^{-1}\tau_0}).$$

We must show that the dimension of the tangent space to $(X_{U'})^{\tau_0, i}$ at any k -valued point x is equal to this number. Let (B, λ', ρ', W) be the moduli object corresponding to x , and let $S = \text{Spec } k[\epsilon]/\epsilon^2$. Then, by Grothendieck's theorem, specifying a tangent vector to $(X_{U'})^{\tau_0, i}$ at x is equivalent to specifying the following data:

1. For each τ , a lift $\omega_{p_{\tau}}$ of $\text{Lie}(B/k)_{p_{\tau}}^*$ from $H_{\text{DR}}^1(B/k)_{p_{\tau}}$ to $(H_{\text{cris}}^1(B/k)_S)_{p_{\tau}}$, and
2. a lift \tilde{W} of W to a subspace of $(H_{\text{cris}}^1(B/k)_S)_{p_{\tau_0}}$ that is contained in $\omega_{p_{\tau_0}}$ and in $\text{Fr}(H_{\text{cris}}^1(B^{(p)}/k)_S)_{p_{\tau_0}}$.

The space of possible lifts of W that are contained in $\mathrm{Fr}(H_{\mathrm{cris}}^1(B^{(p)}/k)_S)_{p\tau_0}$ has dimension $is_{\sigma^{-1}\tau_0}$. (Recall that $\mathrm{Lie}(B/k)_{p\sigma^{-1}\tau}^*$ has dimension $r_{\sigma^{-1}\tau_0} - i$, so that $\mathrm{Fr}(H_{\mathrm{DR}}^1(B^{(p)}/k)_{p\tau_0})$ and $\mathrm{Fr}(H_{\mathrm{cris}}^1(B^{(p)}/k)_S)_{p\tau_0}$ have dimension $s_{\sigma^{-1}\tau_0} + i$.) Once we have fixed such a lift, the space of $\omega_{p\tau_0}$ containing that lift has dimension $r_{\tau_0}(s_{\tau_0} - i)$, as $\mathrm{Lie}(B/k)_{p\tau_0}^*$ has dimension $r_{\tau_0} + i$.

On the other hand, $\mathrm{Lie}(B/k)_{p\sigma^{-1}\tau_0}^*$ has dimension $r_{\sigma^{-1}\tau_0} - i$, so the space of possible $\omega_{p\sigma^{-1}\tau_0}$ has dimension $(r_{\sigma^{-1}\tau_0} - i)(s_{\sigma^{-1}\tau_0} + i)$. For τ not equal to either τ_0 or $\sigma^{-1}\tau_0$, the space of possible $\omega_{p\tau}$ has dimension $r_{\tau}s_{\tau}$.

Summing these, we find that the tangent space at x has dimension

$$\left(\sum_{\tau} r_{\tau}s_{\tau}\right) - i(i + r_{\tau_0} - r_{\sigma^{-1}\tau_0}),$$

as desired. \square

Corollary 4.13 *The map $\hat{X}_{U,\tau_0,i}^{\mathrm{red}} \rightarrow (X_{U'})^{\tau_0,i}$ induces an isomorphism on étale cohomology.*

Proof. This is immediate from Proposition 4.8. \square

In summary, we have constructed a cycle $(X_U)_{\tau_0,i}$ on X_U , and a natural desingularization $\tilde{X}_{U,\tau_0,i}$. The geometry of this desingularization is closely related to that of $X_{U'}$; in particular there is a scheme $(X_{U'})^{\tau_0,i}$ defined in terms of the universal abelian variety on $X_{U'}$, that is “nearly isomorphic” to $\tilde{X}_{U,\tau_0,i}$, in the sense that there exists a scheme $\hat{X}_{U,\tau_0,i}^{\mathrm{red}}$ and a diagram:

$$\tilde{X}_{U,\tau_0,i} \leftarrow \hat{X}_{U,\tau_0,i}^{\mathrm{red}} \rightarrow (X_{U'})^{\tau_0,i}$$

in which both maps are bijections on points and isomorphisms on étale cohomology. In particular the étale cohomology groups of $\tilde{X}_{U,\tau_0,i}$ and $(X_{U'})^{\tau_0,i}$ are naturally isomorphic via these maps.

5 Cohomology

We now explore the implications of the previous section for the étale cohomology of Shimura varieties. Let $N = \sum_{\tau} r_{\tau}s_{\tau}$ be the dimension of X_U , and $r = i(i + r_{\tau_0} - r_{\sigma^{-1}\tau_0})$ be the codimension of $(X_U)_{\tau_0,i}$. Then $X_{U'}$ has dimension $N - 2r$. For the purposes of this section, we consider $X_U, X_{U'}$, etc. as schemes over $\overline{\mathbb{F}}_p$.

Fix an ℓ different from p , and let ξ be a finite dimensional $\overline{\mathbb{Q}}_{\ell}$ -representation of $G(\mathbb{A}_{\mathbb{Q}}^f)$. As in [HT], III.2, this determines a lisse $\overline{\mathbb{Q}}_{\ell}$ -sheaf \mathcal{L}_{ξ} on X_U . Let ξ' be the representation of $G'(\mathbb{A}_{\mathbb{Q}}^f)$ induced by ξ and our fixed isomorphism of $G'(\mathbb{A}_{\mathbb{Q}}^f)$ with $G(\mathbb{A}_{\mathbb{Q}}^f)$. Then we also have a lisse sheaf $\mathcal{L}_{\xi'}$ on $X_{U'}$. If $\hat{\pi}$ (resp. $\hat{\pi}'$) denotes the map $\hat{X}_{U,\tau_0,i}^{\mathrm{red}} \rightarrow X_U$ (resp. the map $\hat{X}_{U,\tau_0,i}^{\mathrm{red}} \rightarrow X_{U'}$), then there is a natural isomorphism $\hat{\pi}^* \mathcal{L}_{\xi} \cong (\hat{\pi}')^* \mathcal{L}_{\xi'}$.

We will construct, for each j , a map

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_{\xi}(r)).$$

Note that this takes the middle degree cohomology of $X_{U'}$ to the middle degree cohomology of X_U .

In order to construct this map, let us consider the following situation. Let X and Y be smooth over $\overline{\mathbb{F}}_p$, of dimensions N and $N - r$, respectively. Let \mathcal{F} be a lisse sheaf on X , and $\pi : Y \rightarrow X$ a proper map of $\overline{\mathbb{F}}_p$ -schemes.

Let θ_X and θ_Y denote the structure maps

$$\begin{aligned} \theta_X : X &\rightarrow \text{Spec } \overline{\mathbb{F}}_p \\ \theta_Y : Y &\rightarrow \text{Spec } \overline{\mathbb{F}}_p. \end{aligned}$$

We have natural isomorphisms:

$$\begin{aligned} R\theta_X^! \mathbb{Q}_\ell &\cong \mathbb{Q}_\ell[2N](N) \\ R\theta_Y^! \mathbb{Q}_\ell &\cong \mathbb{Q}_\ell[2(N - r)](N - r). \end{aligned}$$

Since $R\theta_Y^! = R\pi^! R\theta_X^!$, it follows that $R\pi^! \mathbb{Q}_\ell \cong \mathbb{Q}_\ell[-2r](-r)$.

It follows that for any lisse sheaf \mathcal{F} on X , $R\pi^! \mathcal{F}$ is naturally isomorphic to $\pi^* \mathcal{F}[-2r](-r)$; this isomorphism is simply the tensor product of the above isomorphism with \mathcal{F} .

Since π is proper, we have $\pi_! = \pi_*$. We thus have a unit map

$$R\pi_* R\pi^! \mathcal{F} \rightarrow \mathcal{F}$$

and therefore a morphism

$$R\pi_* \pi^* \mathcal{F} \rightarrow \mathcal{F}[2r](r)$$

in the derived category of sheaves on X . This induces a map

$$\nu_\pi : H_{\text{ét}}^j(Y, \pi^* \mathcal{F}) \cong H_{\text{ét}}^j(X, R\pi_* \pi^* \mathcal{F}) \rightarrow H_{\text{ét}}^{j+2r}(X, \mathcal{F}(r)).$$

Note that if X and Y are proper, ν_π is simply the Poincaré dual of the natural map $H_{\text{ét}}^{2n-2r-j}(X, \mathcal{F}) \rightarrow H_{\text{ét}}^{2n-2r-j}(Y, \pi^* \mathcal{F})$, suitably Tate twisted. On the other hand, if π is a closed immersion, ν_π is simply the Gysin map.

This construction is compatible with cycle maps in the following sense: we have a commutative diagram

$$\begin{array}{ccc} A^j(Y) & \rightarrow & A^{j+r}(X) \\ \downarrow & & \downarrow \\ H^{2j}(Y, \mathbb{Q}_\ell(2j)) & \rightarrow & H^{2j+2r}(X, \mathbb{Q}_\ell(2j + 2r)) \end{array}$$

where the vertical maps are cycle class maps and the map $A^j(Y) \rightarrow A^{j+r}(X)$ is the proper pushforward π_* of cycles on Y to cycles on X .

Although the map ν_π is difficult to describe directly, we do have the following result:

Lemma 5.1 *Let c_π denote the class of $H_{\text{ét}}^{2r}(Y, \mathbb{Q}_\ell(r))$ associated to the cycle class $\pi^! \pi_* [Y] \in A^r(Y)$, where $[Y]$ denotes the fundamental class of Y in $A^0(Y)$ and $\pi^!$ denotes the refined Gysin homomorphism $A^r(X) \rightarrow A^r(Y)$ of [Fu], Definition 8.1.2. Let $\eta_{\mathcal{F}}$ be the composition of ν_π with the natural map*

$$H^{j+2r}(X, \mathcal{F}(r)) \rightarrow H^{j+2r}(Y, \pi^* \mathcal{F}(r))$$

Then $\eta_{\mathcal{F}}$ is given by cup product with the cohomology class c_π .

Proof. The map $\eta_{\mathcal{F}}$ is induced by the morphism (in the derived category)

$$\xi_{\mathcal{F}} : \pi^* \mathcal{F} \rightarrow \pi^* \mathcal{F}[-2r](r)$$

that is the composition of the sequence of morphisms:

$$\begin{aligned} \pi^* \mathcal{F} &\rightarrow \pi^* R\pi_* \pi^* \mathcal{F} \\ &\rightarrow \pi^* R\pi_* \pi^! \mathcal{F}[2r](r) \\ &\rightarrow \pi^* \mathcal{F}[2r](r). \end{aligned}$$

If we identify $\pi^* \mathcal{F}$ with $\pi^* \mathcal{F} \otimes \mathbb{Q}_\ell$, the map $\xi_{\mathcal{F}}$ is simply $\text{id} \otimes \xi_{\mathbb{Q}_\ell}$. It follows that if $a \in H_{\text{ét}}^j(Y, \pi^* \mathcal{F})$, and $b \in H^j(Y, \mathbb{Q}_\ell)$, then $\eta_{\mathcal{F}}(a \cup b) = a \cup \eta_{\mathbb{Q}_\ell}(b)$. Taking $b = 1$ we see that $\eta_{\mathcal{F}}(a) = a \cup \eta_{\mathbb{Q}_\ell}(1)$. It thus remains to compute $\eta_{\mathbb{Q}_\ell}(1)$.

We have a commutative diagram:

$$\begin{array}{ccccc} A^0(Y) & \xrightarrow{\pi_*} & A^r(X) & \xrightarrow{\pi^!} & A^r(Y) \\ \downarrow & & \downarrow & & \downarrow \\ H_{\text{ét}}^0(Y, \mathbb{Q}_\ell) & \rightarrow & H_{\text{ét}}^{2r}(X, \mathbb{Q}_\ell(r)) & \rightarrow & H_{\text{ét}}^{2r}(Y, \mathbb{Q}_\ell(r)) \end{array}$$

in which the vertical arrows associate cohomology classes to cycles. Note that the composition of the two bottom maps is the map $H_{\text{ét}}^0(Y, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^{2r}(Y, \mathbb{Q}_\ell(r))$ induced by $\eta_{\mathbb{Q}_\ell}$. Since $[Y] \in A^0(Y)$ maps to $1 \in H_{\text{ét}}^0(Y, \mathbb{Q}_\ell)$, the commutativity of the above diagram implies that $\eta_{\mathbb{Q}_\ell}(1)$ is the cohomology class associated to $\pi^! \pi_* [Y]$, as claimed. \square

Remark 5.2 If π is a closed immersion, the class $\pi^! \pi_* [Y]$ is simply the self-intersection of Y in X , considered as a cycle of codimension r on Y .

We now return to the situation considered at the beginning of this section. The map $\hat{\pi}'$ induces a map

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^j(\hat{X}_{U, \tau_0, i}^{\text{red}}, (\hat{\pi}')^* \mathcal{L}_{\xi'}).$$

Composing this with the map

$$\nu_{\hat{\pi}} : H_{\text{ét}}^j(\hat{X}_{U, \tau_0, i}^{\text{red}}, \hat{\pi}^* \mathcal{L}_{\xi}) \rightarrow H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_{\xi}(r)),$$

we obtain maps:

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_{\xi}(r)).$$

It is easy to verify that these maps are compatible with the action of prime-to- p Hecke operators on these cohomology spaces.

Our main result is then:

Theorem 5.3 *The maps:*

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_{\xi}(r))$$

are injective.

The proof of this will occupy the remainder of this section, and the next.

Consider the class $c_{\hat{\pi}} \in H_{\text{ét}}^{2r}(\hat{X}_{U, \tau_0, i}^{\text{red}}, \mathbb{Q}_{\ell}(r))$. The map $\hat{X}_{U, \tau_0, i}^{\text{red}} \rightarrow (X_{U'})^{\tau_0, i}$ constructed in the previous section allows us to view this as a cohomology class on $(X_{U'})^{\tau_0, i}$.

The Leray spectral sequence for the natural map

$$\pi' : (X_{U'})^{\tau_0, i} \rightarrow X_{U'}$$

is a spectral sequence

$$E_2^{j,k} = H_{\text{ét}}^j(X_{U'}, R^k \pi'_* \mathbb{Q}_{\ell}) \rightarrow H_{\text{ét}}^{j+k}((X_{U'})^{\tau_0, i}, \mathbb{Q}_{\ell}).$$

It degenerates at E_2 by weight considerations. In particular this yields a surjection

$$H_{\text{ét}}^{2r}((X_{U'})^{\tau_0, i}, \mathbb{Q}_{\ell}) \rightarrow H_{\text{ét}}^0(X_{U'}, R^{2r} \pi'_* \mathbb{Q}_{\ell}).$$

Denote this surjection by α .

Let V be the complement of the (possibly empty) cycle $(X_{U'})_{\tau_0, 1}$, and let $V^{\tau_0, i}$ be the preimage of V in $X^{\tau_0, i}$. Then $V^{\tau_0, i}$ is a Grassmannian bundle over V , with fibers isomorphic to $G(i, 2i + r_{\tau_0} - r_{\sigma^{-1}\tau_0})$. These fibers have dimension r .

Consider the map

$$R^{2r} \pi'_* \mathbb{Q}_{\ell} \rightarrow j_* j^* R^{2r} \pi'_* \mathbb{Q}_{\ell},$$

where j is the inclusion of V in $X_{U'}$. Note that by the proper base change theorem, the stalk of $j^* R^{2r} \pi'_* \mathbb{Q}_{\ell}$ at a point x of V is isomorphic to $H_{\text{ét}}^{2r}(Z_x, \mathbb{Q}_{\ell})$, where $Z_x = (\pi')^{-1}(x)$, and is therefore one-dimensional. The map

$$H_{\text{ét}}^{2r}((X_{U'})^{\tau_0, i}, \mathbb{Q}_{\ell}) \rightarrow H_{\text{ét}}^{2r}(Z_x, \mathbb{Q}_{\ell})$$

given by applying α and then passing to the stalk at x is the same as the map induced by the inclusion of Z in $(X_{U'})^{\tau_0, i}$.

Let W denote the universal (τ_0, i) -constrained bundle on $(X_{U'})^{\tau_0, i}$. It has rank i , and its restriction W_x to Z_x for any point x of V can be identified with the tautological subbundle on $G(i, 2i + r_{\tau_0} - r_{\sigma^{-1}\tau_0})$.

Denote by $c_i(W)$ the top Chern class of W , and consider the class C defined by

$$C = (-1)^r c_i(W)^{i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}.$$

For x in V , the intersection $C \cap Z_x$ is $(-1)^r c_i(W_x)^{i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}$, which is the class of a point on Z_x .

Consider the class in $H_{\text{ét}}^{2r}((X_{U'})^{\tau_0, i}, \mathbb{Q}_{\ell}(r))$ arising from the cycle class C . For each x in V , its restriction to $H_{\text{ét}}^{2r}(Z_x, \mathbb{Q}_{\ell}(r))$ is the fundamental class

of $H_{\text{ét}}^{2r}(Z_x, \mathbb{Q}_\ell(r))$. Thus the image under α of this class is an element of $H_{\text{ét}}^0(X_{U'}, R^{2r}\pi'_*\mathbb{Q}_\ell(r))$ that generates each stalk of $j^*R^{2r}\pi'_*\mathbb{Q}_\ell(r)$ as a \mathbb{Q}_ℓ -vector space. It follows that $j_*j^*R^{2r}\pi'_*\mathbb{Q}_\ell(r)$ is isomorphic to the constant sheaf \mathbb{Q}_ℓ , and that the map

$$R^{2r}\pi'_*\mathbb{Q}_\ell(r) \rightarrow j_*j^*R^{2r}\pi'_*\mathbb{Q}_\ell(r) \cong \mathbb{Q}_\ell$$

is *split*. In particular we obtain, for each j , a surjection

$$\beta_j : H_{\text{ét}}^j(X_{U'}, R^{2r}\pi'_*\mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^j(X_{U'}, \mathbb{Q}_\ell(-r)).$$

Also note that by the projection formula,

$$R^{2r}\pi'_*(\pi')^*\mathcal{L}_{\xi'} \cong R^{2r}\pi'_*\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell} \mathcal{L}_{\xi'},$$

and therefore $\mathcal{L}_{\xi'}(-r)$ is a direct summand of $R^{2r}\pi'_*(\pi')^*\mathcal{L}_{\xi'}$. We therefore obtain for each j a surjection

$$\beta_{j,\xi'} : H_{\text{ét}}^j(X_{U'}, R^{2r}\pi'_*(\pi')^*\mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}(-r)).$$

The Leray spectral sequence induces an increasing filtration $\text{Fil}_j^m(\mathcal{L}_{\xi'})$ on $H_{\text{ét}}^j((X_{U'})^{\tau_0, i}, (\pi')^*\mathcal{L}_{\xi'})$, such that

$$\text{Fil}_j^m(\mathcal{L}_{\xi'}) / \text{Fil}_j^{m-1}(\mathcal{L}_{\xi'}) = H_{\text{ét}}^m(X_{U'}, R^{j-m}\pi'_*(\pi')^*\mathcal{L}_{\xi'}).$$

Let b be a class in $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$, and let c be a class in $H_{\text{ét}}^{2r}(X_{U'}, \mathbb{Q}_\ell(r))$. Then $(\pi')^*b$ is a class in Fil_j^0 . The product $c \cup b$ then lies in $\text{Fil}_{j+2r}^{2r}(\mathcal{L}_{\xi'}(r))$, and hence maps onto $H_{\text{ét}}^{j+2r}(X_{U'}, R^{2r}\pi'_*(\pi')^*\mathcal{L}_{\xi'}(r))$. This in turn maps via $\beta_{j,\xi'}$ onto $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$.

We thus obtain a map from $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$ to itself. It can be described in terms of c in the following way: $\alpha(c)$ is an element of $H_{\text{ét}}^{2r}(X_{U'}, R^{2r}\pi'_*\mathbb{Q}_\ell(r))$; this maps onto $H_{\text{ét}}^0(X_{U'}, \mathbb{Q}_\ell)$ via β_0 . Thus $\beta_0(\alpha(c))$ is a class of $H_{\text{ét}}^0(X_{U'}, \mathbb{Q}_\ell)$; the endomorphism of $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$ described above is simply multiplication by this class.

The upshot of all of this is:

Proposition 5.4 *If, for a particular choice of X_U, τ_0 , and i , the corresponding $\beta_0\alpha(c_{\hat{\pi}})$ is nonvanishing, then Theorem 5.3 holds for X_U, τ_0 , and i (and all \mathcal{L}_ξ).*

Proof. Consider the map

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^{j+2r}((X_{U'})^{\tau_0, i}, (\pi')^*\mathcal{L}_{\xi'}(r))$$

that is obtained from the map

$$H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'}) \rightarrow H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_\xi(r))$$

of Theorem 5.3 by composing with the natural map

$$H_{\text{ét}}^{j+2r}(X_U, \mathcal{L}_\xi(r)) \rightarrow H_{\text{ét}}^{j+2r}(\hat{X}_{U, \tau_0, i}^{\text{red}}, \hat{\pi}^*\mathcal{L}_\xi(r)),$$

and identifying $H_{\text{ét}}^{j+2r}(\hat{X}_{U,\tau_0,i}^{\text{red}}, \hat{\pi}^* \mathcal{L}_{\xi}(r))$ with $H_{\text{ét}}^{j+2r}((X_{U'})^{\tau_0,i}, (\pi')^* \mathcal{L}_{\xi'}(r))$. To establish Theorem 5.3, it suffices to show this map is injective.

By Lemma 5.1, this map takes an element b of $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$ to $c_{\hat{\pi}} \cup \hat{\pi}^* b$. This lies in $\text{Fil}_j^{2r}(\mathcal{L}_{\xi'}(r))$, and maps via $\beta_{j,\xi'}$ to the element $\beta_0(\alpha(c_{\hat{\pi}}))b$ of $H_{\text{ét}}^j(X_{U'}, \mathcal{L}_{\xi'})$. This element is clearly nonzero if b is. \square

6 The Thom-Porteus formula

In this section we complete the proof of Theorem 5.3 by computing $\beta_0 \alpha(c_{\hat{\pi}})$. The key ingredient is the Thom-Porteus formula, which will give us an expression for the cycle class of $X_{U,\tau_0,i}$ (in the Chow ring of X_U) in terms of a polynomial in Chern classes of bundles on X_U .

Before we state this formula we will need a bit of notation. For X a scheme, let $A^*(X) = \bigoplus_r A^r(X)$ denote the Chow ring of X . For an element c in $A^*(X)$ we denote by $\Delta_q^{(p)}(c)$ the determinant of the p by p matrix

$$\begin{pmatrix} c_q & c_{q+1} & \cdots & c_{q+p-1} \\ c_{q-1} & c_q & \cdots & c_{q+p-2} \\ \vdots & \vdots & \cdots & \vdots \\ c_{q-p+1} & c_{q-p+2} & \cdots & c_q \end{pmatrix}.$$

Here c_r is the r th graded part of c ; the determinant $\Delta_q^{(p)}(c)$ therefore lies in $A^{pq}(X)$.

Then the Thom-Porteus formula states:

Theorem 6.1 ([Fu], 14.IV.4) *Let X be a Cohen-Macaulay scheme, purely of dimension N , and $\varsigma : E \rightarrow F$ a map of vector bundles on X . Let $D_k(\varsigma)$ denote the subscheme of X defined by the condition $\text{rank } \varsigma \leq k$. Suppose that $D_k(\varsigma)$ has the “expected codimension”; that is, that the codimension of $D_k(\varsigma)$ is equal to $(e-k)(f-k)$ where e and f are the ranks of E and F . Then, as elements of $A^{(e-k)(f-k)}(X)$, we have:*

$$[D_k(\varsigma)] = \Delta_{f-k}^{(e-k)}(c(F)c(E)^{-1}).$$

We apply this to $X_{U,\tau_0,i}$. For each τ , let \mathcal{E}_{τ} denote the bundle $\text{Lie}(A/X_U)_{p_{\tau}}^*$.

Proposition 6.2 *As elements of $A^r(X_U)$, we have:*

$$[X_{U,\tau_0,i}] = \Delta_i^{(i+r\tau_0-r\sigma^{-1}\tau_0)}(c(\mathcal{E}_{\tau_0})^{-1}c(F_{\text{abs}}^* \mathcal{E}_{\sigma^{-1}\tau_0})),$$

where F_{abs} is the absolute Frobenius.

Proof. The cycle $X_{U,\tau_0,i}$ is the locus where the map

$$\text{Ver} : \mathcal{E}_{\tau_0} \rightarrow \text{Lie}(\mathcal{A}^{(p)}/X_U)_{p_{\tau_0}}^*$$

has rank less than (or equal to) $r_{\sigma^{-1}\tau_0} - i$. It has the expected codimension (equal to r), so the Thom-Porteus formula yields:

$$[X_{U,\tau_0,i}] = \Delta_i^{(i+r\tau_0-r_{\sigma^{-1}\tau_0})} (c(\mathcal{E}_{\tau_0})^{-1} c(\text{Lie}(\mathcal{A}^{(p)}/X_U)_{p_{\tau_0}}^*)).$$

The result then follows from the isomorphism:

$$\text{Lie}(\mathcal{A}^{(p)}/X_U)_{p_{\tau_0}}^* \cong F_{\text{abs}}^* \text{Lie}(\mathcal{A}/X_U)_{p_{\sigma^{-1}\tau_0}}^* = F_{\text{abs}}^* \mathcal{E}_{\sigma^{-1}\tau_0}.$$

□

This allows us to express $\hat{\pi}^! \hat{\pi}_* [\hat{X}_{U,\tau_0,i}^{\text{red}}]$ in terms of Chern classes of bundles on $\hat{X}_{U,\tau_0,i}^{\text{red}}$. In particular we have:

$$\hat{\pi}_* [\hat{X}_{U,\tau_0,i}^{\text{red}}] = [X_{U,\tau_0,i}].$$

Since for any bundle F on X_U we have $\hat{\pi}^! c(F) = c(\hat{\pi}^* F)$ (see [Fu], Theorem 6.3 and the paragraph before example 8.1.1), it follows that we have:

$$\hat{\pi}^! \hat{\pi}_* [\hat{X}_{U,\tau_0,i}^{\text{red}}] = \Delta_i^{(i+r\tau_0-r_{\sigma^{-1}\tau_0})} (c(\hat{\mathcal{E}}_{\tau_0})^{-1} c(F_{\text{abs}}^* \hat{\mathcal{E}}_{\sigma^{-1}\tau_0})),$$

where $\hat{\mathcal{E}}_{\tau}$ is the restriction of \mathcal{E}_{τ} to $\hat{X}_{U,\tau_0,i}^{\text{red}}$.

The next step is to express this in terms of classes of bundles pulled back from $(X_{U'})^{\tau_0,i}$. Let ι denote the natural purely inseparable map $\hat{X}_{U,\tau_0,i}^{\text{red}} \rightarrow (X_{U'})^{\tau_0,i}$, and recall that W denotes the tautological (rank i) bundle on $(X_{U'})^{\tau_0,i}$.

Let \mathcal{B} be the universal abelian variety on $X_{U'}$, and let \mathcal{E}'_{τ} be the bundle $\text{Lie}(\mathcal{B}/X_{U'})_{p_{\tau}}$.

Proposition 6.3 *There are exact sequences:*

$$\begin{aligned} 0 \rightarrow \iota^* W \rightarrow (\hat{\pi}')^* \mathcal{E}'_{\tau_0} \rightarrow \hat{\mathcal{E}}_{\tau_0} \rightarrow 0 \\ 0 \rightarrow F_{\text{abs}}^* (\hat{\pi}')^* \mathcal{E}'_{\sigma^{-1}\tau_0} \rightarrow F_{\text{abs}}^* \hat{\mathcal{E}}_{\sigma^{-1}\tau_0} \rightarrow \iota^* W \rightarrow 0. \end{aligned}$$

Proof. Let $\hat{\mathcal{A}}$ be the restriction of \mathcal{A} to $\hat{X}_{U,\tau_0,i}^{\text{red}}$, and $\hat{\mathcal{B}}$ be the pullback of \mathcal{B} to $\hat{X}_{U,\tau_0,i}^{\text{red}}$. The universal isogeny $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$ then induces a map

$$\text{Lie}(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\tau_0}}^* \rightarrow \hat{\mathcal{E}}_{\tau_0}.$$

The kernel of this map is $\iota^* W$; counting dimensions after pulling back to any closed point shows it is surjective. The first exact sequence above then follows from the isomorphism

$$\text{Lie}(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}}) \cong (\hat{\pi}')^* \mathcal{E}'_{\tau_0}.$$

For the second exact sequence, consider the commutative diagram:

$$\begin{array}{ccc} F_{\text{abs}}^* (H_{\text{DR}}^1(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\sigma^{-1}\tau_0}}) & \rightarrow & H_{\text{DR}}^1(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\tau_0}} \\ \downarrow & & \downarrow \\ F_{\text{abs}}^* (H_{\text{DR}}^1(\hat{\mathcal{A}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\sigma^{-1}\tau_0}}) & \rightarrow & H_{\text{DR}}^1(\hat{\mathcal{A}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\tau_0}} \end{array}$$

where the vertical maps are induced by the universal isogeny $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}}$, and the horizontal maps are induced by relative Frobenius.

Note that the left-hand vertical map is an isomorphism. Thus the kernel of the bottom map (which is equal to $F_{\text{abs}}^* \hat{\mathcal{E}}_{\sigma^{-1}\tau_0}$) is isomorphic to the kernel of the composition of the upper horizontal map and the right-hand vertical map. The kernel of the former is $F_{\text{abs}}^* \text{Lie}(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\sigma^{-1}\tau_0}}^*$; the kernel of the latter is ι^*W . We thus obtain an exact sequence:

$$0 \rightarrow F_{\text{abs}}^* \text{Lie}(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}})_{p_{\sigma^{-1}\tau_0}}^* \rightarrow F_{\text{abs}}^* \hat{\mathcal{E}}_{\sigma^{-1}\tau_0} \rightarrow \iota^*W \rightarrow 0.$$

(Exactness on the right follows by pulling back to closed points and counting dimensions.) The second exact sequence again follows from the isomorphism

$$\text{Lie}(\hat{\mathcal{B}}/\hat{X}_{U,\tau_0,i}^{\text{red}}) \cong (\hat{\pi}')^* \text{Lie}(\mathcal{B}/X_{U'}).$$

□

By the multiplicativity of the total Chern class, we have:

$$\hat{\pi}'^! \hat{\pi}'_* [\hat{X}_{U,\tau_0,i}^{\text{red}}] = \Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})} (c(\iota^*W)^2 c((\hat{\pi}')^* \mathcal{E}'_{\tau_0})^{-1} c(F_{\text{abs}}^* (\hat{\pi}')^* \mathcal{E}'_{\sigma^{-1}\tau_0})).$$

This means that $c_{\hat{\pi}'}$, when considered as a cohomology class on $(X_{U'})^{\tau_0,i}$, is the cohomology class associated to the element:

$$\Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})} (c(W)^2 c((\pi')^* \mathcal{E}'_{\tau_0})^{-1} c((\pi')^* F_{\text{abs}}^* \mathcal{E}'_{\sigma^{-1}\tau_0})) \in A^r((X_{U'})^{\tau_0,i}).$$

Consider $\beta_0(\alpha(c_{\hat{\pi}'}))$. It is a section of the constant sheaf \mathbb{Q}_ℓ on $(X_{U'})$. We have shown that for any x in V , $\beta_0(\alpha(c_{\hat{\pi}'}))_x$ is obtained by pulling back $c_{\hat{\pi}'}$ to an element of $H_{\text{ét}}^{2r}((\pi')^{-1}(x), \mathbb{Q}_\ell(r))$, and applying the canonical isomorphism of this space with \mathbb{Q}_ℓ . In other words, $\beta_0(\alpha(c_{\hat{\pi}'}))_x$ is the element of $H_{\text{ét}}^{2r}((\pi')^{-1}(x), \mathbb{Q}_\ell(r))$ obtained by intersecting

$$\Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})} (c(W)^2 c((\pi')^* \mathcal{E}'_{\tau_0})^{-1} c((\pi')^* F_{\text{abs}}^* \mathcal{E}'_{\sigma^{-1}\tau_0}))$$

with $Z = (\pi')^{-1}(x)$ and then taking the associated cohomology class.

Any bundle pulled back from $X_{U'}$ via π' becomes the trivial bundle when restricted to Z , and thus has trivial total Chern class in $A^*(Z)$. On the other hand, Z is a Grassmannian of i planes in a $2i + r_{\tau_0} - r_{\sigma^{-1}\tau_0}$ dimensional space, and W restricts to the tautological subbundle W_Z on this Grassmannian.

Thus, $\beta_0(\alpha(c_{\hat{\pi}'}))$ is the cohomology class in $H_{\text{ét}}^{2r}(Z, \mathbb{Q}_\ell(r))$ associated to the element

$$\Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})} (c(W_Z)^2) \in A^r(Z).$$

Proposition 6.4 *Let $[P]$ be the class of a point in $A^r(Z)$. Then we have:*

$$\Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})} (c(W_Z)^2) = (-1)^r \binom{2i + r_{\tau_0} - r_{\sigma^{-1}\tau_0}}{i} [P].$$

Proof. We have a tautological exact sequence:

$$0 \rightarrow W_Z \rightarrow \mathcal{O}_Z^{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}} \rightarrow Q \rightarrow 0$$

of vector bundles on Z . Dualizing yields a sequence:

$$0 \rightarrow Q^* \rightarrow \mathcal{O}_Z^{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}} \rightarrow W_Z^* \rightarrow 0.$$

Let M be any endomorphism of $\overline{\mathbb{F}}_p^{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}$ with distinct eigenvalues. We obtain a map $\varsigma_M : Q^* \rightarrow W_Z^*$ by including Q^* in $\mathcal{O}_Z^{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}$, applying the endomorphism M , and then projecting to W_Z^* .

The subscheme $D_0(\varsigma_M)$ of points of Z on which ς_M is the zero map is easily seen to be reduced, and equal to the union of those points of Z that correspond to $i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}$ -dimensional subspaces of $\overline{\mathbb{F}}_p^{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}$ that are stable under M . Any such space is the direct sum of precisely $i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}$ of the $2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}$ distinct eigenspaces of M . Thus we have:

$$[D_0(\varsigma_M)] = \binom{2i+r_{\tau_0}-r_{\sigma^{-1}\tau_0}}{i}[P].$$

On the other hand, the Thom-Porteus formula tells us that we have:

$$[D_0(\varsigma_M)] = \Delta_i^{(i+r_{\tau_0}-r_{\sigma^{-1}\tau_0})}(c(W_Z^*)c(Q^*)^{-1}).$$

The result follows by putting these two together, and using the basic identities:

$$\begin{aligned} c(W_Z^*)c(Q^*) &= 1, \\ c_j(W_Z) &= (-1)^j c_j(W_Z^*). \end{aligned}$$

□

It follows that $\beta_0(\alpha(c_{\bar{\pi}}))$ is a non-vanishing section of the constant sheaf \mathbb{Q}_ℓ on $X_{U'}$. Theorem 5.3 thus follows from Proposition 5.4.

7 Jacquet-Langlands correspondences

We now use the above characteristic p results to study the cohomology of Shimura varieties in characteristic zero. As the Shimura varieties we consider are not necessarily proper, we will first need some results beyond the standard theory of vanishing cycles to accomplish this.

Let $S = \text{Spec } W(\overline{\mathbb{F}}_p)$; let s and η denote the closed point and a geometric generic point of S , respectively. Let X be a smooth S -scheme, and let \overline{X} be a compactification of X with the following properties:

- \overline{X} is smooth and proper over S ,
- the complement $\overline{X} \setminus X$ is a divisor D with normal crossings, and

- if D_1, \dots, D_i are irreducible components of D , then the intersection $D_1 \cap \dots \cap D_i$ is either empty or smooth over S .

Under these hypotheses, we have:

Lemma 7.1 *Let $\overline{\mathcal{F}}$ be a lisse sheaf on \overline{X} , and let \mathcal{F} denote its restriction to X . Then the specialization maps*

$$H_{\text{ét}}^j(X_\eta, \mathcal{F}_\eta) \rightarrow H_{\text{ét}}^j(X_s, \mathcal{F}_s)$$

are isomorphisms.

Proof. We work by induction on the number of irreducible components of D . In the base case D is empty and the above result is immediate from the theory of vanishing cycles.

Suppose the result is true for D having k components. Let $X^k = \overline{X} \setminus (D_1 \cup \dots \cup D_k)$, and let $D_{k+1}^k = D_{k+1} \setminus (D_1 \cup \dots \cup D_k)$. Then the specialization maps:

$$H_{\text{ét}}^j(X_\eta^k, \mathcal{F}_\eta) \rightarrow H_{\text{ét}}^j(X_s^k, \mathcal{F}_s)$$

$$H_{\text{ét}}^j((D_{k+1}^k)_\eta, \mathcal{F}_\eta) \rightarrow H_{\text{ét}}^j((D_{k+1}^k)_s, \mathcal{F}_s)$$

are isomorphisms.

These fit into a Gysin sequence:

$$\begin{array}{ccccccc} \rightarrow & H_{\text{ét}}^j((D_{k+1}^k)_\eta, \mathcal{F}_\eta) & \rightarrow & H_{\text{ét}}^{j+2}(X_\eta^k, \mathcal{F}_\eta) & \rightarrow & H_{\text{ét}}^{j+2}(X_\eta^{k+1}, \mathcal{F}_\eta) & \rightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \rightarrow & H_{\text{ét}}^j((D_{k+1}^k)_s, \mathcal{F}_s) & \rightarrow & H_{\text{ét}}^{j+2}(X_s^k, \mathcal{F}_s) & \rightarrow & H_{\text{ét}}^{j+2}(X_s^{k+1}, \mathcal{F}_s) & \rightarrow \end{array}$$

and hence the result holds for X^{k+1} as well. \square

Call such a compactification of X a *good* compactification. In [FC], Faltings-Chai show that the toroidal compactifications of the moduli spaces of principally polarized abelian varieties are good compactifications, and assert (with no details) that their methods carry over to arbitrary PEL Shimura varieties. As yet unpublished work of Kai-Wen Lan ([La], Thm 6.4.1.1) shows that the unitary Shimura varieties X_U admit good (toroidal) compactifications \overline{X}_U . Moreover, the sheaves \mathcal{L}_ξ extend to lisse sheaves on \overline{X}_U . We thus have a natural isomorphism:

$$H_{\text{ét}}^j((X_U)_\eta, (\mathcal{L}_\xi)_\eta) \cong H_{\text{ét}}^j((X_U)_s, (\mathcal{L}_\xi)_s).$$

We can use this, together with Theorem 5.3, to “transfer” automorphic representations from one algebraic group to another. Fix two places τ_0, τ'_0 of F^+ , with $r_{\tau'_0} \leq r_{\tau_0}$, and fix an i such that $1 \leq i \leq \min(r_{\tau'_0}, n - r_{\tau_0})$. Then there exists a \mathcal{V}' such that $\mathcal{V}'(\mathbb{A}_{\mathbb{Q}}^\infty)$ is isomorphic to $\mathcal{V}(\mathbb{A}_{\mathbb{Q}}^\infty)$, but whose invariants at infinity satisfy:

- $r_{\tau_0}(\mathcal{V}') = r_{\tau_0} + i$,
- $r_{\tau'_0}(\mathcal{V}') = r_{\tau'_0} - i$,

- $r_\tau(\mathcal{V}') = r_\tau(\mathcal{V})$ for τ outside $\{\tau_0, \tau'_0\}$.

Fix such a \mathcal{V}' , and let G' be the corresponding unitary group. Also fix an identification of $\mathcal{V}(\mathbb{A}_\mathbb{Q}^\infty)$ with $\mathcal{V}'(\mathbb{A}_\mathbb{Q}^\infty)$; this yields an identification of $G(\mathbb{A}_\mathbb{Q}^\infty)$ with $G'(\mathbb{A}_\mathbb{Q}^\infty)$.

Theorem 7.2 *Let π' be an automorphic representation of G' , and suppose that there exists a representation ξ' of $G'(\mathbb{A}_\mathbb{Q}^\infty)$ over $\overline{\mathbb{Q}}_\ell$ such that π'_∞ is cohomological for ξ' . Suppose also that there exist good compactifications for unitary Shimura varieties attached to G and G' . Then there exists an automorphic representation π of G such that $\pi_v = \pi'_v$ for all finite places v of \mathbb{Q} , and such that π_∞ is cohomological for the representation ξ of $G(\mathbb{A}_\mathbb{Q}^\infty)$ that corresponds to ξ' .*

Proof. Let U' be a compact open subgroup of $G'(\mathbb{A}_\mathbb{Q}^\infty)$, such that π' has a nonzero U' -fixed vector. Let U be the corresponding subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$.

Fix, for each p , an embedding of $W(\overline{\mathbb{F}}_p)$ as a subring of \mathbb{C} . This determines a Frobenius element Frob_p of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, up to inertia. By Čebotarev, we can find a p such that p is unramified in F and split in E , such that U_p is a maximal compact subgroup of $G(\mathbb{Q}_p)$, and such that $\text{Frob}_p \tau'_0 = \tau_0$. Also choose an auxiliary prime l different from p .

Associated to these choices we have Shimura varieties X_U and $X_{U'}$ over $W(\overline{\mathbb{F}}_p)$. Let N be an integer divisible by all the primes of bad reduction of X_U , and let \mathbb{T}_U be the Hecke algebra (over $\overline{\mathbb{Q}}_\ell$) of prime-to- Np Hecke operators for G .

Let $s : \text{Spec } \overline{\mathbb{F}}_p \rightarrow \text{Spec } W(\overline{\mathbb{F}}_p)$ be the closed point of $\text{Spec } W(\overline{\mathbb{F}}_p)$. Theorem 5.3 yields, for all j , an injection

$$H_{\text{ét}}^j((X_{U'})_s, (\mathcal{L}_{\xi'})_s) \rightarrow H_{\text{ét}}^{j+2r}((X_U)_s, (\mathcal{L}_\xi)_s)$$

that is compatible with the action of the Hecke algebra \mathbb{T}_U . This yields \mathbb{T}_U -equivariant injections

$$H_{\text{ét}}^j((X_{U'})_\eta, (\mathcal{L}_{\xi'})_\eta) \rightarrow H_{\text{ét}}^{j+2r}((X_U)_\eta, (\mathcal{L}_\xi)_\eta),$$

by the above lemma, where η is a geometric generic point of $\text{Spec } W(\overline{\mathbb{F}}_p)$.

The representation π' determines a maximal ideal \mathfrak{m} of \mathbb{T}_U , and our hypotheses guarantee that for some j , $H_{\text{ét}}^j((X_{U'})_\eta, (\mathcal{L}_{\xi'})_\eta)_\mathfrak{m}$ will be nonzero. Then $H_{\text{ét}}^{j+2r}((X_U)_\eta, (\mathcal{L}_\xi)_\eta)_\mathfrak{m}$ is nonzero as well. There is therefore an automorphic representation π of G , such that:

- π_∞ is cohomological for ξ ,
- π has a U -fixed vector, and
- for any Hecke operator in \mathbb{T}_U , the Hecke eigenvalue for π is the same as that for π' .

It follows that π_v is isomorphic to π'_v for any finite place v not dividing Np . By Cebotarev, the Galois representations associated to π and π' coincide. It then follows that $\pi_v = \pi'_v$ for all finite places v . \square

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