

Birational Contractions of $\overline{M}_{3,1}$ and $\overline{M}_{4,1}$

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In this paper we study the birational geometry of $\overline{M}_{g,1}$ for some small values of g . In particular, we construct birational contractions of these spaces in which certain divisors of interest are contracted.

The effective divisors of a variety X generate a cone inside the Neron Severi space $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We call the closure of this cone the cone of effective divisors of X and denote it by $\overline{NE}^1(X)$. Studying this cone in the case where X is the moduli spaces of curves has produced many interesting results, including Harris and Mumford's proof that \overline{M}_g is of general type for $g \geq 24$ [?]. A key element of their proof is the computation of the class of certain divisors on \overline{M}_g , called **Brill-Noether divisors**.

Definition 1. *When $g + 1 = (r + 1)(g - d + r)$, the closure of the locus of curves in \overline{M}_g possessing a g_d^r is a divisor in \overline{M}_g , called a **Brill-Noether divisor**.*

In their work, they discovered that all of the Brill-Noether divisors lie on a single ray in $\overline{NE}^1(\overline{M}_g)$, an observation that led Harris and Morrison to conjecture that this ray is extremal. The Harris-Morrison Slope Conjecture has recently been proven false in [4] and subsequently in [5], but the statement is known to hold for certain small values of g . In particular, it holds for $g \leq 11$, $g+1$ composite. In each of these cases, the statement can be proved by use of the contraction theorem, which states that the set of exceptional divisors of a birational contraction $X \dashrightarrow Y$ span a simplicial face of $\overline{NE}^1(X)$ (see [13]). In other words, the Slope Conjecture has been shown to hold for small values of g by constructing explicit birational models for the moduli space in which the Brill-Noether divisor is contracted (see, for example, [11]). Moreover, these models arise naturally as geometric invariant theory quotients.

The purpose of this paper is to carry out a pointed analogue of the discussion above in some low genus cases. In [10], Logan introduced the notion of **pointed Brill-Noether divisors**.

Definition 2. *Let $Z = (a_0, \dots, a_r)$ be an increasing sequence of nonnegative integers with $\alpha = \sum_{i=0}^r a_i - i$. Let $BN_{d,Z}^r$ be the closure of the locus of pointed curves $(C, p) \in M_{g,1}$ possessing a g_d^r on C with ramification sequence Z at*

p . When $g + 1 = (r + 1)(g - d + r) + \alpha$, this is a divisor in $\overline{M}_{g,1}$, called a **pointed Brill-Noether divisor**.

Our main result is the following:

Theorem 0.1. *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN_{3,(0,3)}^1$. Similarly, there is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN_{3,(0,2)}^1$.*

We prove this by explicitly constructing these birational models as GIT quotients. As a consequence, we identify extremal rays of the effective cone.

Corollary 0.2. *For $g = 3, 4$, there is an extremal ray of $\overline{NE}^1(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor.*

The outline of the paper is as follows. In section 1 we provide some background on geometric invariant theory, which is the main tool that we use in the subsequent sections. In section 2, we develop a tool for studying GIT quotients of families of curves on surfaces. In particular, we construct a large class of divisors on these spaces that are invariant under the automorphism group of the surface, called Hessians. In sections 3 and 4 we then examine separately curves on \mathbb{P}^2 and on $\mathbb{P}^1 \times \mathbb{P}^1$, yielding our result in the cases of $g = 3$ and 4.

We plan on discussing similar results for genus 5 and 6 in a later paper.

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1 Background on Geometric Invariant Theory

The birational contractions that we construct arise naturally as GIT quotients. In this section, we provide the necessary background to understand these constructions.

Let X be a smooth projective variety over \mathbb{C} and G a reductive algebraic group acting on X . If we want to examine the orbits of this action, it is natural to ask for the existence of a quotient variety. In other words, we would like there to be a projective variety $X//G$ such that, if $X \rightarrow Y$ is a G -equivariant morphism of varieties, then it factors through a unique map $X//G \rightarrow Y$. Unfortunately, such a quotient may not necessarily exist. It is often possible, however, to find an open set $U \subseteq X$ such that $U//G$ exists. It has long been understood that this open subset is not unique; it depends on

the choice of a G -ample line bundle. In particular, if \mathcal{L} is such a line bundle, then we define the set of **semistable** points to be

$$X^{ss}(\mathcal{L}) = \{x \in X \mid \exists s \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ for some } n \text{ such that } s(x) \neq 0\}$$

and the set of **stable** points to be

$$X^s(\mathcal{L}) = \{x \in X^{ss}(\mathcal{L}) \mid G \cdot x \text{ is closed in } X^{ss}(\mathcal{L}) \text{ and the stabilizer of } x \text{ is finite}\}.$$

$X^{ss}(\mathcal{L})//G$ is a projective variety that satisfies the categorical definition of a quotient. Notice that, if X is irreducible, and if both $X^{ss}(\mathcal{L})$ and $X^{ss}(\mathcal{L}')$ are nonempty, then the quotients $X^{ss}(\mathcal{L})//G$ and $X^{ss}(\mathcal{L}')//G$ are birational. We will make use of this observation to construct birational models for certain spaces by varying the choice of line bundle.

Let \mathcal{L} be a line bundle on X . We consider G -linearizations of \mathcal{L} , or actions of G on \mathcal{L} that are linear on the fibers of \mathcal{L} and commute with the projection to X . The action of G on \mathcal{L} induces an action of G on $H^0(X, \mathcal{L})$, which allows us to talk about the G -invariant sections of \mathcal{L} . \mathcal{L} is G -effective if some tensor power of it admits a G -invariant section. In other words, \mathcal{L} is G -effective if $X^{ss}(\mathcal{L}) \neq \emptyset$. An ample G -effective line bundle is called G -ample. If we let $Pic^G X$ be the group of isomorphism classes of G -linearized line bundles on X , then there is a homomorphism $Pic^G X \rightarrow Pic X$ whose kernel is isomorphic to the group of characters on G . In the examples we will be considering, the group G admits no nontrivial characters, and so we will often refer to the line bundle without specifying its linearization.

The tensor product of two G -ample line bundles is also G -ample, so the G -ample line bundles form a cone inside $Pic^G X$. Since the quotient $X^{ss}(\mathcal{L})//G$ depends only on the algebraic equivalence class of \mathcal{L} , we may restrict our attention to the cone spanned by the G -ample line bundles in $NS_{\mathbb{Q}}^G$. Following Dolgachev and Hu, we will call this the **G -ample cone**, denoted $C^G(X)$. The properties of this cone are studied extensively in [3] and [14]. The following theorem is a summary of some of the major results of those papers:

Theorem 1.1. [3] [14] *The G -ample cone is divided into a finite number of convex cones, called chambers, by a finite number of codimension 1 homogeneous spaces, called walls. These satisfy the following properties:*

1. \mathcal{L} lies on a wall if and only if $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$;
2. \mathcal{L} lies on a boundary of $C^G(X)$ if and only if $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L}) = \emptyset$;
3. \mathcal{L} and \mathcal{L}' lie in the same chamber if and only if $X^s(\mathcal{L}) = X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}') = X^s(\mathcal{L}')$;
4. If $\mathcal{L}(t) = \mathcal{L}^t \otimes \mathcal{L}'^{(1-t)}$, then $X^{ss}(\mathcal{L}) \cap X^{ss}(\mathcal{L}') \subseteq X^{ss}(\mathcal{L}(t))$;

5. If \mathcal{L} lies on a wall and \mathcal{L}' lies in an adjacent chamber, then the inclusion $X^{ss}(\mathcal{L}') \subseteq X^{ss}(\mathcal{L})$ induces a morphism $X^{ss}(\mathcal{L}')//G \rightarrow X^{ss}(\mathcal{L})//G$ that is an isomorphism over $X^s(\mathcal{L})//G$.

The case of $G = \mathbb{C}^*$ has been studied in detail. Let X^G be the set of fixed points of the action, and let $X^G = X_1 \cup X_2 \cup \dots \cup X_r$ be its decomposition into irreducible components. Let

$$X_i^+ = \{x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in X_i\}$$

$$X_i^- = \{x \in X \mid \lim_{t \rightarrow \infty} t \cdot x \in X_i\}.$$

We can think of X_i^+ as the points of X that “flow from” X_i , and X_i^- as those that “flow to” X_i . The partition of X as $X = X_1^+ \cup \dots \cup X_r^+ = X_1^- \cup \dots \cup X_r^-$ is known as the Bialynicki-Birula decomposition of X with respect to the action of \mathbb{C}^* [1]. Every point of $X \setminus X^G$ is contained in exactly one subset of the following form:

$$C_{ij} = (X_i^+ \setminus X_i) \cap (X_j^- \setminus X_j)$$

Using the C_{ij} 's, we can define a partial ordering on the X_i 's by saying that X_i is *directly* less than X_j if $C_{ij} \neq \emptyset$, and X_i is less than X_j if there is a sequence $X_i = X_{i_0}, X_{i_1}, \dots, X_{i_k} = X_j$ such that X_{i_n} is directly less than $X_{i_{n+1}}$ for all n . (In other words, X_i is less than X_j if there are points that “flow from” X_i to X_j .) An open subset $U \subseteq X$ is then the semistable locus of some G -ample line bundle if and only if it is of the form

$$U = \cup_{i \in A^-, j \in A^+} C_{ij}$$

where A^-, A^+ is a partition of $\{1, \dots, r\}$ such that, if $i \in A^-$ and X_j is less than X_i , then $j \in A^-$.

In the case that $G \neq \mathbb{C}^*$, less is known, but it is often possible to reduce to the case of a \mathbb{C}^* action. In [14], Thaddeus considers the following situation: let $\mathcal{L}_+, \mathcal{L}_-$ be G -ample line bundles and define $\mathcal{L}(t) = \mathcal{L}_+^t \otimes \mathcal{L}_-^{1-t}$. Suppose that there exists a t_0 with $X^{ss}(\mathcal{L}(t)) = X^{ss}(\mathcal{L}_+)$ for all $1 > t > t_0$, and $X^{ss}(\mathcal{L}(t)) = X^{ss}(\mathcal{L}_-)$ for all $-1 < t < t_0$. This is the case, for example, when \mathcal{L}_+ and \mathcal{L}_- are in adjacent chambers in the G -ample cone, and the line between them crosses a wall precisely at $\mathcal{L}(t_0)$. In this case, he lets $Y = \mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-)$, and $Z = Y//G$. There is then a family of \mathbb{C}^* actions on Z such that $X^{ss}(\mathcal{L}(t))//G = Z//\mathbb{C}_t^*$, and $X^{ss}(t) = q(p^{-1}(Z^{ss}(t)))$. Here, $q : Y \rightarrow X$ and $p : Y \rightarrow Z$ are the natural projection maps. Using this, he can define subsets of X analogous to the flowlines of a \mathbb{C}^* action:

$$X^\pm = X^{ss}(0) \setminus X^{ss}(\mp)$$

$$X^0 = X^{ss}(0) \setminus (X^{ss}(+) \cup X^{ss}(-))$$

and then proves:

Theorem 1.2. [14] *Let $x \in X^0$ be a smooth point of X . Suppose that $G \cdot x$ is closed in $X^{ss}(0)$ and that $G_x \cong \mathbb{C}^*$. Then the natural map $X^{ss}(\mathcal{L}(\pm))/G \rightarrow X^{ss}(\mathcal{L})(0)/G$ is an isomorphism outside of $X^\pm/G(\pm)$. Over a neighborhood of x in $X^0/G(0)$, $X^{/pm}/G(\pm)$ are fibrations with fibers weighted projective spaces.*

Note: This theorem requires the hypothesis that we are working in characteristic zero.

Given a G -ample line bundle \mathcal{L} , it is often difficult to describe the set $X^{ss}(\mathcal{L})$. Fortunately, Mumford's numerical criterion is a useful tool for determining whether or not a point is in this set. First, the line bundle \mathcal{L} determines an embedding of X in some projective space $\mathbb{P}(V)$, where V is a complex vector space, which in turn determines a map $G \rightarrow GL(V)$. Now, define a 1-parameter subgroup to be a homomorphism $\lambda : \mathbb{C}^* \rightarrow G$. Since the image of \mathbb{C}^* is an abelian subgroup of $GL(V)$, there exists a basis $\{b_1, b_2, \dots, b_n\}$ for V with respect to which $\lambda(t)$ acts on V by the diagonal matrix

$$\lambda(t) = \begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{a_n} \end{pmatrix}.$$

We will refer to the a_i 's as the weights of the \mathbb{C}^* action. For a point $x \in X$, lift it to V , and write $x = \sum_{i=1}^n x_i b_i$ with respect to this basis. Define $\mu_\lambda(x)$ to be the smallest value of a_i such that the corresponding x_i is nonzero. Then x is semistable if and only if $\mu_\lambda(x) \leq 0$ for every 1-parameter subgroup λ of G , and x is stable if and only if $\mu_\lambda(x) < 0$ for every nontrivial 1-parameter subgroup λ of G . We will use this criterion often to determine the (semi)stability of points in our space.

2 Hessians

We now turn our attention to families of curves on surfaces. We do this because, in both of the cases we will be considering, the general curve embeds canonically into a surface. In particular, the general genus 3 curve admits a canonical embedding into \mathbb{P}^2 . Similarly, the general genus 4 curve admits a canonical embedding into \mathbb{P}^3 whose image is contained in a unique quadric surface. This surface is generically isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Let Y be a smooth projective surface over \mathbb{C} , \mathcal{L}' an effective line bundle on Y , and $Z = \mathbb{P} H^0(Y, \mathcal{L}')$. Let

$$X = \{(C, p) \in Z \times Y \mid C(p) = 0\}.$$

We denote the various maps as in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \times Y \xrightarrow{\pi_1} Y \\ \downarrow f & & \downarrow \pi_2 \\ Z & \xrightarrow{id} & Z \end{array}$$

In other words, we will denote by $\pi_1 : Z \times Y \rightarrow Y$ and $\pi_2 : Z \times Y \rightarrow Z$ the natural projections, i the inclusion map $i : X \rightarrow Z \times Y$, and f the projection map $f : X \rightarrow Z$. If \mathcal{L}' is base-point free, then X is a projective space bundle over Y , so it is smooth, and if $h^0(\mathcal{L}') \geq 2$, then $PicX \cong PicY \times \mathbb{Z}$. There is a natural action of $Aut(Y)$ on X , given by

$$A(C, p) = (C \circ A^{-1}, A(p)).$$

We will later study the geometric invariant theory quotients of X by the action of this group. Our first step is to identify a collection of G -effective divisors on X , called Hessians.

If C is a curve on Y and \mathcal{L} is another line bundle on Y , then for every point $p \in C$ there are $n + 1 = h^0(C, \mathcal{L}|_C)$ different orders of vanishing of sections $s \in H^0(C, \mathcal{L}|_C)$. When written in increasing order,

$$a_0^{\mathcal{L}}(p) < \dots < a_n^{\mathcal{L}}(p)$$

these integers are called the **vanishing sequence** of \mathcal{L} at p . The **weight** of \mathcal{L} at p is defined to be $w^{\mathcal{L}}(p) = \sum_{i=0}^n a_i^{\mathcal{L}}(p) - i$. A point p is said to be an **\mathcal{L} -flex** if the weight of \mathcal{L} at p is nonzero. In other words, p is an \mathcal{L} -flex if the vanishing sequence of \mathcal{L} at p is anything other than $0 < 1 < \dots < n$. The divisor of \mathcal{L} -flexes consists of all the \mathcal{L} -flexes, each counted with multiplicity equal to its weight. In other words, it is $\sum_{p \in C} w^{\mathcal{L}}(p)p$. The divisor of \mathcal{L} -flexes corresponds to a section $W_{\mathcal{L}}$ of a certain line bundle that is called the **Wronskian** of \mathcal{L} . We say that a curve H on Y is an \mathcal{L} -Hessian if the restriction of H to C is precisely the divisor of \mathcal{L} -flexes.

Returning to our family of curves $f : X \rightarrow Z$ above, suppose that \mathcal{L} is a line bundle on Y such that the pushforward $f_*(i \circ \pi_1)^* \mathcal{L}$ is locally free of rank $n + 1$. We define a relative \mathcal{L} -Hessian to be a divisor $H \subseteq X$ whose restriction to each fiber is the divisor of $f_*(i \circ \pi_1)^* \mathcal{L}$ -flexes. In [2], Cukierman shows that the class of the relative \mathcal{L} -Hessian is

$$(n + 1)c_1 \pi_1^* \mathcal{L} + \binom{n + 1}{2} c_1 \Omega_{X/Z}^1 - c_1 f^* f_* \pi_1^* \mathcal{L}$$

For the remainder of this section, we will follow the proof in [2] to determine this class more explicitly in our case. We see that if I is the ideal sheaf of X in $Z \times Y$, then we have the exact sequence

$$0 \rightarrow I/I^2 \rightarrow \pi_1^* \Omega_Y^1|_X \rightarrow \Omega_{X/Z}^1 \rightarrow 0$$

so we have

$$c_1 \Omega_{X/Z}^1 = c_1 \pi_1^* \Omega_Y^1|_X - c_1 I/I^2.$$

Also, X is the scheme of zeros of a section of the line bundle $E = \pi_1^* \mathcal{L}' \otimes \pi_2^* \mathcal{O}_Z(1)$ on $Z \times Y$. Note that $I/I^2 \cong E^* \otimes \mathcal{O}_X = E^*|_X$. It follows that

$$\begin{aligned} c_1 \Omega_{X/Z}^1 &= c_1 \pi_1^* \Omega_Y^1|_X + c_1 E \\ &= c_1 \pi_1^* \Omega_Y^1|_X + c_1 \pi_1^* \mathcal{L}' + c_1 \pi_2^* \mathcal{O}_Z(1). \end{aligned}$$

Now, note that

$$\pi_{2*}(\pi_1^* \mathcal{L} \otimes E^*) = H^0(Y, \mathcal{L}) \oplus H^0(Y, \mathcal{L} \otimes \mathcal{L}'^*) \otimes \mathcal{O}_Z(-1)$$

so the class of the relative \mathcal{L} -Hessian is

$$\begin{aligned} (n+1)c_1 \pi_1^* \mathcal{L} + \binom{n+1}{2} (c_1 \pi_1^* \Omega_Y^1|_X + c_1 \pi_1^* \mathcal{L}' + c_1 \pi_2^* \mathcal{O}_Z(1)) \\ - c_1 (H^0(Y, \mathcal{L}) \oplus H^0(Y, \mathcal{L} \otimes \mathcal{L}'^*) \otimes \mathcal{O}_Z(-1)). \end{aligned}$$

These Hessians provide a large class of G -invariant sections to work with when we are trying to determine the GIT quotients of X .

In particular, suppose that $Y = \mathbb{P}^2$, and $\mathcal{L}' = \mathcal{O}_Y(d)$ for some $d \geq 4$. By the above, we see that for every m and d , a relative $\mathcal{O}_Y(m)$ -Hessian H_m exists. Since $\Omega_Y^1 = \mathcal{O}_Y(-3)$, if $m < d$, H_m is cut out by a G -invariant section W_m of

$$\mathcal{O}_X((n+1)m + \binom{n+1}{2}(d-3), \binom{n+1}{2}),$$

where $n+1 = h^0(Y, \mathcal{L}) = \binom{m+2}{2}$. In particular, H_1 is cut out by a section $W_1 \in H^0(\mathcal{O}_X(3(d-2), 3))$. W_1 vanishes at (C, p) if C is smooth at p and the tangent line to C at p intersects C with multiplicity at least 3, or if p is a singular point of C . Similarly, H_2 is defined by a section of $W_2 \in H^0(\mathcal{O}_X(15d-33, 15))$. W_2 vanishes at (C, p) if C is smooth at p and the osculating conic to C at p intersects C with multiplicity at least 6, or if p is a singular point of C .

Notice, however, that if

$$0 < a_1(p) < a_2(p)$$

is a vanishing sequence for $\mathcal{O}_C(1)$ at p , and $a_2(p) \neq 2a_1(p)$, then

$$0 < a_1(p) < \min(2a_1(p), a_2(p)) < \max(2a_1(p), a_2(p)) < a_1(p) + a_2(p) < 2a_2(p)$$

is a vanishing sequence for $\mathcal{O}_C(2)$ at p . Thus, $w^{\mathcal{O}_C(2)}(p) \geq 4w^{\mathcal{O}_C(1)}(p) - 3$ (with equality in the case that $2a_1(p) \neq a_0(p) + a_2(p)$). From this, we see that if $(C, p) \in H_1$, then $(C, p) \in H_2$ as well, and so $H_2 = H_1 \cup H'_2$ is reducible. The points of $H'_2 \cap C$ are classically known as the **sextatic points** of C , and H'_2 is cut out by a G -invariant section W'_2 of $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$. From the equation above, we see that if $p \in H_1 \cap C$, then H_2 intersects C with the same multiplicity as H_1 only if $w^{\mathcal{O}_C(1)}(p) = 1$. Thus, $H'_2 \cap C$ contains not only the sextatic points of C , but also those points of C where $w^{\mathcal{O}_C(1)}(p) > 1$. These include singular points and points where the tangent line to C is a **hyperflex** (a line that intersects C at p with multiplicity ≥ 4).

Similarly, suppose that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{L}' = \mathcal{O}_Y(d, d)$. Note that, for every (m_1, m_2, d) with $m_i < d$, a relative $\mathcal{O}_Y(m_1, m_2)$ -Hessian H'_{m_1, m_2} exists. In this case, our formulas show that the rank of $f_*(i \circ \pi_1)^* \mathcal{O}_Y(m_1, m_2)$ is

$$n + 1 = h^0(\mathcal{O}_Y(m_1, m_2)) = (m_1 + 1)(m_2 + 1).$$

Also, since $\Omega_Y^1 = \mathcal{O}_Y(-2, -2)$, we see that H'_{m_1, m_2} is cut out by a section $W'_{m_1, m_2} \in H^0(\mathcal{O}_X(a_1, a_2, b))$ for

$$a_1 = (n + 1)m_1 + \binom{n + 1}{2}(d - 2)$$

$$a_2 = (n + 1)m_2 + \binom{n + 1}{2}(d - 2)$$

$$b = \binom{n + 1}{2}.$$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ has a natural involution, we know that W'_{m_1, m_2} cannot be G -invariant if $m_1 \neq m_2$. Notice, however, that $W'_{m_1, m_2} \otimes W'_{m_2, m_1}$ is a G -invariant section of $\mathcal{O}_X(a, a, b)$ for

$$n + 1 = (m_1 + 1)(m_2 + 1)$$

$$a = (n + 1)(m_1 + m_2) + 2 \binom{n + 1}{2}(d - 2)$$

$$b = 2 \binom{n + 1}{2}.$$

We will use W_{m_1, m_2} to denote the G -invariant section described here, and H_{m_1, m_2} to denote its zero locus.

In particular, $W_{0,1} \in H^0(\mathcal{O}_X(2(d-1), 2(d-1), 2))$. It vanishes at a point (C, p) if C intersects one of the two lines through p with multiplicity at least 2 (or, equivalently, if the osculating $(1, 1)$ curve is a pair of lines). Similarly, $W_{1,1} \in H^0(\mathcal{O}_X(2(3d-4), 2(3d-4), 6))$. It vanishes at a point (C, p) if there is a curve of bidegree $(1, 1)$ that intersects C with multiplicity 4 or more at p .

2.1 Curves in \mathbb{P}^2

Recall that the general genus 3 curve admits a canonical embedding into \mathbb{P}^2 as a plane quartic. To construct a birational model for $\overline{M}_{3,1}$, therefore, we consider GIT quotients of the universal family over the space of plane quartics. The image of the Weierstrass divisor in this model is precisely the Hessian H_1 . Our goal therefore is to exhibit a GIT quotient in which this locus is contracted. Our methods work equally well if we consider, more generally, plane curves of any degree d .

Specifically, following the set-up of the previous section, let $Y = \mathbb{P}^2$ and $\mathcal{L}' = \mathcal{O}_Y(d)$ for $d \geq 4$. Then $\pi_2 : X \rightarrow \mathbb{P} H^0(\mathbb{P}^2, \mathcal{O}(d))$ is the family of all plane curves of degree d . By the above, we know that $\text{Pic} X \cong \mathbb{Z} \times \mathbb{Z}$. We will write $\mathcal{O}_X(a, b)$ to denote $(i \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes f^*(\mathcal{O}_Z(b))$. The automorphism group of \mathbb{P}^2 is $G = PSL(3, \mathbb{C})$. Recall that our goal is to study the GIT quotients of X by the action of this group.

Proposition 2.1. $\mathcal{O}_X(3(d-2), 3)$ lies on a boundary of $C^G(X)$.

Proof. Let $\mathcal{L} = \mathcal{O}_X(3(d-2), 3)$. It suffices to show that $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L}) = \emptyset$. It is clear that $X^{ss}(\mathcal{L}) \neq \emptyset$, since W_1 is a G -invariant section of \mathcal{L} .

To show that $X^s(\mathcal{L}) = \emptyset$, we invoke the numerical criterion. Let $(C, p) \in X$. By change of coordinates, we may assume that $p = (0, 0, 1)$ and the tangent line to C at p is $x_0 = 0$. So, if

$$C = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k,$$

then $a_{0,0,d} = a_{0,1,d-1} = 0$. Under the embedding determined by \mathcal{L} , we write (C, p) in terms of the basis of monomials of the form

$$x_2^{3(d-2)} \prod_{\alpha=1}^3 a_{i_\alpha, j_\alpha, k_\alpha}.$$

Now consider the 1-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

Then $\lambda(t)$ acts on the monomial above with weight $3(d-2) + \sum_{\alpha=1}^3 i_\alpha - k_\alpha$. We want to know when this weight is negative. Let i, k be the values of i_α, k_α that minimize $i_\alpha - k_\alpha$. Then

$$3(d-2) + \sum_{\alpha=1}^3 i_\alpha - k_\alpha \geq 3(d-2) + 3(i-k)$$

which is negative whenever $i-k-2 < -d = -i-j-k$, or $2i+j < 2$. This only occurs when both $i=0$ and $j < 2$, in other words, when either $a_{0,0,d}$ or $a_{0,1,d-1}$ is nonzero. By assumption, however, this is not the case, so $(C,p) \notin X^s(\mathcal{L})$. Since (C,p) was arbitrary, it follows that $X^s(\mathcal{L}) = \emptyset$. \square

Proposition 2.2. *Let $\mathcal{L} = \mathcal{O}_X(12(d-\frac{9}{4}), 12)$. Then $X^{ss}(\mathcal{L}) = X \setminus (H_1 \cap H'_2)$, and $X^s(\mathcal{L}) \subseteq X \setminus H_1$. In particular, since $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$, \mathcal{L} lies on a wall in $C^G(X)$.*

Proof. First, notice that if $(C,p) \notin H'_2$, then $(C,p) \in X^{ss}(\mathcal{L})$, since W'_2 is a G -invariant section of \mathcal{L} that does not vanish at (C,p) . Thus, $X \setminus (H_1 \cap H'_2) \subseteq X^{ss}(\mathcal{L})$.

Now, suppose that $(C,p) \in H_1$. As before, by change of coordinates, we may assume that $p = (0,0,1)$ and the tangent line to C at p is $x_0 = 0$. Since $(C,p) \in H_1$, either p is a singular point of C or this tangent line intersects C at p with multiplicity at least 3. Thus, if we write

$$C = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k,$$

then $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ (if p is singular) or $a_{0,2,d-2} = 0$ (if p is a flex). Again, under the embedding determined by \mathcal{L} , we write (C,p) in terms of the basis of monomials of the form

$$x_2^{12d-27} \prod_{\alpha=1}^{12} a_{i_\alpha, j_\alpha, k_\alpha}.$$

We first examine the case where p is a flex. In this case, consider the 1-parameter subgroup:

$$\lambda(t) = \begin{pmatrix} t^{-5} & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^4 \end{pmatrix}.$$

Then $\lambda(t)$ acts on the monomial above with weight $4(12d-27) + \sum_{\alpha=1}^{12} 5i_\alpha - j_\alpha - 4k_\alpha$. We want to know when this weight is negative. Let i, j, k be the

values of $i_\alpha, j_\alpha, k_\alpha$ that minimize $5i_\alpha - j_\alpha - 4k_\alpha$. Then

$$\begin{aligned} & 4(12d - 27) + \sum_{\alpha=1}^{12} 5i_\alpha - j_\alpha - 4k_\alpha \\ & \geq 4(12d - 27) + 12(5i - j - 4k) = 12(9i + 3j - 9) \end{aligned}$$

which is negative when $3i + j < 3$. This happens when both $i = 0$ and $j < 3$, but, by assumption, this is not the case. Thus $(C, p) \notin X^s(\mathcal{L})$.

Notice furthermore that if the tangent line to C at p is a hyperflex, then $a_{0,3,d-3} = 0$ as well, and so the expression $3i + j - 3$ above is strictly positive (rather than simply nonnegative), and thus $(C, p) \notin X^{ss}(\mathcal{L})$.

Next we look at the case where p is a singular point. Consider the 1-parameter subgroup:

$$\lambda(t) = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^2 \end{pmatrix}.$$

Then $\lambda(t)$ acts on the monomial above with weight $2(12d - 27) + \sum_{\alpha=1}^{12} i_\alpha + j_\alpha - 2k_\alpha$. We want to know when this weight is nonpositive. Let i, j, k be the values of $i_\alpha, j_\alpha, k_\alpha$ that minimize $i_\alpha + j_\alpha - 2k_\alpha$. Then

$$2(12d - 27) + \sum_{\alpha=1}^{12} i_\alpha + j_\alpha - 2k_\alpha \geq 2(12d - 27) + 12(i + j - 2k) = 12(3i + 3j - \frac{9}{2})$$

which is nonpositive when $i + j \leq \frac{3}{2}$. This happens when one of i, j is 0 and the other is at most 1. Since p is a singular point of C , however, this is not the case. Thus $(C, p) \notin X^{ss}(\mathcal{L})$. It follows that $X^s(\mathcal{L}) \subseteq X \setminus H_1$.

If $(C, p) \in H_1 \cap H'_2$, then either p is a singular point of C , or C is smooth at p and the tangent line to C at p is a hyperflex. From our observations above, we may therefore conclude that $X^{ss}(\mathcal{L}) \subseteq X \setminus (H_1 \cap H'_2)$. It follows that $X^{ss}(\mathcal{L}) = X \setminus (H_1 \cap H'_2)$. □

Corollary 2.3. *If $\mathcal{L} = \mathcal{O}_X(a, b)$ for any a, b such that $d - 2 > \frac{a}{b} > d - \frac{9}{4}$, then $X^{ss}(\mathcal{L}) = X \setminus H_1$.*

Proof. If $(C, p) \notin H_1$, then $(C, p) \in X^{ss}(\mathcal{O}_X(12(d - \frac{9}{4}), 12)) \cap X^{ss}(\mathcal{O}_X(3(d - 2), 3))$. It follows that $(C, p) \in X^{ss}(\mathcal{L})$, so $X \setminus H_1 \subseteq X^{ss}(\mathcal{L})$. It remains to show that, if $(C, p) \in H_1$, then $(C, p) \notin X^{ss}(\mathcal{L})$. For this, mirror the proof above with the same 1-parameter subgroups. □

We are left to consider the behavior of our quotient as we cross over the wall defined by $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$ in the G -ample cone. Let $\mathcal{L}(0) = \mathcal{O}_X(12(d - \frac{9}{4}), 12)$, and $\mathcal{L}_+, \mathcal{L}_-$ be line bundles lying in the chambers adjacent to $\mathcal{L}(0)$ in the G -ample cone. In particular, $\mathcal{L}_+ = \mathcal{O}_X(a, b)$ for some a, b such that $d - 2 > \frac{a}{b} > d - \frac{9}{4}$. Similarly, $\mathcal{L}_- = \mathcal{O}_X(a, b)$ for some a, b such that $d - \frac{9}{4} > \frac{a}{b} > d - 3$. Our first task is to determine X^- and X^0 in this situation.

Proposition 2.4. *With the set-up above, $X^- = H_1 \setminus (H_1 \cap H'_2)$. X^0 is the set of all pointed curves (C, p) with the following description: C is a reducible curve, consisting of a cuspidal cubic plus $d - 3$ copies of the “tangent line” to C at the cusp (this is the unique line in \mathbb{P}^2 whose unique point of intersection with the cubic is the cusp). The point p is the unique smooth flex point of the cuspidal cubic.*

Proof. We have already seen that $X^{ss}(0) = X \setminus (H_1 \cap H'_2)$ and $X^{ss}(+) = X \setminus H_1$. Thus, $X^- = H_1 \setminus (H_1 \cap H'_2)$.

To prove the statement about X^0 , let $(C, p) \in X^0$. Notice that, since $X^0 \subseteq X^-$, p is a smooth point of C and the tangent line to C at p intersects C with multiplicity exactly 3. Since $(C, p) \notin X^{ss}(-)$, there must be a nontrivial 1-parameter subgroup $\lambda : \mathbb{C} \rightarrow PSL(3, \mathbb{C})$ such that $\lambda(t)$ acts on (C, p) with strictly positive weight. Choose a basis that diagonalizes λ :

$$\lambda(t) = \begin{pmatrix} t^{r_0} & 0 & 0 \\ 0 & t^{r_1} & 0 \\ 0 & 0 & t^{r_2} \end{pmatrix}.$$

By change of basis, we may assume that $r_0 \leq r_1 \leq r_2$. Since the image of λ is contained in $PSL(3, \mathbb{C})$, we know that $r_0 + r_1 + r_2 = 0$. Furthermore, since λ is nontrivial, we have that r_0, r_1, r_2 are not all zero, which means that $r_0 < 0$ and $r_2 > 0$. Now, writing C in terms of this basis

$$C = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k,$$

we can write (C, p) in terms of the basis consisting of monomials of the form

$$\prod_{\alpha=1}^a x_{l_\alpha} \prod_{\beta=1}^b a_{i_\beta, j_\beta, k_\beta}.$$

Again, we pick values that minimize everything, and reduce to the problem of figuring out whether

$$ar_l - b(r_0 i + r_1 j + r_2 k)$$

is negative. Notice that this is the same as asking whether

$$\frac{a}{b}r_l < r_0i + r_1j + r_2k.$$

We divide this into cases, depending on p .

Case 1 – $p = (0, 0, 1)$: In this case, $r_l = r_2$, so we are trying to determine when $\frac{a}{b}r_2 < r_0i + r_1j + r_2k$. Recall that $\frac{a}{b} < d - \frac{9}{4}$. Now, if $\frac{a}{b}r_2 \geq r_0 + (d-1)r_2$, then $(d - \frac{9}{4})r_2 > r_0 + (d-1)r_2$, so $-\frac{5}{4}r_2 > r_0$, or $r_1 > \frac{1}{4}r_2$. This means that $\frac{a}{b}r_2 < (d - \frac{9}{4})r_2 < 3r_1 + (d-3)r_2$. It follows that $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ or $a_{0,2,d-2} = a_{0,3,d-3} = 0$. But we know that p is a smooth point of C and the tangent line to C at p intersects C with multiplicity exactly 3, so neither of these is a possibility.

Case 2 – p lies on the line $x_0 = 0$, but not on the line $x_1 = 0$: In this case, $r_l = r_1$, so we are trying to determine when $\frac{a}{b}r_1 < r_0i + r_1j + r_2k$. If $r_1 > 0$, then since $r_1 \leq r_2$, we have $\frac{a}{b}r_1 < (d - \frac{9}{4})r_1 \leq dr_1 \leq r_1j + r_2(d-j)$, so we see that $a_{0,0,d} = a_{0,1,d-1} = \dots = a_{0,d,0} = 0$. This means that p lies on a linear component of C , and therefore $(C, p) \notin X^-$.

On the other hand, if $r_1 \leq 0$, then since $r_2 \geq -2r_1$, we see that $\frac{a}{b}r_1 \leq (d-3)r_1 \leq (d-1)r_1 + r_2 \leq r_1j + (d-j)r_2 + r_2$ for $j \leq d-1$, so $a_{0,0,d} = a_{0,1,d-1} = \dots = a_{0,d-1,1} = 0$. This means that either p lies on a linear component of C or the only point of C lying on the line $x_0 = 0$ also lies on the line $x_1 = 0$. Given our assumptions, neither of these is possible.

Case 3 – p does not lie on the line $x_0 = 0$: In this case, $r_l = r_0$, so we are trying to determine when $\frac{a}{b}r_0 < r_0i + r_1j + r_2k$. Since $r_0 < 0$ and $r_1 < r_0 < r_2$, we see that $\frac{a}{b}r_0 < (d-3)r_0 = (d-2)r_0 + r_1 + r_2 < r_0i + r_1j + r_2k$ for $i \leq d-2, k \neq 0$. Notice that, if $\frac{a}{b}r_0 < (d-1)r_0 + r_2$, then C is of the form

$$C = \sum_{i+j=d} a_{i,j,0} x_0^i x_1^j.$$

In other words, C is d distinct lines. In this case, the tangent line to every point of C is a component of C itself, so $(C, p) \notin X^-$ for any p . We therefore see that $\frac{a}{b}r_0 \geq (d-1)r_0 + r_2$, so $(d - \frac{9}{4})r_0 \geq (d-1)r_0 + r_2$. But then $\frac{5}{4}r_0 \leq -r_2 = r_0 + r_1$, so $r_0 \leq 4r_1$. It follows that $\frac{a}{b}r_0 < (d-3)r_0 \leq (d-4)r_0 + 4r_1 \leq r_0i + r_1j$ for $j \geq 4$.

We see that C is of the form

$$C = x_0^{d-3}(a_{d,0,0}x_0^3 + a_{d-1,1,0}x_0^2x_1 + a_{d-2,2,0}x_0x_1^2 + a_{d-3,3,0}x_1^3 + a_{d-1,0,1}x_0^2x_2).$$

Thus, C consists of a cuspidal cubic together with $d-3$ copies of the line whose unique point of intersection with the cubic is the cusp. The point p is the unique flex point of the cuspidal cubic.

It is clear that this $(C, p) \in X^-$, since the tangent line to C at p intersects C with multiplicity exactly 3. To see that $(C, p) \notin X^{ss}(-)$, consider the 1-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^5 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-4} \end{pmatrix}.$$

Finally, notice that since all cuspidal plane cubics are projectively equivalent, X^0 must be the set of *all* cuspidal cubics together with $d - 3$ copies of the line whose unique point of intersection with the cubic is the cusp, with marked point the unique flex point of the cubic. □

Corollary 2.5. *The map $X^{ss}(-)//G(-) \rightarrow X^{ss}(0)//G(0)$ contracts the locus $H_1 \setminus (H_1 \cap H_2')$ to a point. Outside of this locus, the map is an isomorphism.*

Proof. Let $C = x_2^{d-3}(x_0x_2^2 - x_1^3)$, and $p = (0, 0, 1)$. Then $(C, p) \in X^0$. Since all cuspidal plane cubics are projectively equivalent, $G \cdot (C, p) = X^0$, so $G \cdot (C, p)$ is closed in $X^{ss}(0)$ and $X^0/G(0)$ is a point. Notice that the stabilizer of (C, p) must fix $p = (0, 0, 1)$, the cusp, which is $(1, 0, 0)$, and the intersection of its linear component with the tangent line through p , which is $(0, 1, 0)$. Thus, the stabilizer of (C, p) must consist solely of diagonal matrices. A quick check shows that the stabilizer of (C, p) is the one-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^5 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-4} \end{pmatrix}$$

which is isomorphic to \mathbb{C}^* . The conclusion follows by the theorem of Thaddeus above. □

We are particularly interested in the case where $d = 4$, because in this case $X^{ss}(-)//G(-)$ is a birational model for $\overline{M}_{3,1}$. In particular, we have the following:

Proposition 2.6. *There is a birational contraction $\beta : \overline{M}_{3,1} \dashrightarrow X^{ss}(-)//G(-)$.*

Proof. It suffices to exhibit a morphism $\beta^{-1} : V \rightarrow \overline{M}_{3,1}$, where $V \subseteq X^{ss}(-)//G(-)$ is open with complement of codimension ≥ 2 and β^{-1} is an isomorphism onto its image. To see this, let $U \subseteq X^{ss}(-)$ be the set of all moduli stable pointed curves $(C, p) \in X^{ss}(-)$. Notice that the complement of U is strictly contained in the discriminant locus Δ , which is an irreducible hypersurface in $X^{ss}(-)$. Thus, in the quotient, we have the containment $(X^{ss}(-) \setminus U)//G(-) \subset \Delta//G(-) \subset X^{ss}(-)//G(-)$, and $\Delta//G(-)$

is irreducible. Notice, furthermore, that both Δ and $X^{ss}(-)\setminus U$ are G -invariant, so if $(C, p) \in X^s(-)\setminus\Delta$ (respectively, $(C, p) \in X^s(-) \cap \Delta \cap U$), then the orbit of (C, p) does not intersect Δ (respectively, $X^{ss}(-)\setminus U$). Since this point is stable, this means that the image of (C, p) is not contained in $\Delta//G(-)$ (respectively, $(X^{ss}(-)\setminus U)//G(-)$). Thus, the containments $(X^{ss}(-)\setminus U)//G(-) \subset \Delta//G(-)$ and $\Delta//G(-) \subset X^{ss}(-)//G(-)$ are strict. It follows that the complement of U has codimension ≥ 2 .

By the universal property of the moduli space, since $U \rightarrow Z$ is a family of moduli stable curves, it admits a unique map $U \rightarrow Z \rightarrow \overline{M}_{3,1}$. This map is certainly G -equivariant, so it factors uniquely through a map $U//G(-) \rightarrow \overline{M}_{3,1}$. Since every degree 4 plane curve is canonical, two such curves are isomorphic if and only if they differ by an automorphism of \mathbb{P}^2 . It follows that this map is an isomorphism onto its image. □

Corollary 2.7. *Let $W \subset \overline{M}_{3,1}$ be the Weierstrass divisor and $BN \subset \overline{M}_{3,1}$ the pullback of the (non-pointed) Brill-Noether divisor from \overline{M}_3 . Then the divisors W , BN , and Δ_i for $i \geq 1$ span a simplicial face of $\overline{NE}^1(\overline{M}_{3,1})$.*

Proof. The composition $\overline{M}_{3,1} \dashrightarrow X^{ss}(-)//G(-) \rightarrow X^{ss}(0)//G(0)$ is a birational contraction. By the above, the Weierstrass divisor is contracted by this map, so it suffices to show that BN and the Δ_i 's are contracted as well. The image of BN is the set of double conics, which is just a point. Similarly, the image of Δ_i is contained in the singular locus Δ , which is an irreducible hypersurface. Since the generic point of Δ is an irreducible nodal curve, the image of Δ_i is codimension 2 or greater for all $i \geq 1$. The result follows from the contraction theorem. □

2.2 Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

We now turn to the case of genus 4 curves. Recall that the general genus 4 curve admits a canonical embedding into $\mathbb{P}^1 \times \mathbb{P}^1$ as a curve of bidegree $(3, 3)$. In a similar way to the previous section, we will use this observation to construct a birational model for $\overline{M}_{4,1}$. Specifically, we consider GIT quotients of the universal family over the space of such curves. Here, the Hessian $H_{0,1}$ is again the image of a pointed Brill-Noether divisor. As above, our goal is to find a GIT quotient in which this locus is contracted.

Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{L}' = \mathcal{O}_Y(d, d)$ for $d \geq 3$. Here, $\pi_2 : X \rightarrow Z = \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d, d))$ is the family of all curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d, d) . By the above, we know that $\text{Pic}X \cong \mathbb{Z}^3$. We will write $\mathcal{O}_X(a_1, a_2, b)$ to denote $(i \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_1, a_2)) \otimes \pi_2^*(\mathcal{O}_Z(b))$. Recall that $G = \text{Aut}(\mathbb{P}^1 \times$

$\mathbb{P}^1) = (PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z} \cong PSO(4, \mathbb{C})$. Our goal, as before, is to study the GIT quotients of X by the action of this group.

Proposition 2.8. $\mathcal{O}_X(2(d-1), 2(d-1), 2)$ lies on a boundary of $C^G(X)$.

Proof. Let $\mathcal{L} = \mathcal{O}_X(2(d-1), 2(d-1), 2)$. It suffices to show that $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L}) = \emptyset$. It is clear that $X^{ss}(\mathcal{L}) \neq \emptyset$, since $W_{0,1}$ is a G -invariant section of \mathcal{L} .

To show that $X^s(\mathcal{L}) = \emptyset$, we invoke the numerical criterion. Let $(C, p) \in X$. By change of coordinates, we may assume that $p = (0, 1; 0, 1)$. So, if

$$C = \sum_{i_0+i_1=j_0+j_1=d} a_{i_0, i_0, j_0, j_1} x_0^{i_0} x_1^{i_1} y_0^{j_0} y_1^{j_1},$$

then $a_{0,d,0,d} = 0$. Under the embedding determined by \mathcal{L} , we write (C, p) in terms of the basis of monomials of the form

$$x_1^{2(d-1)} y_1^{2(d-1)} \prod_{\alpha=1}^2 a_{i_{0\alpha}, i_{1\alpha}, j_{0\alpha}, j_{1\alpha}}.$$

Now consider the 1-parameter subgroup

$$\lambda(t) = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \in PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C}).$$

Then $\lambda(t)$ acts on the monomial above with weight $4(d-1) + \sum_{\alpha=1}^2 i_{0\alpha} - i_{1\alpha} + j_{0\alpha} - j_{1\alpha}$. We want to know when this weight is negative. Again, find values that minimize this expression. Then

$$4(d-1) + \sum_{\alpha=1}^2 i_{0\alpha} - i_{1\alpha} + j_{0\alpha} - j_{1\alpha} x_1^{2(d-1)} y_1^{2(d-1)} \geq 4(d-1) + 2((i_0 - i_1) + (j_0 - j_1))$$

which is negative whenever $(i_0 - i_1) + (j_0 - j_1) < -2(d-1) = -i_0 - i_1 - j_0 - j_1 + 2$, or $i_0 + j_0 < 1$. This only occurs when $i_0 = j_0 = 0$, in other words, when $a_{0,d,0,d}$ is nonzero. By assumption, however, this is not the case, so $(C, p) \notin X^s(\mathcal{L})$. Since (C, p) was arbitrary, it follows that $X^s(\mathcal{L}) = \emptyset$. \square

Proposition 2.9. Let $\mathcal{L} = \mathcal{O}_X(2(3d-4), 2(3d-4), 6)$. Then $X^{ss}(\mathcal{L}) = X \setminus (H_{0,1} \cap H_{1,1})$, and $X^s(\mathcal{L}) \subseteq X \setminus H_{0,1}$. In particular, since $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$, \mathcal{L} lies on a wall in the $C^G(X)$.

Proof. First, notice that if $(C, p) \notin H_{1,1}$, then $(C, p) \in X^{ss}(\mathcal{L})$, since $W_{1,1}$ is a G -invariant section of \mathcal{L} that does not vanish at (C, p) . Thus, $X \setminus (H_{0,1} \cap H_{1,1}) \subseteq X^{ss}(\mathcal{L})$.

Now, suppose that $(C, p) \in H_{0,1}$. As before, by change of coordinates, we may assume that $p = (0, 1 : 0, 1)$. Since $(C, p) \in H_{0,1}$, C intersects one of the two lines through p with multiplicity at least 2. Without loss of generality, we may assume that this line to be $x_0 = 0$. Thus, if we write

$$C = \sum_{i_0+i_1=j_0+j_1=d} a_{i_0, i_1, j_0, j_1} x_0^{i_0} x_1^{i_1} y_0^{j_0} y_1^{j_1},$$

then $a_{0,d,0,d} = a_{0,d,1,d-1} = 0$. Under the embedding determined by \mathcal{L} , we write (C, p) in terms of the basis of monomials of the form

$$x_1^{2(3d-4)} y_1^{2(3d-4)} \prod_{\alpha=1}^6 a_{i_{0\alpha}, i_{1\alpha}, j_{0\alpha}, j_{1\alpha}}.$$

Now, consider the 1-parameter subgroup:

$$\lambda(t) = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix} \right) \in PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C}).$$

Then $\lambda(t)$ acts on the monomial above with weight $6(3d-4) + \sum_{\alpha=1}^6 i_{0\alpha} - i_{1\alpha} + 2j_{0\alpha} - 2j_{1\alpha}$. We want to know when this weight is negative. Let i_0, i_1, j_0, j_1 be the values of $i_{0\alpha}, i_{1\alpha}, j_{0\alpha}, j_{1\alpha}$ that minimize $i_{0\alpha} - i_{1\alpha} + 2j_{0\alpha} - 2j_{1\alpha}$. Then

$$6(3d-4) + \sum_{\alpha=1}^6 i_0 - i_1 + 2j_0 - 2j_1$$

$$\geq 6(3d-4) + 6(i_0 - i_1 + 2j_0 - 2j_1) = 12(i_0 + 2j_0 - 2)$$

which is negative when $i_0 + 2j_0 \leq 2$. This happens when both $i_0 \leq 1$ and $j_0 = 0$, but, by assumption, this is not the case. Thus $(C, p) \notin X^s(\mathcal{L})$. We therefore see that $X^s(\mathcal{L}) \subseteq X \setminus H_{0,1}$.

Notice that if $(C, p) \in H_{0,1} \cap H_{1,1}$, this means that the osculating $(1, 1)$ curve to C at p is the pair of lines through that point, and this curve intersects C with multiplicity at least 4. This means that either $a_{0,d,1,d-1} = 0$ or $a_{2,d-2,0,d} = 0$, in addition to the conditions above, which would imply that the expression $i_0 + 2j_0 - 2$ above is zero for at most one term, and strictly positive for all of the others. After a moment's thought, one can see that if the weights of the 1-parameter subgroup are changed slightly, then all of the weights become strictly positive, and so $(C, p) \notin X^{ss}(\mathcal{L})$. It follows that $X^{ss}(\mathcal{L}) = X \setminus (H_{0,1} \cap H_{1,1})$. □

Corollary 2.10. *If $\mathcal{L} = \mathcal{O}_X(a, a, b)$ for any a, b such that $d-1 > \frac{a}{b} > d - \frac{4}{3}$, then $X^{ss}(\mathcal{L}) = X \setminus H_{0,1}$.*

Proof. If $(C, p) \notin H_{0,1}$, then $(C, p) \in X^{ss}(\mathcal{O}_X(2(3d-4), 2(3d-4), 6))$ and $(C, p) \in X^{ss}(\mathcal{O}_X(2(d-1), 2(d-1), 2))$. It follows that $(C, p) \in X^{ss}(\mathcal{L})$, so $X \setminus H_{0,1} \subseteq X^{ss}(\mathcal{L})$. It remains to show that, if $(C, p) \in H_{0,1}$, then $(C, p) \notin X^{ss}(\mathcal{L})$. For this, mirror the proof above with the same 1-parameter subgroup. □

Again, we want to use Thaddeus' theorem to study the GIT quotients of the space X . We will let $\mathcal{L}(0) = \mathcal{O}_X(2(3d-4), 2(3d-4), 6)$. Let \mathcal{L}_+ and \mathcal{L}_- be line bundles lying in the chambers adjacent to $\mathcal{L}(0)$ in the $C^G(X)$. In particular, $\mathcal{L}_+ = \mathcal{O}_X(a, a, b)$ for some a, b such that $d-1 > \frac{a}{b} > d - \frac{4}{3}$. Similarly, $\mathcal{L}_- = \mathcal{O}_X(a, a, b)$ for some a, b such that $d - \frac{4}{3} > \frac{a}{b} > d - \frac{3}{2}$. Our first task is to determine X^- and X^0 in this situation.

Proposition 2.11. *With the set-up above, $X^- = H_{0,1} \setminus (H_{0,1} \cap H_{1,1})$. X^0 is the set of all pointed curves (C, p) admitting the following description: C is a reducible curve consisting of a smooth curve of bidegree $(1, 2)$ (or $(2, 1)$), together with $d-1$ copies of the tangent line to this curve through a point that has a tangent line, and $d-2$ copies of the other line through this same point. The marked point p is the unique other point on the smooth $(1, 2)$ curve that has a tangent line. (Here, the term "line" means a $(0, 1)$ or $(1, 0)$ curve).*

Proof. We have already seen that $X^{ss}(0) = X \setminus (H_{0,1} \cap H_{1,1})$ and $X^{ss}(+) = X \setminus H_{0,1}$. Thus, $X^- = H_{0,1} \setminus (H_{0,1} \cap H_{1,1})$.

To prove the statement about X^0 , let $(C, p) \in X^0$. Notice that, since $X^0 \subseteq X^-$, exactly one of the two lines through p intersects C with multiplicity exactly 2. Since $(C, p) \notin X^{ss}(-)$, there must be a nontrivial 1-parameter subgroup $\lambda : \mathbb{C} \rightarrow (PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$ such that $\lambda(t)$ acts on (C, p) with strictly positive weight. Choose a basis that diagonalizes λ :

$$\lambda(t) = \left(\left(\begin{array}{cc} t^{-r_0} & 0 \\ 0 & t^{r_0} \end{array} \right), \left(\begin{array}{cc} t^{-r_1} & 0 \\ 0 & t^{r_1} \end{array} \right) \right) \in PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C}).$$

By change of basis, we may assume that r_0 and r_1 are nonnegative, and, using the involution, we may assume that $r_0 \leq r_1$. Since λ is nontrivial, we have that r_0 and r_1 are not both zero, so $r_1 > 0$. Now, writing C in terms of this basis

$$C = \sum_{i_0+i_1=j_0+j_1=d} a_{i_0, i_0, j_0, j_1} x_0^{i_0} x_1^{i_1} y_0^{j_0} y_1^{j_1},$$

we can write (C, p) in terms of the basis consisting of monomials of the form

$$\prod_{\alpha=1}^a x_{l_\alpha} y_{m_\alpha} \prod_{\beta=1}^b a_{i_{0\beta}, i_{1\beta}, j_{0\beta}, j_{1\beta}}.$$

Again, we pick values that minimize everything, and reduce to the problem of figuring out whether

$$b(r_0(i_0 - i_1) + r_1(j_0 - j_2)) - a((-1)^l r_0 + (-1)^m r_1)$$

is negative. Notice that this is the same as asking whether

$$\frac{a}{b}((-1)^l r_0 + (-1)^m r_1) > r_0(i_0 - i_1) + r_1(j_0 - j_1).$$

We divide this into cases, depending on p .

Case 1 – $p = (0, 1 : 0, 1)$: In this case, $l = m = 1$, so we are trying to determine when $\frac{a}{b}(-r_0 - r_1) > r_0(i_0 - i_1) + r_1(j_0 - j_1)$. Recall that $\frac{a}{b} < d - \frac{4}{3}$, so $\frac{a}{b}(r_0 + r_1) < (d - \frac{4}{3})(r_0 + r_1) < (d - 1)(r_0 + r_1) \leq (d - 2)r_0 + dr_1$. Now, if $\frac{a}{b}(r_0 + r_1) \geq dr_0 + (d - 2)r_1$, then $(d - \frac{4}{3})(r_0 + r_1) > dr_0 + (d - 2)r_1$, so $r_1 \geq 2r_0$. This means that $\frac{a}{b}(r_0 + r_1) < (d - \frac{4}{3})(r_0 + r_1) < (d - 4)r_0 + dr_1$. It follows that $a_{0,d,0,d} = a_{1,d-1,0,d} = 0$, and either $a_{0,d,1,d-1} = 0$ or $a_{2,d-2,0,d} = 0$. But we know that exactly one of the two lines through p intersects C with multiplicity exactly 2, so neither of these is a possibility.

Case 2 – p lies on the line $y_0 = 0$, but not the line $x_0 = 0$: In this case, $l = 1$ and $m = 0$, so we are trying to determine when $\frac{a}{b}(r_0 - r_1) > r_0(i_0 - i_1) + r_1(j_0 - j_1)$. Here, $\frac{a}{b}(r_0 - r_1) > (d - \frac{4}{3})(r_0 - r_1) \geq d(r_0 - r_1) \geq kr_0 - dr_1$ for all $k \leq d$. It follows that $a_{k,d-k,0,d} = 0$ for all values of k , which means that $y_0 = 0$ is a linear component of C . Thus p lies on a linear component of C , so $(C, p) \notin X^-$, which is impossible.

Case 3 – p lies on the line $x_0 = 0$, but not on the line $y_0 = 0$: In this case, $l = 0$ and $m = 1$, so we are trying to determine when $\frac{a}{b}(-r_0 + r_1) > r_0(i_0 - i_1) + r_1(j_0 - j_1)$. Note that $\frac{a}{b}(-r_0 + r_1) > (d - 2)(-r_0 + r_1) \geq -dr_0 + kr_1$ for all $k \leq d - 2$. We therefore see that $a_{0,d,k,d-k} = 0$ for all $k \leq d - 2$. If $a_{0,d,d,0} \neq 0$, then every point of C that lies on the line $x_0 = 0$ also lies on the line $y_0 = 0$, a contradiction. We therefore see that $a_{0,d,d,0} = 0$ as well, but this means that p lies on a linear component of C , and therefore $(C, p) \notin X^-$.

Case 4 – p does not lie on either of the lines $x_0 = 0$ or $y_0 = 0$: In this case, $l = m = 0$, so we are trying to determine when $\frac{a}{b}(r_0 + r_1) > r_0(i_0 - i_1) + r_1(j_0 - j_1)$. Now note that $\frac{a}{b}(r_0 + r_1) > (d - 2)(r_0 + r_1)$, so $a_{k_0,d-k_0,k_1,d-k_1} = 0$ if k_0 and k_1 are both less than d . Furthermore, since $r_0 < r_1$, $(d - 2)(r_0 + r_1) > dr_0 + (d - 4)r_1$, so $a_{d,0,k,d-k} = 0$ for $k < d - 1$. Now, if $(d - \frac{3}{2})(r_0 + r_1) \leq (d - 6)r_0 + dr_1$, then $3r_0 \leq r_1$, so $\frac{a}{b}(r_0 + r_1) > (d - \frac{3}{2})(r_0 + r_1) \geq dr_0 + (d - 2)r_0$. It follows that either $a_{d,0,d-1,1} = 0$, in which case C is a product of $2d$ lines, which is impossible, or $a_{k,d-k,d,0} = 0$ for all $k < d - 2$. We therefore see that C is of the form

$$C = x_0^{d-2} y_0^{d-1} (a_{d,0,d,0} x_0^2 y_0 + a_{d,0,d-1,1} x_0^2 y_1 + a_{d-1,1,d,0} x_0 x_1 y_0 + a_{d-2,2,d,0} x_1^2 y_0).$$

Thus, C consists of three components. One is a curve of bidegree $(2, 1)$. The other two components consist of multiple lines through one of the points on

this curve that has a tangent line. The point p is forced to be the unique other such point.

It is clear that this $(C, p) \in X^-$, since by definition, one of the lines through p intersects C with multiplicity greater than 1, and it is impossible for it to intersect a smooth curve of bidegree $(2, 1)$ with higher multiplicity than 2, or for the other line through p to intersect the curve with multiplicity at all. To see that $(C, p) \notin X^{ss}(-)$, consider the 1-parameter subgroup

$$\lambda(t) = \left(\left(\begin{array}{cc} t^{-1} & 0 \\ 0 & t \end{array} \right), \left(\begin{array}{cc} t^{-2} & 0 \\ 0 & t^2 \end{array} \right) \right).$$

Finally, notice that all such curves are in the same orbit of the action of G , so X^0 must be the set of *all* such curves. To see this, note that if we fix the two points that have tangent lines to be $(1, 0 : 1, 0)$ and $(0, 1 : 0, 1)$, then the curve is determined uniquely by the third point of intersection of the curve with the diagonal. Since $PSL(2, \mathbb{C})$ acts 3-transitively on points of \mathbb{P}^1 , we obtain the desired result. \square

Corollary 2.12. *The map $X^{ss}(-)//G(-) \rightarrow X^{ss}(0)//G(0)$ contracts the locus $H_{0,1} \setminus (H_{0,1} \cap H_{1,1})$ to a point. Outside of this locus, the map is an isomorphism.*

Proof. Let $C = x_1^{d-2}y_1^{d-1}(x_0^2y_1 + x_1^2y_0)$, and $p = (0, 1 : 0, 1)$. Then $(C, p) \in X^0$. As we have seen, X^0 is the orbit of (C, p) , so $G \cdot (C, p)$ is closed in $X^{ss}(0)$ and $X^0/G(0)$ is a point. Notice that the stabilizer of (C, p) must fix $p = (0, 1 : 0, 1)$, and the other ramification point, which is $(1, 0 : 1, 0)$. Thus, the stabilizer of (C, p) must consist solely of pairs of diagonal matrices. A quick check shows that the stabilizer of (C, p) is the one-parameter subgroup

$$\lambda(t) = \left(\left(\begin{array}{cc} t^{-1} & 0 \\ 0 & t \end{array} \right), \left(\begin{array}{cc} t^{-2} & 0 \\ 0 & t^2 \end{array} \right) \right)$$

which is isomorphic to \mathbb{C}^* . Again, the conclusion follows from the theorem of Thaddeus above. \square

Our main interest is the case where $d = 3$. As above, this is because in this case $X^{ss}(-)//G(-)$ is a birational model for $\overline{M}_{4,1}$. In particular, we have the following:

Proposition 2.13. *There is a birational contraction $\beta : \overline{M}_{4,1} \dashrightarrow X^{ss}(-)//G(-)$.*

Proof. As above, it suffices to exhibit a morphism $\beta^{-1} : V \rightarrow \overline{M}_{4,1}$, where $V \subseteq X^{ss}(-)//G(-)$ is open with complement of codimension ≥ 2 and β^{-1}

is an isomorphism onto its image. Again, we let $U \subseteq X^{ss}(-)$ be the set of all moduli stable pointed curves $(C, p) \in X^{ss}(-)$. The proof in this case is exactly like that in the case of \mathbb{P}^2 , as the discriminant locus $\Delta \subseteq X$ is again an irreducible G -invariant hypersurface.

By the universal property of the moduli space, since $U \rightarrow Z$ is a family of moduli stable curves, it admits a unique map $U \rightarrow Z \rightarrow \overline{M}_{4,1}$. This map is certainly G -equivariant, so it factors uniquely through a map $U/G(-) \rightarrow \overline{M}_{4,1}$. Since every curve of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is canonical, two such curves are isomorphic if and only if they differ by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. It follows that this map is an isomorphism onto its image. \square

Corollary 2.14. *Let $L \subset \overline{M}_{4,1}$ be the pointed Brill-Noether divisor consisting of pointed curves (C, p) such that p is a ramification point of a limit g_3^1 on C . Let $P \subset \overline{M}_{4,1}$ be the pullback of the Petri divisor on \overline{M}_4 . Then the divisors L , P , and Δ_i for $i \geq 1$ span a simplicial face of $\overline{NE}^1(\overline{M}_{4,1})$.*

Proof. The composition $\overline{M}_{4,1} \dashrightarrow X^{ss}(-)//G(-) \rightarrow X^{ss}(0)//G(0)$ is a birational contraction. By the above, the given pointed Brill-Noether divisor is contracted by this map, so it suffices to show that P and the Δ_i 's are contracted as well. The image of P is the set of triple $(1, 1)$ curves, which is just a point. As above, the image of Δ_i is contained in the singular locus Δ , which is an irreducible hypersurface. Since the generic point of Δ is an irreducible nodal curve, the image of Δ_i is codimension 2 or greater for all $i \geq 1$. The result follows from the contraction theorem. \square

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