PROBLEMS IN HYPERBOLIC GEOMETRY

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1. Introduction

The purpose of this document is to list some outstanding questions in the subject of hyperbolic geometry (real, complex, quaternionic, and the Cayley plane), with the primary focus on the latter three. This list was inspired by the problem sessions held at the conference on Complex Hyperbolic Geometry in Luminy (CIRM) in July of 2003.

The questions are roughly grouped by similarity. The name(s) next to the questions are the individual(s) who submitted the question. In the absence of a name, the problem was submitted anonymously.

2. Introduction

2.1. **Notation.** Throughout, we denote X-hyperbolic n-space by $\mathbf{H}^n_X$, where $X = \mathbb{C}$ or $\mathbb{H}$ (and on occasion $\mathbb{R}$). We denote the associated isometry group by $\mathrm{Isom}(\mathbf{H}^n_X)$.

2.2. **Note to reader.** It is my hope that this list will evolve over time. With this in mind, I encourage continued submission of problems, updates on the progress of the problems, inaccuracies of a mathematical and historical natures, etc.

3. The problems

Presently, the problems have been grouped into three (rather artificial) categories: Geometric, Analytic, and Algebraic.

3.1. Geometric.

Problem 1 (John Parker). *A problem about lattices in* $\mathbb{R}^n$.

Suppose that Λ is a Euclidean lattice in $\mathbb{R}^n$. Let $\{v_1, \ldots, v_n\}$ be a Minkowski reduced basis, that is a basis so that $|v_j| \le |v|$ for all $v = x_1v_1 + \cdots x_nv_n$ in Λ with $x_j \neq 0$. Let Λ_j be the sublattice spanned by $\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$. Find a universal lower bound (depending only on n) for

$$\max_j \left(\frac{\mathrm{Vol}(\Lambda)}{|v_j|\,\mathrm{Vol}(\Lambda_j)} \right)$$

where $\mathrm{Vol}(\Lambda)$ is the volume of a fundamental cell of Λ, that is the determinant of the matrix whose columns are the v_j.

One may simplify so that $|v_j| = 1$ for $j = 1, \ldots, n$. The Minkowski reduced condition becomes $|v| \ge 1$ for all $v \in \Lambda$. For $n = 2$ the constant is $\sqrt{3}/2$ and for $n = 3$ the constant is $\sqrt{2/3}$.

The version for $n = 3$ was an ingredient in the estimate for volumes of cusped quaternionic hyperbolic orbifolds given in [Kim-Parker2003].

Problem 2 (Ben McReynolds and Alan Reid). *1-cusped hyperbolic manifolds.*

Prove or disprove the existence of 1-cusped hyperbolic manifolds. This is unknown in $\mathbf{H}^n_{\mathbb{R}}$ for $n \geq 4$, $\mathbf{H}^n_{\mathbb{C}}$ for $n \geq 2$, and $\mathbf{H}^n_{\mathbb{H}}$ for $n \geq 2$. It is not known whether 1-cusped hyperbolic orbifolds exist in every dimension either.

Problem 3 (Yoshi Kamishima and Ben McReynolds). *1-cusped complex hyperbolic 2 orbifolds and 2-manifolds.*

In [McReynolds] it is shown that a every nil 3-manifold arises as a cusp cross-section of a complex hyperbolic 2-orbifold (possibly with many cusps). In particular, every nil 3-manifold is diffeomorphic to a boundary component of a compact 4-orbifold, whose interior admits a complete, complex hyperbolic orbifold structure with finite volume.

Which nil 3-manifolds are the one cusp of a 1-cusped complex hyperbolic 2-orbifold?

Suppose that a Heisenberg 3-manifold N bounds complex hyperbolic 2-orbifold: calculate the Burns-Epstein invariant $\mu(N)$ (i) in terms of the spherical CR-manifold N, also calculate $\mu(N)$ (ii) in terms of complex hyperbolic one. (If it bounds a complex hyperbolic manifold X, then it is known that $\mu(N) = -\chi(X)$.) Can one use this to show the non-existence of 1-cusped complex hyperbolic 2-manifolds.

Problem 4 (Martin Deraux and Pierre Pansu). *Branched coverings of ball quotients.*

It would be interesting to construct holomorphic branched covers $X \to B$ over compact ball quotients, with ramification locus a smooth totally geodesic complex hypersurface $S \subset B$. Some sparse examples have been constructed in dimension two but, as far as I know, this has not been done in dimension three or higher. The Mostow-Siu surfaces [Mostow-Siu1980] or the three-folds described in [Deraux] are not exactly of this nature, since they are described only locally as branched covers of the ball. One can check that any branched cover as above would admit a Kähler metric of negative curvature, and that it cannot be a ball quotient itself (the Chern number calculations given in [Mostow-Siu1980] carry over to higher dimensions). There are compact ball quotients with smooth totally geodesic complex hypersurfaces in any dimension (see for instance one of many constructions by J. Millson). It is then natural to try to use such a hypersurface as the branch locus of a cyclic cover. There is, however, an obvious topological obstruction for doing this : if there exists a branched cover, the integer homology class of S must be divisible (see [Hirzebruch1969], where this condition is also proved to be sufficient). Are there compact ball quotients of arbitrary dimension that contain a smooth totally geodesic hypersurface with a divisible integral homology class?

Problem 5 (Daniel Allcock). *Lattices generated by reflections.*

For which n is there a lattice in $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}})$ or $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}})$ which is generated by complex or quaternionic reflections?

Vinberg [Vinberg1984] showed that beyond a certain n there are no lattices in $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}})$ generated by reflections.

Problem 6 (Richard Schwartz). *Closed hyperbolic 3-manifolds with spherical CR structures.*

Give an example of a closed hyperbolic 3 manifold which does not admit a complete spherical CR structure. Its enough to find one which does not admit a symplectically fillable contact structure. More generally, classify the closed hyperbolic 3-manifolds which admit complete spherical CR structures.

Problem 7 (Richard Schwartz)**.** *Quasi-isometry interchanging a $\mathbb{C}$-slice and an $\mathbb{R}$-slice.*

Is there a quasi-isometry of the complex hyperbolic plane which interchanges a $\mathbb{C}$-slice with an $\mathbb{R}$-slice? I think that this may be a well known question, but I've never heard an answer. One place to look for such a map would be in the "circle of modular groups" produced by [Falbel-Koseleff2002] and [Gusevskii-Parker2002].

Update (H. M. Reimann)**.** Quasi-isometries of complex hyperbolic space extend to quasiconformal mappings on the boundary. (If I understand correctly) $\mathbb{R}$–slices intersect the sphere in horizontal curves and $\mathbb{C}$–slices in in curves transversal to the contact structure. The question of mapping a horizontal curve onto a transversal curve by a quasiconformal mapping has been studied for several years (I do not know who was first in considering it). The general guess seems to be that this is impossible but so far there is no proof for this.

Problem 8 (Richard Schwartz)**.** *Translation length of loxodromic elements in deformations.*

In the deformations of the complex reflection triangle groups, one observes experimentally that the translation lengths of the loxodromic elements is monotone in the parameter. Prove the conjecture that this monotonicity always holds.

Problem 9 (John Parker)**.** *Hausdorff dimension of limit sets of complex hyperbolic quasi-Fuchsian groups.*

In [Falbel-Koseleff2002] and [Gusevskii-Parker2002] a deformation of the modular group in complex hyperbolic space interpolating between $\mathbb{C}$-Fuchsian and $\mathbb{R}$-Fuchsian groups was constructed. We know that the limit set of an $\mathbb{R}$-Fuchsian group is an $\mathbb{R}$-circle which has Hausdorff dimension 1; the limit set of a $\mathbb{C}$-Fuchsian group is a $\mathbb{C}$-circle (chain) which has Hausdorff dimension 2. Find an expression for the Hausdorff dimension of the limit set of the other groups in the deformation.

Problem 10 (Richard Schwartz)**.** *Hausdorff dimension of the limit set of the golden triangle group.*

What is the Hausdorff dimension of the golden triangle group, also known as the last ideal complex hyperbolic triangle group? Dimension bounds would also be interesting.

Problem 11 (Pierre Pansu)**.** *Representation spaces of surface groups in isometry groups of rank one symmetric spaces.*

When do these spaces have the expected dimension, i.e. $-\chi d$ where χ denotes the Euler characteristic of the surface and d the dimension of the Lie group? When this holds, can one describe a piece of the representation space in terms of bending? Of variants of bending (Apanasov)? Does the representation space admit natural complex structures?

Problem 12. *Quasi-conformal deformation theory of discrete complex hyperbolic groups.*

Develop a working theory of quasi-conformal deformations of complex hyperbolic discrete groups.[1]

3.2. **Analytic.**

Problem 13 (Pierre Pansu). *Isoperimetric problems.*

Which compact domains in complex hyperbolic space minimize boundary volume among domains of equal volume? Which compact domains in 3D Heisenberg group minimize boundary volume (3D Hausdorff measure) among domains of equal volume (4D Hausdorff measure with respect to Carnot-Caratheodory metric)?

Problem 14 (Pierre Pansu). *L^p-homology.*

Poincaré duality holds in the L^p setting: it relates $L^{p'}$ cohomology to L^p homology, $p' = p/p - 1$. Vanishing of torsion in L^p homology amounts to a kind of linear filling property: an exact L^p cycle bounds an L^p chain with a linear estimate on the L^p norm. Can one see concretely how this works on examples, using explicit cycles?

Problem 15 (Pierre Pansu). *Complex hyperbolic manifolds of minimal volume.*

The volume of a complex hyperbolic manifold, compact of with finite volume, is proportional to its signature. Nevertheless, it is hard to produce examples with small signature. Mumford did it in dimension 2. What about higher dimensions? The algebro geometric characterization of complex hyperbolic manifolds (Bogomolov, Yau) helps.

Problem 16. *Minimal pinching constant for Riemannian manifolds quasi-isometric to the complex hyperbolic plane.*

Can there be a Riemannian manifold, better than 1/4-pinched, which is quasi-isometric to the complex hyperbolic plane? Pansu's result says this is impossible if this manifold is better than 1/16-pinched.[2]

Problem 17 (Yoshi Kamishima). *Action of the quaternionic isometry group on the boundary sphere.*

The real, complex or quaternionic hyperbolic space has the natural compactification on which the hyperbolic isometric groups extend to a real analytic action. According to the real, complex hyperbolic spaces, each boundary sphere admits a conformally flat structure, or spherical Cauchy-Riemann (CR) structure, on which the isometry group of those extends to the conformal group action and the CR-action respectively. Similarly, the boundary sphere S^{4n+3} of the quaternionic hyperbolic space $\mathbf{H}^{n+1}_{\mathbb{H}}$ in quaternionic dimension $n+1$ supports a geometric structure (we call this *spherical Q C-C geometry* $(\mathrm{PSp}(n+1,1), S^{4n+3})$). Of course, the action of quaternionic isometry group extends to an analytic action on the sphere S^{4n+3} (a restriction of projective transformation of the quaternionic projective space $P\mathbb{H}^{n+1}$).

How does this extended action behave analytically on the boundary sphere S^{4n+3}? In the real, complex case, it is characterized as conformal, CR actions on the sphere S^n, S^{2n+1} respectively.

Update (H. M. Reimann). This question can be posed for arbitrary symmetric spaces of non compact type and their boundaries. An answer is given by Lemma 1 in [CDKR2002].

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[1] The submitter says this is another holy grail in the subject.

[2] The submitter of this problem thinks it is originally due to Gromov.

Problem 18 (H. M. Reimann). *Equivariant differential operators.*

Let $G \cong \mathrm{PSU}(n,1)$ be the group of biholomorphic transformations of the complex hyperbolic space B and its boundary $S = \partial B$ and let MAN be the stabilizer subgroup of a point $P \in S$.

An equivariant (linear) differential operator D is an operator $D : C^\infty(G \times_{MAN} V) \to C^\infty(G \times_{MAN} W)$ between sections of homogeneous vector bundles which commutes with the group action. Assume that the representations of MAN on V and on W are irreducible. Then the first order operator can be classified (see [Korányi-Reiman2000] where this is shown in greater generality).

Question:

1.) Can the second order (linear) equivariant operators be classified?
Can any equivariant second order operator be represented as a sum of products of first order equivariant operators?
2.) Can the classification of equivariant first order differential operators be extended to the case where the representation of MAN on V or on W is not irreducible?

Problem 19 (H. M. Reimann). *Quasiconformal mappings.*

On the boundary $S = \partial B$ of the complex hyperbolic space the quasiconformal mappings are defined with respect to the Carnot–Caratheodory distance, which is adapted to the contact structure (see e.g. [Korányi-Reiman1995]).

1.) Can every quasiconformal mapping $f : \partial B \to \partial B$ be extended to a symplectic mapping of B onto itself (symplectic with respect to the Bergman–Kähler form on B)?
The answer is affirmative for those mappings f which can be obtained from a quasiconformal deformation [Korányi-Reiman1998].
2.) Quasiconformal mappings can be defined between boundaries of smooth strictly pseudoconvex domains. The Levi form gives rise to a Carnot–Caratheodory metric. Denote by
$$E_k = \{z \in \mathbb{C}^n : |z|^2 + \mathrm{Re}\sum_{j=1}^{n} k_j\, z_j{}^2 - 1 < 0\}$$
the "real" ellipsoids in $\mathbb{C}^n$ (with $0 \le k_j < 1$; $j = 1, ...n$).
Problem:
Determine a non–trivial lower bound for the constant K of quasiconformality of any quasiconformal mapping $f : \partial E_0 \to \partial E_k$.
3.) A quasiconformal group is a group of quasiconformal mappings $f : S \to S$ with uniformly bounded distortion K. By a theorem of Sullivan-Tukia, on the sphere S^2 in $\mathbb{R}^3$ the quasiconformal groups are quasiconformally conjugate to groups of Möbius transformations (isometries).
Is the theorem still true for quasiconformal groups on the boundary S of complex hyperbolic space B^2?
For the classical result consult the reference [Martin1988].

3.3. **Algebraic.**

Problem 20 (Alan Reid). *Subgroup separability in lattices in* $\mathrm{Isom}(\mathbf{H}^n_X)$.

For a group G and subgroup H of G, we say that H and $g \in G \setminus H$ can be *separated* if there exists a finite index subgroup $K < G$ such that $H < K$ and

$g \notin K$. We say that H is *separable* if for any $g \in G \setminus H$, we can separate g and H. We say that G is *LERF* or *subgroup separable* if every finitely generated subgroup of G is separable.

Do there exist any lattices in $\mathrm{Isom}(\mathbf{H}^n_X)$ or $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}})$ which are LERF (for $X = \mathbb{R}$, it is known for $n \leq 2$)?

We could ask for a weaker condition, namely that we can separate geometrically finite subgroups. This condition is called *GFERF* (see [Agol-Long-Reid2001] where it was shown that the Bianchi groups are GFERF). This condition is more accessible in some cases.

It is known that if a lattice has the congruence subgroup property, then it cannot be LERF. It is unknown whether or not lattices in $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}})$ have the congruence subgroup property.

Problem 21 (Ben McReynolds)**.** *Engulfing subgroups of hyperbolic lattices.*

We say that a subgroup $H < G$ is *engulfed*[3] if there exists a proper, finite index subgroup $K < G$ with $H < K$. Darren Long [Long1988] has conjectured that for a Kleinian group Γ, every finitely generated subgroup H with $\Lambda H \neq S^2$ is engulfed. More generally, we can ask which subgroups of lattices in $\mathrm{Isom}(\mathbf{H}^n_X)$ are engulfed?

Problem 22. *Arithmeticity of lattices in* $\mathrm{PU}(n,1)$.

The arithmeticity of lattices in $\mathrm{PO}(n,1)$, $\mathrm{PU}(n,1)$, and $\mathrm{PSp}(n,1)$ has been of interest for sometime. In [Gromov-Piatetski-Shapiro1988], nonarithmetic lattices where constructed for every n in $\mathrm{PO}(n,1)$. In [Corlette1992] and [Gromov-Schoen1992], it was shown that every lattice in $\mathrm{PSp}(n,1)$ is arithmetic for $n \geq 2$. Deligne and Mostow [Deligne-Mostow1993] constructed nonarithmetic lattices in $\mathrm{PU}(2,1)$ and $\mathrm{PU}(3,1)$. Do nonarithmetic lattice in $\mathrm{PU}(n,1)$ exist for all $n \geq 4$? If so, construct infinitely many non-commensurable nonarithmetic lattices for each $n \geq 2$.[4]

Problem 23 (Richard Schwartz)**.** *Discreteness for groups with large word length for elliptic elements.*

Suppose that G, a subgroup $\mathrm{PU}(2,1)$, is a finitely generated group, possibly with some efficiency conditions imposed on the generating set. Can one find, in terms of G and the generators, an a priori value of N such that G is discrete provided that words of length less than N are all non-elliptic elements?

Problem 24 (Richard Schwartz)**.** *Space of discrete, type preserving representations for finite index subgroups of the modular group.*

Choose a finite index subgroup of the modular group and give a complete characterization of the space of discrete, type-preserving representations into $\mathrm{PU}(2,1)$. One should start with index 2, obviously.

Problem 25 (Yoshi Kamishima)**.** *CAT(0)-dimension of nilpotent groups.*

As we know for a Heisenberg group Δ of rank 3, there exists an aspherical manifold $\mathbf{H}^2_{\mathbb{C}}/\Delta$, a quotient of complex hyperbolic 2-space by Δ. Thus, it is nonpositive— it is a CAT(0)-space. By definition, the CAT(0)-dimension of Δ is less than or equal to 4 (minimum dimension of non-positively curved complete metric space on which Δ acts properly as isometries). Moreover, if we can check

[3]Engulfing finitely generated subgroups sits between residual finiteness and LERFness.

[4]In [Deligne-Mostow1993], there is only one example, up to commensurability in $\mathrm{PU}(3,1)$. In $\mathrm{PU}(2,1)$, there are only finitely many known commensurability classes of nonarithmetic lattices.

that the CAT(0)-dimension of Δ is not 3, then the CAT(0)-dimension of the Heisenberg nilpotent group Δ would be 4 (if X/Δ is a closed aspherical nonpositive curved manifold, then Δ must be virtually free abelian of rank 3 so it seems the CAT(0)-space for the Heisenberg group does not seem to be of dimension 3).

(a) Is the CAT(0)-dimension of the Heisenberg nilpotent group of rank n is $n+1$?
(b) What can we say about the CAT(0)-dimension for any nilpotent group of rank n. (By definition, CAT(0)-dimension of the Bieberbach group Γ of rank n is n, because $\mathbb{R}^n/\Gamma$ is compact, $\mathbb{R}^n$ is CAT(0) space, $\dim \mathbb{R}^n$ is equal to the cohomological dimension of Γ).

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