Elliptic involutive structures and generalized Higgs algebroids

Eric O. Korman

Department of Mathematics
University of Pennsylvania

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The overall theme of my thesis is on the module theory of certain Lie algebroids. This consists of three main parts:

▶ Higher direct images of Lie algebroid modules along a submersion.
▶ Modules over the distribution determined by a transversely holomorphic foliation.
▶ Modules over generalized Higgs algebroids.
   ▶ Characteristic classes and a Grothendieck-Riemann-Roch theorem for Higgs bundles.

Motivation:

▶ Refined index theorems (e.g. the flat index theorem of Bismut and Lott).
▶ Generalized geometry.
▶ Physics (coisotropic branes and supersymmetric field theory).
Part I: Higher direct images of Lie algebroid modules
   Higher direct image construction
   Leray-Hirsch theorem

Part II: Elliptic involutive structures
   Examples

Part III: Generalized Higgs algebroids
   Higgs bundles proper
- $M$ a smooth manifold.
- $A_M \xrightarrow{\rho} T_CM$ a complex Lie algebroid over $M$.
- $\Omega^\bullet_{A_M}(M; E) = \Gamma(M; \Lambda^\bullet A^*_M \otimes E)$.
- $\nabla^{A_M;E} : \Omega^\bullet_{A_M}(M; E) \to \Omega^{\bullet+1}_{A_M}(M; E)$ an $A_M$-connection on a vector bundle $E \to M$.
- An $A_M$-module is a vector bundle $E \to M$ together with a flat $A_M$-connection.
- $H^\bullet_{A_M}(M; E)$ is the cohomology of the complex $(\Omega^\bullet_{A_M}(M; E), \nabla^{A_M;E})$.
- $\text{Pic}_{A_M}(M)$ is the isomorphism classes of rank one $A_M$-modules.
Set-up:

- Smooth submersion $M \xrightarrow{\pi} B$ compatible with $A_M \to T_C M, A_B \to T_C B$:

$$
\begin{array}{c}
0 \to A_{M/B} \to A_M \xrightarrow{\pi_A} \pi^* A_B \to 0 \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 \to T_C(M/B) \to T_C M \xrightarrow{\pi^*} \pi^* T_C B \to 0.
\end{array}
$$

- $E \to M$ an $A_M$-module.

**Theorem**

There is a canonical $\mathbb{Z}$-graded $A_B$-module $H^\bullet_{A_{M/B}}(M/B; E)$ obtained by taking the vertical Lie algebroid cohomology of $E$. 
The connection is constructed from viewing $\nabla^{A_M;E}$ as an $A_B$-superconnection on the infinite rank bundle $\Gamma(M; \Lambda^\bullet A^*_M / B \otimes E)$ (Bismut construction).

**Theorem (Projection formula)**

If $E$ is an $A_M$-module and $F$ an $A_B$-module then

$$H^\bullet_{A_M/B} (M/B; \pi^* F \otimes E) \simeq F \otimes H^\bullet_{A_M/B} (M/B; E).$$
Theorem (Twisted Leray-Hirsch)

Let $E$ be an $A_M$-module and $F$ an $A_B$-module. If there exist $\alpha_1, \ldots, \alpha_d \in H_{A_M}^\bullet (M; E)$ that give a trivialization of $H_{A_{M/B}}^\bullet (M/B; E)$ then

$$H_{A_M}^\bullet (M; \pi^* F \otimes E) \simeq H_{A_B}^\bullet (B; F) \otimes H_{A_{M/B}}^\bullet (M/B; E).$$
Definition

An elliptic involutive structure (EIS) on $M$ is an elliptic complex Lie algebroid with injective anchor. It is determined by:

- An involutive complex distribution $V \subset T_{\mathbb{C}}M$ with $V + \overline{V} = T_{\mathbb{C}}M$. Or
- A subbundle $V^\perp \subset T^*_{\mathbb{C}}M$ that generates a differential ideal and has $V^\perp \cap T^*M = 0$.

Extreme cases:

- $V = T_{\mathbb{C}}M$. Modules are flat vector bundles.
- $V = T^{0,1}M$ for some complex structure on $M$. Modules are holomorphic vector bundles.
Generalizations of classical results\(^1\)

**Theorem (Newlander-Nirenberg)**

*If \( V \) is an EIS then locally there exist on \( M \) real coordinates \((t^1, \ldots, t^d)\) and complex coordinates \((z^1, \ldots, z^n)\) such that*

\[
V = \text{span} \left\{ \frac{\partial}{\partial t^i}, \frac{\partial}{\partial z^j} \right\}.
\]

**Theorem (Poincaré lemma)**

*If \( U \subset \mathbb{R}^d \) is open and convex and \( W \subset \mathbb{C}^n \) open and pseudo-convex, then*

\[
H^k_{T_{\mathbb{C}} U \oplus T^{0,1} W(U \times W)} = \begin{cases} 
\mathbb{C}; & k = 0 \\
0; & k \geq 1.
\end{cases}
\]

\(^1\)see Trèves or Berhanu-Cordaro-Hounie
Corollary

\[ H^\bullet_{V}(M) \cong H^\bullet(\mathcal{O}_V), \]

where \( \mathcal{O}_V \) is the structure sheaf of \( V \), i.e. the sheaf of germs of \( C^\infty \) functions annihilated by all vector fields in \( V \).
Theorem

For $V$ an EIS over $M$, there is an equivalence

$$(V\text{-modules}) \leftrightarrow \text{(locally free sheaves of } \mathcal{O}_V\text{-modules)}$$

$$(E, \nabla^V_E) \mapsto \mathcal{O}_V(E)$$

$$C^\infty_M \otimes_{\mathcal{O}_V} \mathcal{E} \leftarrow \mathcal{E}.$$

Corollary

For a $V$-module $E$, we have

$$H^\bullet_V(M; E) \simeq H^\bullet(\mathcal{O}(E)).$$
$S^{2n+1}$ inherits an EIS $V$ from the fibration $S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ by

\[ V^\perp = \pi^*(T^{1,0}\mathbb{C}P^n)^*. \]

▶ Leray-Hirsch applies and

\[ H^\bullet_V(S^{2n+1}; \wedge^\bullet V^\perp) \cong H^\bullet(S^1; \mathbb{C}) \otimes H^{\bullet,\bullet}(\mathbb{C}P^n). \]

▶ Pic$_V(S^{2n+1}) \cong \mathbb{C}$ with $n \in \mathbb{Z} \subset \mathbb{C}$ corresponding to $\pi^*O(n)$.

\[ 0 \to \text{Pic}(\mathbb{C}P^n) \to \text{Pic}_V(S^{2n+1}) \to \text{Vect}_{\text{flat}}^1(S^1) \to 0. \]

▶ Even though the complex structure on $\mathbb{C}P^n$ is stable, the EIS on $S^{2n+1}$ is not stable. Space of infinitesimal deformations is

\[ H^1_V(S^{2n+1}; T_{\mathbb{C}}S^{2n+1}/V) \cong H^{0,0}(\mathbb{C}P^n; T^{1,0}\mathbb{C}P^n) \cong \mathfrak{sl}(n+1, \mathbb{C}). \]
A compact simply connected semi-simple Lie group inherits an EIS $V$ from the fibration $G \xrightarrow{\pi} G/T$. $V$ is the distribution determined by a Borel subalgebra.

- $H^\bullet_V(G; \Lambda^\bullet V^\perp) \simeq H^\bullet(T; \mathbb{C}) \otimes H^{0,\bullet}(G/T)$.
- $\text{Pic}_V(G) \simeq t^*_\mathbb{C}$, with integral weights corresponding to holomorphic line bundles on $G/T$.
- Again, the complex structure on $G/T$ is stable, but the EIS has infinitesimal deformations given by

$$H^1_V(G; T_\mathbb{C}G/V) \simeq t^*_\mathbb{C} \otimes H^{0,0}(G/T; T^{1,0}G/T) \neq 0.$$  

This EIS descends to an EIS on certain homogeneous spaces, e.g. $SU(n+1)/SU(n) (= S^{2n+1})$, $Spin(2n)/SU(n)$. 

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Elliptic involutive structures and generalized Higgs algebroids
$E \xrightarrow{\pi} M$ flat complex vector bundle. The flat connection gives an integrable distribution $H \subset TE$. Have an EIS

$$V = T^{0,1} \ker \pi_* \oplus H_\mathbb{C}.$$ 

$V$ induces an EIS on $\mathbb{P}(E)$. 
Twisted generalized Higgs algebroids

Definition

A twisted generalized Higgs algebroid (TGHA) is an elliptic Lie algebroid $A \xrightarrow{\rho} T\mathbb{C}M$ such that $K := \ker \rho$ is abelian.

Remarks:

- $A$ determines an EIS $V := \rho(A)$.
- $K$ is naturally a $V$-module and the obstruction to splitting

$$0 \to K \to A \to V \to 0$$

is a class in $H^2_V(M; K)$. We call $A$ an (untwisted) generalized Higgs algebroid (GHA) if this class vanishes.
Let $A$ be a GHA with a splitting $A \cong K \rtimes V$. An $A$-module is equivalent to the data of

1. A $V$-module $E \to M$.
2. A Higgs field $\theta \in H^0_V(M; \text{End } E \otimes K^*)$ with $\theta \wedge \theta = 0$.

If $A$ is a TGHA, then by our results on EIS’s, $A$ is locally untwisted. Can work on the $V$-gerbe on $K^*$ determined by the twisting class in $H^2_V(M; K)$. 
Examples:

- The Lie algebroid determined by a generalized complex structure is a TGHA.
- Higgs bundles over a complex manifold $X$ are modules over the GHA $T^{1,0}X \ltimes T^{0,1}X$. More generally, $K$-valued Higgs bundles are modules over $K \ltimes X$, with $K$ a holomorphic vector bundle.
Now specialize to a Higgs bundle \((E, \theta)\) over a complex manifold \(X\). We have the class \(at(E) \in H^{1,1}(X; \text{End} E)\) and \(\theta \in H^{1,0}(X; \text{End} E)\).

**Definition**

For \(j \geq 0\), let

\[
a_{2j+1}(E, \theta) = \text{tr}(at(E)^j \theta) \in H^{j+1,j}(X).
\]
If $X$ is Kähler, the non-abelian Hodge theorem says that certain Higgs bundles correspond to certain flat vector bundles. Flat vector bundles have characteristic classes in $c_{2j+1} \in H^{2j+1}(X; \mathbb{R})$, which vanish in degrees 3 and higher for Kähler manifolds (Reznikov’s theorem).

**Theorem**

*The classes $a_{2j+1}$ and $c_{2j+1}$ are compatible with the non-abelian Hodge theorem and the Hodge decomposition: if $(E, \theta)$ has corresponding flat connection $\nabla$ then*

- $\text{Re} \ a_1(E, \theta) = c_1(E, \nabla) \in H^1(X; \mathbb{R})$.
- $a_{2j+1}(E, \theta) = 0$ for $j \geq 1$. 
Using these classes, we have a secondary index theorem for our direct image construction along a projection:

**Theorem**

*Suppose $X$ is a complex manifold, $Y$ is Kähler and $(E, \theta)$ is a Higgs bundle over $X \times Y$. Then*

$$a_{2j+1}(\text{ind} (\bar{\partial}_X; E + \theta)) = \int_Y e(TY) a_{2j+1}(E, \theta).$$
Thank you for your attention!