A hyperholomorphic line bundle on certain hyperkähler manifolds not admitting an $S^1$-action

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Background

Hitchin [4] and Haydys [3] have shown that if a hyperkähler manifold $M$ has an isometric $S^1$-action satisfying

$$L_{\omega_j} \omega_i = 0, \quad L_{\omega_j} \omega_i = -\omega_k, \quad L_{\omega_j} \omega_i = \omega_j$$

for $\omega_j \neq 0$, $d\omega_j = 0$, $d(J\omega_j) = \omega_j + (J\omega_j) g$, $d(K\omega_j) = \omega_j + (K\omega_j) g$, $\Omega$ is of type $(1,1)$ in each complex structure.

Theorem 1 [6]

Suppose $M$ has a 1-form $\alpha$ satisfying

$$d\alpha = 0, \quad d(J\alpha) = \omega_j + (J\alpha) g, \quad d(K\alpha) = \omega_j + (K\alpha) g,$$

for $\alpha_j$, $\alpha_k$ of type $(1,1)$ in each complex structure. Then $\omega_j = d(\alpha_j)$ is of type $(1,1)$ in each complex structure.

We use the differential form description since in the infinite dimensional setting $X$ may not exist.

The proof uses the vanishing of the Nijenhuis tensor for each complex structure, and is very different from the proof in [4] in the case that $F_1 = F_2 = F_3$ and $X$ is Killing (i.e. $\nabla (\alpha_j)$ is slow).

Holomorphic line bundle on twistor space

The twistor space $Z \to CP^3$ has a meromorphic vertical symplectic (2,0) form

$$\omega = \frac{1}{2} (\omega_j + i\omega_k) + 2\omega_j + \frac{1}{2} (\omega_j - i\omega_k),$$

and vector field $Y = X + i\omega_j$.

The holomorphic Lie algebraic on $Z$ is determined by $\hat{\mathfrak{c}}$ Cech description.

Following Hitchin, find singular $(1,0)$-forms $\phi_j \in \mathfrak{so}^{\mathbb{C}}(U_j)$ of $\{U_j\}$ a cover of $Z$ so that

- $\phi_j = d\alpha_j$, $\phi_j$ non-singular so give a class in $H^2(\mathfrak{so})$, the group that classifies holomorphic Lie algebraic extensions of $\mathfrak{so}$ by $\mathfrak{g}$.
- Chern class of the Lie algebraic is $2\omega_j - 2d(\alpha_j) \in H^1(\mathfrak{so})$.

Hyperfíkher reduction

Examples of hyperkähler manifolds with 1-forms satisfying 2 come from hyperkähler reduction. Suppose $M$ is a hyperkähler manifold with a hamiltonian $G$-action with moment map $\mu_G: M \to \mathfrak{g}^* \otimes \mathbb{R}$.

- $\alpha$ is a $G$-basic 1-form satisfying (1).
- The automatically locally constant functions $J_\alpha(Y'''), K_\alpha(Y')$ on $\mu_G^{-1}(0)$ are constant, where $Y'''$ denotes the action vector field of $Y$ on $\mathfrak{g}$.
- $K_{\alpha}(g) \in \mathfrak{g}$ is the corresponding linear functions on $\mathfrak{g}$.
- $\Omega \in \mathfrak{so}(\mu_G^{-1}(0) \otimes \mathfrak{g})$ is the curvature of the canonical connection on the principal $G$-bundle $\mu_G^{-1}(0) \to M/G$.

Theorem 2: Reduction of eq. (1)

If $\hat{\alpha}$ denotes the induced 1-form on $M/G = \mu_G^{-1}(0)/G$ we have

$$d\hat{\alpha} = 0, \quad d(J\hat{\alpha}) = \omega_j + (J\hat{\alpha}) g, \quad d(K\hat{\alpha}) = \omega_j + (K\hat{\alpha}) g.$$

Fix

- A compact Lie group $G$ with Lie algebra $\mathfrak{g}$ and $A$-invariant inner product.
- $\tau_j$, $\tau_k$, $\tau_i \in \mathfrak{g}$ such that the intersection of the centralizers is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.
- $A_{\tau_i,\tau_j,\tau_k} = \{ T_i + iT_j + IT_k, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10} \} \to \mathfrak{g}$.

Example 1: Moduli space of parabolic Higgs bundles [5, 8]

Fix

- Closed Riemann surface $\Sigma$ and a divisor $P = p_1 + \cdots + p_s$.
- Rank $r$ vector bundle $E \to \Sigma$ with trivial determinant and hermitian metric singular at each puncture $p_j$.
- Flag at each puncture $p_j$ with parabolic weights.
- Prescribed eigenvalues $\lambda = (\lambda^0, k = 1, \ldots, n, j = 1, \ldots, n) \subset \mathbb{C}$ for residues of Higgs fields at the punctures.

The moduli space of parabolic Higgs fields is obtained by hyperkähler reduction of the infinite dimensional affine space

$$C = \text{simple aff}(\mu(E))-\text{connections} \times \mathfrak{so}(\mathfrak{sl}(\mathfrak{E}(E)))/\mathfrak{sl}(\mathfrak{E}(E))$$

by the action of the gauge group $G = \mathfrak{so}(\mathfrak{sl}(\mathfrak{E}(E)))$.

Have a 1-form $\alpha$ on $C$ defined by

$$\alpha = (\omega_j + K_{\alpha}(g) \xi_j) \times \mathfrak{so}(\mathfrak{sl}(\mathfrak{E}(E)))/\mathfrak{sl}(\mathfrak{E}(E)) \simeq T_{A_{\alpha}} C.$$

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