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M408D First Midterm Exam, summer 2007

1. Evaluate the following limits and improper integrals, or write "DNE" if the limit does not exist. Show the STEPS.

a) $\lim_{x \rightarrow 2} \frac{\ln(x^2) - 2\ln(2)}{x^2 - 4}$

L'Hopital $\lim_{x \rightarrow 2} \frac{\frac{1}{x^2} \cdot 2x}{2x} = \lim_{x \rightarrow 2} \frac{1}{x^2} = \boxed{\frac{1}{4}}$

b) $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x$

trick $\lim_{x \rightarrow \infty} e^{\ln\left(1 - \frac{2}{x}\right)^x} = \lim_{x \rightarrow \infty} e^{x \ln\left(1 - \frac{2}{x}\right)} = \lim_{x \rightarrow \infty} e^{\frac{\ln\left(1 - \frac{2}{x}\right)}{\frac{1}{x}}}$

Lopital's: $\lim_{x \rightarrow \infty} \frac{\ln\left(1 - \frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{+1}{1 - \frac{2}{x}} \left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{2}{x}} = 1$

So $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x = e^{\lim_{x \rightarrow \infty} \frac{\ln\left(1 - \frac{2}{x}\right)}{\frac{1}{x}}} = e^1 = \boxed{e}$

c) $\lim_{x \rightarrow 5} \frac{x^2 - 6x + 10}{x^2 - 25}$

$x^2 - 6x + 10 \Big|_5 = 25 - 30 + 10 = 5$

$x^2 - 25 \Big|_5 = 0$

$\lim_{x \rightarrow 5} \frac{x^2 - 6x + 10}{x^2 - 25} = \infty$ so $\boxed{\text{DNE}}$

d) $\int_0^{\infty} 3xe^{-x^2} dx$

$= \lim_{b \rightarrow \infty} \int_0^b 3xe^{-x^2} dx = \lim_{b \rightarrow \infty} \frac{3}{2} \int_0^b e^{-u} du$

$u = x^2$

$du = 2x dx$

$= \lim_{b \rightarrow \infty} \frac{3}{2} (-e^{-u} \Big|_0^b)$

$= \lim_{b \rightarrow \infty} \frac{3}{2} (1 - e^{-b})$

$= \boxed{\frac{3}{2}}$

factorial > exponential > n > log

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2. A sequence $\{a_n\}$ is said to "grow faster" than $\{b_n\}$ if $\lim_{n \rightarrow \infty} b_n/a_n = 0$. Put the following sequences in order of growth rate, from fastest to slowest. Show SIMPLIFICATIONS and the PRINCIPLES you use for each comparison (e.g. "a grow faster than b by L'Hospital's Rule").

$a_n = \log_{10}(1000n) = \log_{10}(1000) + \log_{10}(n) = 3 + \log_{10}(n)$

$b_n = \frac{1}{256} 2^n$

$c_n = \sqrt{n+999}$

$d_n = \sqrt{n!}$

$e_n = e^{3 \ln n} = e^{\ln(n^3)} = n^3$

check what's faster
 a_n or d_n .

$\lim_{n \rightarrow \infty} \frac{e^n}{a_n}$

$\lim_{n \rightarrow \infty} \frac{\sqrt{n+999}}{3 + \log_{10}(n)}$

$\lim_{n \rightarrow \infty} \frac{1}{2} (n+999)^{-1/2}$

is faster $\log_{10}(n)$

$d_n > b_n > e_n > c_n > a_n$

3. Which of the following series converge, and which diverge? In each case, give a one sentence explanation (e.g. "diverges by comparison to $\sum 1/k$ ").

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a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

integral test

$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \int_{\ln(2)}^{\infty} u du = \lim_{b \rightarrow \infty} \left[\frac{1}{2} u^2 \right]_{\ln(2)}^b = \lim_{b \rightarrow \infty} \frac{1}{2} (b^2 - (\ln(2))^2) = \text{DNE}$

Diverges

By integral test

$= \lim_{b \rightarrow \infty} \frac{1}{2} (b^2) \Big|_{\ln(2)}^b = \lim_{b \rightarrow \infty} \frac{1}{2} (b^2 - (\ln(2))^2) = \text{DNE}$

b) $\sum_{n=1}^{\infty} \frac{\sin(n\pi + \frac{\pi}{2})}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

$b_n = \frac{1}{n+1}$ b_n decreasing

$\lim_{n \rightarrow \infty} b_n \rightarrow 0$ as $n \rightarrow \infty$

so series converges

c) $\sum_{n=1}^{\infty} \frac{n^6}{6^n}$

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{(n+1)^6}{6^{n+1}} \cdot \frac{6^n}{n^6} \right| = \lim_{n \rightarrow \infty} \frac{1}{6} \left(\frac{n+1}{n} \right)^6 = \frac{1}{6}$

Because $\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}}{a_n} \right| < 1$ Series converges by ratio test

d) $\sum_{n=1}^{\infty} n! 10^{-n}$

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \cdot \frac{10^{-(n+1)}}{10^{-n}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)}{10} = \infty$ Diverges.

Series Diverges because $\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}}{a_n} \right| \rightarrow \infty$

9/3/12 @ 4
 for all 30
 out of 20 pts.
 (20)

5 4. a) Write down the Taylor series for $\sin x$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

5 b) Write down the 6-th order Taylor polynomial for $f(x) = \sin(-2x^2)$.

$$f(x) = \sin(-2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{2n+1} x^{4n+2}}{(2n+1)!}$$

So 6th order Taylor poly is:

$$-2x^2 + \frac{8}{3!}x^6 = -2x^2 + \frac{4}{3}x^6$$

5 c) Evaluate $f^{(5)}(0)$ and $f^{(6)}(0)$;

$$f^{(5)}(0) = 0 \quad (\text{from Taylor expansion } -2x^2 + \frac{4}{3!}x^6)$$

$$f^{(6)}(0) = \frac{4}{3} \cdot 6! = 6 \cdot 5 \cdot 4 \cdot 2 \cdot 4 = [240] \cdot 4 = [960]$$

5 d) Estimate $\int_0^{0.1} f(x) dx$ to five decimal places.

$$\int_0^{0.1} f(x) dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{2n+1} x^{4n+2}}{(2n+1)!} dx$$

$$= \sum_{n=0}^{\infty} \int_0^{0.1} \frac{(-1)^n (-2)^{2n+1} x^{4n+2}}{(2n+1)!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{2n+1} x^{4n+3}}{(4n+3)(2n+1)!} \Big|_0^{0.1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{2n+1} \left(\frac{1}{10}\right)^{4n+3}}{(4n+3)(2n+1)!} = \sum_{n=0}^{\infty} a_n$$

$$\text{error } |R_0| \leq |a_1| = 2^3 \left(\frac{1}{10}\right)^7$$

$$= \frac{8^3 \left(\frac{1}{10}\right)^7}{7 \cdot 6!}$$

$$< \frac{1}{10000} \cdot 3$$

So the first term is enough to approximate to w/ m 5 decimal places:

$$\int_0^{0.1} f(x) dx \approx \frac{-2 \left(\frac{1}{10}\right)^3}{3} = \frac{-2}{3000} = \boxed{\frac{-1}{1500}}$$

10-10-10

$$\int_0^{0.1} 2x dx = -2 \left(\frac{1}{10}\right)^3 + 8 \left(\frac{1}{10}\right)^7 = -0.000666 \dots$$

$$\approx \boxed{-0.00067}$$

20 5. a) Find the second order Taylor polynomial for $g(x) = \sqrt{x+3}$ around $x=1$.

$$g(x)|_{x=1} = \sqrt{4} = 2$$

$$g'(x) = \frac{1}{2}(x+3)^{-1/2} \quad g'(1) = \frac{1}{2} \left(\frac{1}{2}\right) = \frac{1}{4}$$

$$g''(x) = -\frac{1}{4}(x+3)^{-3/2}, \quad g''(1) = -\frac{1}{4}(4)^{-3/2} = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}$$

2nd order Taylor polynomial $2 + \frac{1}{4}(x-1) - \frac{1}{32} \left(\frac{x-1}{2}\right)^2$

$$= 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$$

b) Find the radius of convergence of the power series $\sum_{j=0}^{\infty} \frac{j^2+2}{3^j} x^j$.

RADIIs of convergence

Root test

$$\begin{aligned} & \lim_{j \rightarrow \infty} \left| \frac{j^2+2}{3^j} x^j \right|^{1/j} \\ &= \lim_{j \rightarrow \infty} \left| \frac{x}{3} \right| (j^2+2)^{1/j} \\ &= \left| \frac{x}{3} \right| \lim_{j \rightarrow \infty} e^{\frac{\ln(j^2+2)}{j}} \quad \text{l'Hospital} \\ &= \left| \frac{x}{3} \right| \lim_{j \rightarrow \infty} e^{\frac{2j}{j^2+2}} \\ &= \left| \frac{x}{3} \right| \lim_{j \rightarrow \infty} e^{\frac{2}{j}} \quad 4 \\ &= \left| \frac{x}{3} \right| e^0 \\ &= \left| \frac{x}{3} \right| \end{aligned}$$

converges for $\left| \frac{x}{3} \right| < 1$ or $|x| < 3$

Radius of convergence is 3

Ratio test

$$\begin{aligned} & \lim_{j \rightarrow \infty} \frac{(j+1)^2+2}{3^{j+1}} x^{j+1} \cdot \frac{3^j}{(j^2+2)x^j} \\ &= \lim_{j \rightarrow \infty} \left| \frac{x}{3} \right| \left(\frac{(j+1)^2+2}{j^2+2} \right) \quad \text{l'Hospital} \\ &= \lim_{j \rightarrow \infty} \left| \frac{x}{3} \right| \frac{2(j+1)}{2j} \quad \text{l'Hospital} \\ &= \left| \frac{x}{3} \right| \lim_{j \rightarrow \infty} 1 \\ &= \left| \frac{x}{3} \right| \quad \text{converges } \left| \frac{x}{3} \right| < 1 \text{ or } |x| < 3 \end{aligned}$$