M408D Second Midterm Exam, July 18, 2007

1. Planes:

a) Find the equation of the plane that goes through the point (2, 3, 7) and has normal vector ⟨−2, 1, −5⟩.

$0 = \langle -2, 1, -5\rangle \cdot (\langle x, y, z\rangle - \langle 2, 3, 7\rangle)$

$0 = -2x + y - 5z - (-4 + 3 - 35)$

$0 = -2x + y - 5z + 36$

$36 = 2x - y + 5z$

b) Find the distance from the point (3, 6, 5) to the plane in part (a).

$V = \langle 3, 6, 5\rangle - \langle 2, 3, 7\rangle = \langle 1, 3, -2\rangle$

projection onto normal vector, $\frac{V \cdot n}{\|n\|} = \frac{\langle 1, 3, -2\rangle \cdot \langle -2, 1, -5\rangle}{\sqrt{4 + 1 + 25}}$

$= \frac{-2 + 3 + 10}{\sqrt{30}} = \frac{9}{\sqrt{30}}$

c) Find the equation of the plane that goes through the three points (2, 0, 3), (11, 1, 0) and (−1, −4, −3).

$u \times v$

$u = \langle 2, 0, 3\rangle - \langle 11, 1, 0\rangle = \langle -9, -1, 3\rangle$

$v = \langle 2, 0, 3\rangle - \langle -1, -4, -3\rangle = \langle 3, 4, 6\rangle$

normal

$n = \begin{vmatrix} i & j & k \\ -9 & -1 & 3 \\ 3 & 4 & 6 \end{vmatrix} = i(-6 - 12) - j(-54 - 9) + k(-36 + 3)$

$= \langle -18, 63, -33\rangle$

$n \cdot \langle x - 2, y, z - 3\rangle = 0$

$\langle -18, 63, -33\rangle \cdot \langle x - 2, y, z - 3\rangle = 0$

$0 = -18x + 63y - 33z + (36 + 99)$

$18x - 63y + 33z = 135$

c) Find the angle between two planes: $x + 2y + 5z = 0$, $8x + 6y - 2z = 4$.

$n_1 = \langle 1, 2, 5\rangle$

$n_2 = \langle 8, 6, 2\rangle$

Angle btwn planes

$\cos\theta = \frac{n_1 \cdot n_2}{\|n_1\|\|n_2\|} = \frac{8 + 12 + 10}{\sqrt{1 + 4 + 25}\sqrt{64 + 36 + 4}}$

$\cos\theta = \frac{30}{\sqrt{30}\sqrt{104}} = \frac{30}{6\sqrt{35}} = \frac{5}{\sqrt{35}}$

$\theta = \arccos\left(\frac{5}{\sqrt{35}}\right)$

3. Parameterized curves: Consider the parameterized curve

$$\mathbf{r}(t) = \langle \cos 2\pi t, \sin 2\pi t, 3t - 1 \rangle.$$

a) Compute the position, velocity, unit tangent vector and speed at time $t = 1$.

Position $r(1) = \langle 1, 0, 2 \rangle$

velocity $r'(t) = \langle -2\pi \sin 2\pi t, 2\pi \cos(2\pi t), 3 \rangle$

$r'(1) = \langle 0, 2\pi, 3 \rangle$

Speed $\|r'(1)\| = \sqrt{4\pi^2 + 9}$

unit tan $T(1) = \frac{r'(1)}{\|r'(1)\|} = \frac{\langle 0, 2\pi, 3 \rangle}{\sqrt{4\pi^2+9}}$

b) Compute the arc-length of the curve from $t = 0$ to $t = 3$.

$L = \int_0^3 \|r'(t)\|\, dt = \int_0^3 \sqrt{4\pi^2 \sin^2(2\pi t) + 4\pi^2 \cos^2(2\pi t) + 9}\ dt$

$= \int_0^3 \sqrt{4\pi^2 + 9}\ dt$

$= \sqrt{4\pi^2+9}\ t \,\Big|_0^3$

$= 3\sqrt{4\pi^2+9}$

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5. The parametric curves $r_1(t) = \langle t^2, 2t, \ln(t^2-3)\rangle$ and $r_2(s) = \langle s+1, s^2-5, s^2-2s-3\rangle$ intersect at the point $(4, 4, 0)$. Find the angle between the two curves at that point.

$r_1(t) = (4,4,0)$ for $t = 2$

$r_2(s) = (4,4,0)$ for $s = 3$

Angle btwn is angle btwn tangent vector $r_1'(t=2)$ & $r_2'(s=3)$

$v = r_1'(t) = \left(2t, 2, \frac{2t}{t^2-3}\right)$, $r_1'(t=2) = \left(4, 2, \frac{4}{1}\right)$

$u = r_2'(s) = (1, 2s, 2s-2)$ $\quad r_2'(s=3) = (1, 6, 4)$

Angle btwn $\quad \frac{u \cdot v}{\|u\|\|v\|} = \cos\theta$

$$\cos\theta = \frac{\langle 4,2,4\rangle \langle 1,6,4\rangle}{\sqrt{16+4+16}\,\sqrt{1+36+16}} = \frac{4+12+16}{\sqrt{36}\,\sqrt{53}} = \frac{32}{6\sqrt{53}} = \frac{16}{3\sqrt{53}}$$

$$\theta = \arccos\left(\frac{16}{3\sqrt{53}}\right).$$

2. Lines

a) Find the equation, in symmetric form, for the line through the point $(2,3,-1)$ in the direction $\langle -4,-6,-2\rangle$.

$\langle x,y,z\rangle = \langle 2,3,-1\rangle + t\langle -4,-6,-2\rangle$

$x = 2-4t$
$y = 3-6t$
$z = -1-2t$

$\frac{x-2}{-4} = \frac{y-3}{-6} = \frac{z+1}{-2}$

b) How far is the point $(3,6,5)$ from this line?

$v = \langle -4,-6,-2\rangle$, $P = \langle 3,6,5\rangle - \langle 2,3,-1\rangle = \langle 1,3,6\rangle$

$d = |P|\sin\theta = \frac{|v\times P|}{|v|} = \frac{1}{\sqrt{56}}\left|\begin{vmatrix} i & j & k \\ -4 & -6 & -2 \\ 1 & 3 & 6 \end{vmatrix}\right|$

$= \frac{1}{\sqrt{56}}\left| i(-6) \dots (-2) + k(\dots)\right|$

$= \frac{\sqrt{36+\dots}}{\sqrt{56}}$

c) Find the equation, in symmetric form, for the line through the two points $(2,-5,-1)$ and $(3,-1,-2)$.

$v = Q - P = \langle 3,-1,-2\rangle - \langle 2,-5,-1\rangle = \langle 1,4,-1\rangle$

$P = \langle 2,-5,-1\rangle$

$\langle x,y,z\rangle = \langle 2,-5,-1\rangle + t\langle 1,4,-1\rangle$

$x = 2+t$ $\quad z = -1-t$
$y = -5+4t$

$x-2 = \frac{y+5}{4} = \frac{z+1}{-1}$

d) Find the equation (in symmetric form) of the line through $(3,6,5)$ that is perpendicular to the lines of parts (a) and (c).

Direction $v_a \times v_c = \langle -4,-6,-2\rangle \times \langle 1,4,-1\rangle$

$= \begin{vmatrix} i & j & k \\ -4 & -6 & -2 \\ 1 & 4 & -1 \end{vmatrix} = i(14) - j(6) + k(-10)$

$= \langle 14,-6,-10\rangle$

$\langle x,y,z\rangle = \langle 3,6,5\rangle + t\langle 14,-6,-10\rangle$

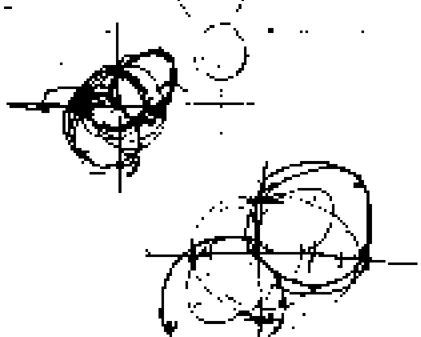
$x = 3+14t$
$y = 6-6t$
$z = 5-10t$

$\frac{x-3}{14} = \frac{y-6}{-6} = \frac{z-5}{-10}$

4. Polar coordinates.

a) Sketch the curve $r = 1 + \sin(2\theta)$. In $[0, 2\pi)$, what values of θ make $r = 0$ and what values of θ make r maximal?

like in class
$r = 1 + \sin 2\theta$



$r = 0$ when $1 + \sin 2\theta = 0$

$\sin 2\theta = -1$

$2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

r maximal

when $\sin 2\theta = 1$

when $2\theta = \frac{\pi}{2}$

$\boxed{\theta = \frac{\pi}{4}}$, $\frac{5\pi}{4}$

b) Find (in polar coordinates!) the points where this curve intersects the circle $r = 1$.

intersection $r = 1$ $1 = 1 + \sin 2\theta$

$0 = \sin 2\theta$

$2\theta = \{0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

c) Write down a definite integral that gives the area, in the first quadrant, inside the curve $1 + \sin(2\theta)$ but outside the circle $r = 1$.

$$\int_0^{\pi/2} \frac{1}{2}\left[(1+\sin 2\theta)^2 - 1^2\right] d\theta$$

d) (Extra Credit) Evaluate this integral.

$$\frac{1}{2}\int_0^{\pi/2} 1 + 2\sin 2\theta + \sin^2 2\theta - 1 \; d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} 2\sin 2\theta + \sin^2 2\theta \; d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} 2\sin 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta \; d\theta$$

$$\frac{1}{2}\left[-\cos 2\theta + \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta \,\Big|_0^{\pi/2}\right]$$

$$\frac{1}{2}\left[-(-1-1) + \frac{\pi}{4} - \frac{1}{8}(0)\right]$$

$$\frac{1}{2}\left[2 + \frac{\pi}{4}\right]$$

$$\boxed{1 + \frac{\pi}{8}}$$