FINAL REVIEW PROBLEMS

(1) Calculate the integral $\int_0^\infty e^{\frac{-x^2}{2}}$.

(2) Prove that the area of a region G with boundary C is $\frac{1}{2}\int_c -ydx + xdy$

(3) Verify Green's Theorem for the disk D with center (0,0) and radius R, for each of the following functions, $F = P\hat{i} + Q\hat{j}$ where P and Q are defined as follows:

a. $P(x,y) = xy^2$, $Q(x,y) = -yx^2$

b. $P(x,y) = x + y$, $Q(x,y) = y$

c. $P(x,y) = Q(x,y) = xy$

d. $P(x,y) = 2y$, $Q(x,y) = x$

(4) On the unit disk D, why does Green's theorem fail for $F = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$.

(5) Is the vector field $F(x,y) = -y\hat{i} + x\hat{j}$ a gradient vector field? If so determine a potential fucntion for it.

(6) Determine a potential function for the gradient vector field, $F(x,y) = x\hat{i} + y\hat{j}$.

(7) Determine a potential function for the gradient vector field, $F(x,y) = y\hat{i} + x\hat{j}$.

(8) Evaluate the line integral $\int_C 2xyzdx + x^2zdy + x^2ydz$ where C is an oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$

(9) Suppose c_1 and c_2 are two paths with the same endpoints and F is a vector field. Show that $\int_{c_1} F \cdot ds = \int_{c_2} F \cdot ds$ is equivalent to $\int_c F \cdot ds = 0$, where c is the closed curved obtained by first moving along c_1 and the moving along c_2 in the opposite direction.

(10) Let c be a smooth path.

a. Suppose F is perpendicular to $c'(t)$ at the point $c(t)$ for all t. Show that $\int_c F \cdot ds = 0$

b. If F is parallel to $c'(t)$, show that $\int_c F \cdot ds = \int_c ||F|| \cdot ds$.

(11) Suppose that $f : \Re^3 \to \Re$ is C^1 and that $c : [a,b] \to \Re^3$ is a piecewise C^1 path. What important fact do you know about $\int_c \bigtriangledown f \cdot ds$?

(12) What is the difference between a line integral and a path integral? What are some ways to interpert each quantity?

(13) Evaluate the path integral $\int_c f \cdot ds$ where $f(x,y,z) = e^{\sqrt{z}}$, and $c : t \to (1,2,t^2)$, $t \in [0,1]$.

(14) Let $f(x,y) = 2x - y$, and consider the path $x = t^4$, $y = t^4$, $-1 \leq t \leq 1$. Compute the integral of f along this path and interpret the answer geometrically.

(15) What is a nice change in coordinate transformation, that takes the unit disk, in polar coordates, to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. How does this change in coordinates affect the area?

(16) Let D be the region $0\le y\le x$ and $0\le x\le 1$. Evaluate $\int\int_D(x+y)dxdy$ by making the change of variables $x=u+v$, $y=u-v$.

(17) Prove that $\int\int f(x)g(y)dxdy=(\int f(x)dx)(\int f(y)dy)$.

(18) Give the definition of an integral curve (flow line) for the vector field F.

(19) Find the integral curve $c(t)$ in the vector field $F(x,y)=y\hat{i}-x\hat{j}$ that passes through the point $(2,0)$.

(20) Find the potential function for the gradient vector field $F(x,y)=y\hat{i}+x\hat{j}$.

(21) Let $c(t)=(t-sint,1-cost)$. Find the velocity, the speed, and the length of one arch.

(22) Suppose that the acceleration of $c(t)$ is always perpendicular to the velocity. What can you say about the magnitude of the velocity?

(23) Give a geometric interpretation of divergence. When is $\bigtriangledown\cdot F\le 0$?

(24) Give a geometric interpretation of curl.

(25) Calculate divergence and curl of $F(x,y,z)=xy\hat{i}+(x+y)\hat{j}+z^3\hat{k}$.

(26) Calculate the work done on a particle traversing the path $c(t)=(t,t^2,t^{-2})$ in the vector field $F(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}$.

(27) Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be $(x,y)\to(e^{4x-y},e^{2x+3y})$. Let $c(t)$ be a path with $c(0)=(0,0)$ and $c'(0)=(1,1)$. What is the tangent vector to the image of $c(t)$ under f at $t=0$?

(28) Find the equation for the plane tangent to the surface $z=f(x,y)=(cosy)(sinx)$ at the point, $(\frac{\pi}{2},0,1)$.

(29) Compute the directional derivative of $f(x,y)=x^y$ at the point $(x_0,y_0)=(e,e)$ in the direction of a vector parallel to $d=5\hat{i}+12\hat{j}$.

(30) Find the tangent plane to the surface $xyz=1$ at the point, $(1,1,1)$.

(31) Find the unit normal to the surface $cos(xy)=e^z-2$ at $(1,\pi,0)$.

(32) Given a location (x,y,z), the temperature meter on a windsurfer's sail calculates the temperature to be $T(x,y,z)=e^{-x^2-3y^2-2z^2}$ (where x, y and z are measured in kilometers). Suppose the windsurfer starts at the locations $(1,0,1)$.

a. In what direction should the windsurfer proceed so the the temperature decrease the temperature most rapidly?

b. If the windsurfer travels e^8 kilometers per second, how fast will the temperature decrease if he proceeds in the direction from part a?

c. Unfortunately, the metal of the sail will crack if cooled at a rate grater than $\sqrt{14}e^2$ degrees per second. Describe the set of possilbe directions in which he may proceed to bring the temperature down at no more than that rate.

(33) Let f and g be functions from $\Re^4 \to \Re$. Suppose f is differentiable and $\bigtriangledown f(x) = g(x)x$. Show that spheres centered at the origin are contained in the level sets for f.

(34) Find all second partials for $f(x, y) = ln(x - y)$.

(35) Show that the function $u(x, y) = x^3 - 3xy^2$ is harmonic.

(36) Suppose $c(t)$ is a vector fuction such that $||c'(t)||$ is constant. Show that the acceleration vector is perpendicular to the velocity vector (i.e. $c'(t)$ is perpendicular to $c''(t)$).

(37) If $r(t) = t\hat{i} + t^2\hat{j} + 4t^3\hat{k}$, what force acts on a particle of mass m moving along r at t=1?

(38) Show that, at a local maximim or minimum of $||r(t)||$, the vector $r'(t)$ is perpendicular to $r(t)$.

(39) Find the arc lenght of the curve $(sin3t, cos3t, 2t^{3/2})$, for $0 \leq t \leq 1$.

(40) What is a vector field? What is a gradient vector field?

(41) Show that the vector field V on R^2 defined by $V(x, y) = y\hat{i} - x\hat{j}$ is NOT a gradient vector field; that is there is no C^1 function f such that $V(x, y) = \bigtriangledown f(x, y)\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$.

(42) What is a flow line for a vector field, F?

(43) Sketch the vector field $F(x, y) = (2y, x)$.

(44) Sketch a few flow lines for the vector field $F(x, y) = (x, x^2)$.

(45) Is $c(t) = (e^2t, log|t|, 1/t)$, $f \neq 0$ a flow line of the vector field $F(x, y, z) = (2x, z, -z^2)$? Why or why not?

(46) Using our definition of the dot product and the vector form of the law of cos ines, prove the equality $< u.v >= ||u||||v||cos\theta$.

(47) In what Euclidean space does the graph of $f(x) = (sinx, cosx)$ live? Sketch th e graph and describe the set of points in set notation. Write in set notation and sketch the level sets for $f = 0$, $f = \pi$, $f = {}^{\pi}4$

(48) What is an upper bound for the magnitude of $< u, v >$?

(49) What is the equation of the plane that contains the two lines $l(t) = t(1, 1, 1)$ and $s(t) = (1, 1, 0) + t(2, 2, 2)$?

(50) What is the distance between the point $(1, 0, 1)$ and the plane $x+2y+z = 1$?

(51) Derive the projection $\bar{p}$ of $\bar{v}$ onto $\bar{u}$.

(52) Prove that the disk $D_r(x, y, z)$ in $\Re^3$ is open.

(53) Does $lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ exist?

(54) Does $lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$ exist? If so, what is the limit?

(55) For the function $f(x, y) = x - y + 2$ write the graph in set notation, sketch the g raph and sketch enough level curves to convince yourself of the graph.

(56) Describe the behavior of the level curves $f(x, y) = c$ as c varies for the func tion $f(x, y) = xy + 1$

(57) What are the level sets for the function $f(x, y, z) = 9x^2 + 4y^2 + z^2$?

(58) Prove that the set $A = \mathbb{R}^4 - \{(0, 1, 0, 0), (-1, 0, 0, 0), (0, 0, 0, 2)\}$ is open.