10/12/17

Last Time: Logarithms
Applications to science + economics

Today: Exponential Growth + Decay
Related Rates (part 2)

Future: HW 07, Mon 10/16
HW 08, Mon 10/23
Exam II, Tues 10/31

1. Find $f'(e)$ when $f(x) = \sqrt{3 + \ln(x)}$

2. Water Drains from a tank. The volume at time $t$ is $V(t) = 5000 \sqrt{1 - \frac{t}{40}}$, $0 \leq t \leq 40$

   a. What is the rate of change at $t = 5$
   b. ... 
   c. ... 
   d. ... 
   e. What is water draining fastest

\[ \frac{d}{dx} \left[ \left(3 + \ln(x)\right)^{\frac{3}{2}} \right] = \frac{1}{2} \left[ 3 + \ln(x) \right]^{-\frac{1}{2}} \cdot \frac{3}{x}, \quad f'(e) = \frac{1}{2e\sqrt{3+\ln(e)}} = \frac{1}{2e\sqrt{3+1}} \]
\[
\frac{1}{2e^{\sqrt{3}+1}}(e) = \frac{1}{2e^{\sqrt{3} \cdot 1}} = \left(\frac{1}{4e}\right)
\]

2. \( V = 5000 \left(1 - \frac{1}{40}t\right)^2 \), what rate is water draining \( \Rightarrow \)

\[\frac{dV}{dt}\]

\[\frac{dV}{dt} = 5000 \cdot 2 \left[1 - \frac{1}{40}t\right]^1 \cdot \left[1 - \frac{1}{40}\right] = 10000 \left(1 - \frac{1}{40}\right) \cdot \left(-\frac{1}{40}\right)\]

\[= -\frac{10000}{400} \left(1 - \frac{1}{40}\right) = -250 \left(1 - \frac{5}{40}\right) = -250 \left(\frac{35}{40}\right) = -250 \left(\frac{7}{8}\right) = -218.75\]

So \( \frac{dV}{dt} \bigg|_{t=5} = -250 \left(1 - \frac{5}{40}\right) = -250 \left(\frac{35}{40}\right) = -250 \left(\frac{7}{8}\right) = -218.75\)

\[\frac{dV}{dt} \bigg|_{t=20} = -250 \left(1 - \frac{20}{40}\right) = -250 \left(\frac{1}{2}\right) = -125\]

\[\frac{dV}{dt} \bigg|_{t=35} = -250 \left(1 - \frac{35}{40}\right) = -250 \left(\frac{1}{8}\right) = -31.25\]

What is this draining fastest? \( t = 0 \)

" " " " " Slowest? \( t = 40 \)
There are instances in the world where the following thing happens:

\[ r y(t) = \frac{dy}{dt} \quad \text{Law of Natural Growth. (or decay)} \]

If \( y(t) \) follows the Law of Natural Growth, then

\[ y(t) = y(0) \cdot e^{rt} = y(0) \cdot e^{rt} \]

\[ r y(t) = 0 \cdot \frac{dy}{dt} \]

\[ r [y(0) e^{rt}] = [y(0) e^{rt}]' \]

\[ y(0) \cdot e^{rt} = y(0) \cdot e^{rt} [r] \]

\[ = y(0) \cdot e^{rt} \cdot r \]

\[ = y(0) \cdot r \cdot e^{rt} \]
Some unknown element has a half-life of 10 days. Assuming the decay follows the law of natural growth and our sample is initially 50 mg, how much is left after 25 days?

Law of natural growth \( \Rightarrow A(t) = A(0) \cdot e^{rt} = 50e^{rt} \)

Goal is to find \( A(25) = 50e^{r(25)} \)

Half life is 10 days mean every 10 days the amount is \( \frac{1}{2} \) of what it was.

\[ \therefore A(0) = 50 \]
\[ A(10) = \frac{25}{50} = 50e^{r(10)} \Rightarrow \frac{1}{2} = e^{r(10)} \]

\[ \frac{1}{2} = e^{r(10)} \Rightarrow \ln \left( \frac{1}{2} \right) = \ln \left( e^{r(10)} \right) \]

\[ \ln \left( \frac{1}{2} \right) = \ln (2^{-1}) = -\ln(2) \]

But \[ \ln \left( \frac{1}{2} \right) = \ln(1)-\ln(2) = -\ln(2) \]

\[ \Rightarrow \ln \left( \frac{1}{2} \right) = \ln \left( e^{r(10)} \right) \]

\[ \Rightarrow \ln \left( \frac{1}{2} \right) = r \cdot 10 \]

\[ \therefore r = \frac{\ln(2)}{10} \]

\[ \therefore A(t) = 50e^{-\frac{\ln(2)}{10} t} , \quad A(25) = 50e^{-\frac{\ln(2)}{10}(25)} = 50e^{-\frac{\ln(2) \cdot 25}{10}} \]

\[ = 50e^{\frac{\ln(2) \cdot (25)}{-10}} \]
\[ A(40) = 50e^{\frac{-5 \ln(2)}{10}} = 50e^{-\ln(2)} = 50e \]

Next Topic: Related Rates:

A square sheet of rubber is being stretched so that each side is increasing at a rate of 2 m/s.

Q: How fast does the area change when side length \( s = 1 \)?

\( s = 2 \) \( s = 5 \) \( s = 10 \) \( s = 100 \)

Q: What do we know about this problem:

\[ \frac{ds}{dt} = 2 \quad ; \quad s = 1 \]

What do I want:

\[ \frac{dA}{dt} \quad \text{when} \quad s = 1. \]

\[ \Rightarrow A = s^2 \quad \text{implicit Diff} \quad \frac{dA}{dt} = 2s \frac{ds}{dt} \]
For Part a) when $s=1$, \[ \frac{dA}{dt} = 2 \cdot (1) \cdot 2 = 4 \]
b) when $s=2$, \[ \frac{dA}{dt} = 2 \cdot (2) \cdot 2 = 8 \]
c) when $s=5$, \[ \frac{dA}{dt} = 2 \cdot (5) \cdot 2 = 20 \]
d) when $s=10$, \[ \frac{dA}{dt} = \cdots \cdot 40 \]
e) when $s=10$, \[ \frac{dA}{dt} = \cdots \cdot 400 \]

A perfectly spherical water balloon is filled at a rate of $5 \text{ ft}^3/\text{min}$. We want to know what rate is the radius increasing when $r = 6$. We know: \( \frac{dV}{dt} = 5, \ r = 6 \)

We need \( \frac{dr}{dt} \)

\[ V = \frac{4}{3} \pi r^3 \]
\[ \frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt} \]

\[ 5 = \frac{4}{3} \pi \cdot 36 \cdot \frac{dr}{dt} \]
\[ \implies \frac{dr}{dt} = \frac{5}{4 \cdot 3 \pi} = \frac{5}{144 \cdot \pi} \]
Side Note: We know that $A = \pi r^2$

\[
\frac{dA}{dr} = 2\pi r \quad \text{— circumference}
\]

\[
v = \frac{4}{3} \pi r^3
\]

\[
\frac{dv}{dr} = 4\pi r^2 \quad \text{— SA of a sphere}
\]
A particle is traveling along the curve \( 16 = xy \) in such a way that the \( x \)-coord is moving left at a rate of 1 unit per second.

\[
\frac{dx}{dt} = -1
\]

How fast is the \( y \)-coord changing when:

- \( x = 10 \), \( \frac{dy}{dt} = \frac{y}{x} = \frac{16}{10} = \frac{16}{10} \)
- \( x = 5 \)
- \( x = 1 \), \( \frac{dy}{dt} = \frac{y}{x} = \frac{16}{1} \)
- \( x = \frac{1}{2} \)
- \( x = \frac{1}{10} \), \( \frac{dy}{dt} = \frac{y}{x} = \frac{160}{\frac{1}{10}} = 1600 \)
- \( x = \frac{1}{100} \).

\[
\frac{d}{dt} \left( x \cdot \frac{dy}{dt} + \frac{dx}{dt} \cdot y \right) = 0
\]

\[
-x + \frac{dy}{dt} \cdot x = 0
\]

\[
\frac{dy}{dt} \cdot x = y
\]

\[
\frac{dy}{dt} = \frac{y}{x}
\]