10/19/17

Last Time: Related Rates

Today: Linear Approximation, Linearization
Differentials
Hyperbolic Functions

Future: HW Due M
Exam II on T.

20 ft Light, 5 ft person 10 feet away moving at 6 m/s. How fast is the top of the shadow moving away from the lightpost?

\[
\frac{\text{d}h}{\text{d}t} = \frac{20}{s} = \frac{x+y}{y} \Rightarrow 4y = \frac{x+y}{s} \Rightarrow 4y = x+y
\]

3y = x
$3y = x$, we know: $\frac{dx}{dt} = 6$

$3 \cdot \frac{dy}{dt} = \frac{dx}{dt}$ \[\Rightarrow 3 \cdot \frac{dy}{dt} = 6 \Rightarrow \frac{dy}{dt} = 2\], \[2\] is the wrong answer.

The shadow is growing at a rate of $2\text{m/s}$.

length

We want is the rate $\text{xyl growth} \Rightarrow \frac{dx}{dt} + \frac{dy}{dt} = 6 + 2 = 8$
Hyperbolic Trig Functions

\[
\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \sinh(0) = \frac{e^0 - e^0}{2} = 0
\]

\[
[\sinh(x)]' = \left[\frac{e^x - e^{-x}}{2}\right]' = \frac{e^x + e^{-x}}{2}
\]

\[
\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \cosh(0) = \frac{e^0 + e^0}{2} = \frac{1 + 1}{2} = 1
\]

\[
[\cosh(x)]' = \left[\frac{e^x + e^{-x}}{2}\right]' = \frac{e^x - e^{-x}}{2} = \sinh(x)
\]

\[
\sinh(0) = \sin(0) = 0, \quad [\sinh(x)]' = \cosh(x)
\]

\[
[\sin(x)]' = \cos(x)
\]

\[
\cosh(0) = \cos(0) = 1, \quad [\cosh(x)]' = \sinh(x)
\]

\[
[\cos(x)]' = -\sin(x)
\]

\[
\left[\frac{\sinh(x) \cosh(x)}{\cosh(3x)}\right]' = \frac{[\sinh(x) \cosh(x)] \cosh(3x) - [\cosh(3x)]' [\sinh(x)]}{\cosh^2(3x)}
\]

\[
= \frac{[\sinh(x)]' \cosh(x) + [\cosh(x)]' \sinh(x)] \cosh(3x) - 3 \sinh(3x) [\sinh(x) \cosh(x)]}{\cosh^2(3x)}
\]

\[
= \frac{\cosh^2(x) + \sin^2(x)}{\cosh^2(3x)} \cosh(3x) - 3 \sinh(3x) [\sinh(x) \cosh(x)]}{\cosh^2(3x)}
\]

\[
= \frac{\cosh^2(x) + \sin^2(x) \cosh(3x) - 3 \sinh(3x) [\sinh(x) \cosh(x)]}{\cosh^2(3x)}
\]
Approximate $\sqrt{4.1}$ without a calculator.

\[
\begin{align*}
 y - 2 &= \frac{1}{4}(x - 4) \\
 y &= \frac{1}{4}x - 4 + 2 \\
 y &= \frac{1}{4}x + 1
\end{align*}
\]

Formula for tangent line is:

- $x_0 = 4$
- $y_0 = 2$
- $m = \text{derivative of } \sqrt{x} \text{ at } x=4$

\[
[\sqrt{x}]' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad \frac{1}{2}
\]

\[
y = \frac{1}{4}x + 1 \quad \Rightarrow \quad \text{at } x=4.1, \quad y = \frac{1}{4} \cdot 4.1 + 1 = \frac{41}{40} + 1 = \frac{41}{40} + \frac{40}{40} = \frac{81}{40} = \text{Linear approximation of } \sqrt{x} \text{ at } x=4.1
\]

Why look at $\sqrt{x}$ when you can look at $y = \frac{1}{4}x + 1$
The linear approximation of \( f(x) \) at \( x = a \) is:

\[
\tilde{f}(x) = f(a) + f'(a)(x-a)
\]

The linearization of \( f(x) \) at \( x = a \) is

\[
L(x) = f(a) + f'(a)(x-a).
\]

Take away: If \( L(x) \) is the linearization of \( f(x) \) at \( x = a \), and \( x \approx a \), then \( f(x) \approx L(x) \).

Find the linearization of \( f(x) = \frac{1}{x+3} \) at \( a = 1 \)

\[
L(x) = f(a) + f'(a)(x-a) = f(1) + f'(1)(x-1)
\]

\[
\begin{align*}
\left[ \frac{1}{x+3} \right]' &= \left[ (x+3)^{-1} \right]' \\
&= -\frac{1}{(x+3)^2} \\
&= -\frac{1}{(x+3)^2} \quad \Rightarrow \quad x = 1 \\
&= -\frac{1}{16} \\
\end{align*}
\]
Find the linearization of \( f(x) = e^{3x} \) at \( x = 0 \).

Use the linearization to approximate \( f(0.5) \).

\[
L(x) = f(a) + f'(a)(x-a)
\]
\[
= f(0) + f'(0)(x-0)
\]
\[
= 1 + 3(x-0)
\]
\[
L(x) = 1 + 3x
\]

\( L(x) \approx f(x) \). \( \Rightarrow f(0.5) \approx L(0.5) = 1 + 3(0.5) = 2 = \frac{5}{2} \)

Very similar idea: Differential.

Given \( y = x^2 + 4x \), find the differential \( dy \).

\[
\frac{dy}{dx} = 2x + 4 \Rightarrow dy = (2x + 4)dx
\]

Differential.

Let \( y = x^3 + x^2 - 2x + 1 \). Suppose \( x \) changes from 2 to 2.05.

\[
\Delta y = [(2.05)^3 + (2.05)^2 - 2(2.05) + 1] - [2^3 + 2^2 - 2(2) + 1] = 0.717825
\]

We can use differentials to approximate \( \Delta y \).

\[
dy = (3x^2 + 2x - 2) \cdot dx = (12 + 4 - 2) \cdot 0.05 = 14 \cdot \frac{5}{100} = 14 \cdot \frac{1}{20} = \frac{7}{10} = 0.7
\]

\[ x = 2 \]

And \( dx = 0.05 \)
A sphere has radius 21 cm with possible error at most 0.05 cm. Approx max error using Differentials.

Exact answer is what? \( V = \frac{4}{3} \pi r^3 \) so

error is \( \frac{4}{3} \pi (21.05)^3 - \frac{4}{3} \pi (20)^3 \) OR \( \frac{4}{3} \pi (21)^3 - \frac{4}{3} \pi (20.95)^3 \)

we use differentials to approximate.

\[ V = \frac{4}{3} \pi r^3, \text{ approx volume error, } \Delta V \]

Instead, we approx with \( dV \).

\[
\frac{dV}{dr} = \frac{4}{3} \pi \cdot 3r^2 \Rightarrow dV = 4 \pi \cdot r^2 \cdot dr
\]

\( r = 21 \)

\( dr = .05 \)

\[ \Delta V \approx dV = 4 \pi \cdot (21)^2 \cdot .05 \]