Today: 4.1, 4.2
Min / Max
Mean Value Theorem

Future: HW09 Due M at 11:30p

Applications of Derivatives

1. Given a function, what can we say about it using Algebra + Pre-Calculus?

Qn: \( f(x) = 3x^4 - 16x^3 + 18x^2 \), \(-1 < x < 4\)

\[ y - \text{intercept} \rightarrow 0 \]
\[ x - \text{intercepts} \rightarrow 3x^4 - 16x^3 + 18x^2 \]
\[ = x^2(8x^2 - 16x + 18) \]
\[ x = 0, \ x = \frac{16 \pm \sqrt{16^2 - 4 \cdot 18}}{6} \]

\[ f(-1) = 3(-1)^4 - 16(-1)^3 + 18(-1)^2 \]
\[ = 3 + 16 + 18 = 37 \]

\[ f(4) = 3(4)^4 - 16(4)^3 + 18(4)^2 \]

It would help if I knew where the min / max's were:
At \( x = b \), \( f(x) \) has a local min.

At \( x = d \), \( f(x) \) is an absolute min.

At \( x = c \), \( f(x) \) is a local max.

At \( x = e \), \( f(x) \) is an absolute max.

At \( x = a \), \( f(x) \) is neither absolute nor local.

If \( f(x) \) has a local min or max, then \( f'(c) = 0 \).

If \( f(x) \) is a continuous function and \( f'(c) = 0 \), then \( c \) is a critical point.

Wonderful Idea: Critical points correspond exactly to local mins and maxes.

Ex.: \( f(x) = x^3 \)

\[ f'(x) = 3x^2 \]

\[ f'(0) = 0 \]

\( x = 0 \) is a critical point, but \( x = 0 \) is not a local min or max.

Local min but \( f'(0) \neq 0 \).
Let \( f(x) \) be a function. The \( x \)-value \( c \) is a critical point (or critical number) if

\[ f'(c) = 0 \quad \text{or} \quad f'(c) \text{ D.N.E.} \]

Ex: Find the critical points (or critical numbers) of

\[ f(x) = x^{\frac{2}{15}}(2-x) \]

\[ f'(x) = \left[x^{\frac{2}{15}}\right]'(2-x) + x^{\frac{2}{15}}(2-x)' \]

\[ = \frac{2}{15}x^{-\frac{13}{15}}(2-x) + x^{\frac{2}{15}}(2-x) \]

\[ = \frac{2}{15}x^{-\frac{13}{15}}(2-x) - x^{\frac{2}{15}} \]

\[ = \frac{2(2-x)}{5x^{\frac{13}{15}}} - x^{\frac{2}{15}} \]

\[ = \frac{4-2x}{5x^{\frac{13}{15}}} - x^{\frac{2}{15}} \]

\[ = \frac{4-2x}{5x^{\frac{13}{15}}} - \frac{5x}{5x^{\frac{13}{15}}} = \frac{4-2x-5x}{5x^{\frac{13}{15}}} = \frac{4-7x}{5x^{\frac{13}{15}}} = f'(x) \]

When is \( f'(x) = 0 \) \( \Rightarrow 4-7x = 0, \quad 4 = 7x, \quad x = \frac{4}{7} \)

When is \( f'(x) \text{ D.N.E.} \) \( 5x^{\frac{13}{15}} = 0, \quad x^{\frac{13}{15}} = 0, \quad x = 0 \)
If \( c \) is a local min or max, then it is a critical point.

If \( c \) is a critical point, \( c \) might be a min or max.

Q: Given a function and an interval, how do we find absolute min and maxes?

Ex: \( f(x) = 2x^2 + 5x - 6 \) on \([-2, 3]\)

Either my \( x \)-value is a critical point OR my \( x \)-value is an endpoint.

1) Find critical points:

\[ f'(x) = 4x + 5 = 0 \]
\[ x = -\frac{5}{4} \]

\[ f\left(-\frac{5}{4}\right) = 2\left(\frac{25}{16}\right) + \frac{25}{4} - 6 = \frac{25}{8} - \frac{25}{4} - 6 = \frac{25 - 50 - 48}{8} = -\frac{73}{8} \]

\[ f(-2) = 2(4) + 5(-2) - 6 = 8 - 10 - 6 = -8 \]

\[ f(3) = 2(9) + 5(3) - 6 = 18 + 15 - 6 = 27 \]

\( f(x) \) has an abs max at \( x = 3 \), \( f(x) = 27 \)

\( \text{abs min at } x = -\frac{5}{4} \), \( f\left(-\frac{5}{4}\right) = -\frac{73}{8} \)
Find absolute min, max of
\[ f(x) = x^{\frac{2}{3}}(3-x) \text{ on } [-1, 3] \]
\[
\begin{align*}
  f'(x) &= \left[ x^{\frac{2}{3}} \right]'(3-x) + x^{\frac{2}{3}}[3-x]' \\
        &= \frac{2}{3}x^{-\frac{1}{3}}(3-x) - x^{\frac{2}{3}} \\
        &= \frac{2(3-x)}{3x^{\frac{2}{3}}} - x^{\frac{2}{3}} \\
        &= \frac{6-2x}{3x^{\frac{2}{3}}} - \frac{3x}{3x^{\frac{2}{3}}} \\
        &= \frac{6-5x}{3x^{\frac{2}{3}}} \\
\end{align*}
\]
\[ f(-1) = 4 \quad \text{Abs Max} \]
\[ f(\frac{4}{3}) \approx 2.03 \]
\[ f(0) = 0 \]
\[ f(3) = -20 \quad \text{Abs Min} \]

\[ f(x) = \frac{\sqrt{x}}{1+x^2} \text{ on } [0, 2] \]
\[
\begin{align*}
  f'(x) &= \frac{[x^{\frac{1}{2}}]'(1+x^2) - \sqrt{x}[1+x^2]'}{(1+x^2)^2} \\
        &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1+x^2) - \sqrt{x} \cdot 2x}{(1+x^2)^2} \\
        &= \frac{1+x^2}{2x^{\frac{1}{2}}(1+x^2)} - \frac{2x^{\frac{3}{2}}}{(1+x^2)^2} \\
        &= \frac{1+x^2}{2x^{\frac{1}{2}}(1+x^2)} - \frac{2x^{\frac{3}{2}}}{(1+x^2)^2} \\
        &= \frac{1+x^2}{2x^{\frac{1}{2}}(1+x^2)} - \frac{4x^2}{2x^{\frac{1}{2}}(1+x^2)} \\
        &= \frac{1+x^2 - 4x^2}{2x^{\frac{1}{2}}(1+x^2)} \\
        &= \frac{1-3x^2}{2x^{\frac{1}{2}}(1+x^2)} \quad \text{Crit points:} \quad x = \sqrt{\frac{1}{3}} \text{ or } -\sqrt{\frac{1}{3}} \text{ or } 0 \\
\end{align*}
\]
\[ f(0) = 0 \quad \text{Abs Min} \]
\[ f(2) = \frac{\sqrt{2}}{5} = .2821 \]
\[ f(\sqrt{\frac{1}{3}}) = .5656 \quad x = .5678 \quad \text{Abs Max} \]
Claim: If \( f(x) \) is continuous on \([a,b]\) AND differentiable on \((a,b)\) AND \( f(a) = f(b) \)

Then there is an \( x \)-value \( c \) such that \( a < c < b \) AND \( f'(c) = 0 \)

Rolle's Theorem

Q:\ What if \( f(a) \neq f(b) \)

Claim: If \( f(x) \) is continuous on \([a,b]\) AND differentiable on \((a,b)\)

slope of this line is: \( \frac{f(b) - f(a)}{b - a} \)

There is an \( x \)-value \( c \) such that \( a < c < b \) AND \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

Mean Value Theorem, MVT
MVT: \( \frac{f(b) - f(a)}{b-a} = f'(c) \)

What happens to MVT if \( f(a) = f(b) \)?

\( 0 = f'(c) \) \( \leftarrow \) Rolle's Theorem

Find the value of \( c \) which satisfies MVT where \( f(x) = 2x^2 - 3x + 1 \), \([0, 2] \).

\[
\frac{f(2) - f(0)}{2 - 0} = \frac{[3] - [1]}{2 - 0} = 1
\]

Qn: when is \( f'(c) = 1 \)

\( f(x) = 4x - 3 = 1 \)

\( 4x = 4 \)

\( x = 1 \)