09/26/17

Last Time: Exam I

Today: Introduction to Differential Equations

Future: HW05 Due Monday

Exam II is Tues, Oct 17th, 3 weeks

We are going to study (briefly) Differential Equations.
A D.E. is an equation which includes a derivative.

\[ y'' + 3y = x^2 \]
\[ y'' + (y')^2 = \sin(x) \]
\[ xy' = \sin(y^3) \]

Ultimate goal is to solve for y.

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In a very silly way, we have been looking at D.E. all semester:

**Ex:** \( y' = \frac{x}{x^2+1} \) \( \Rightarrow y = \frac{1}{2} \ln(x^2+1) + C \)

The D.E. we are studying have both \( y' \) AND \( y \) in the same equation.
Given a D.E., we want to find $y$.

Ex: $y' = 5y$

Guess $y = x^5$, $y' = 5x^4$

Ask: Is $x^5 = x^4$? No. $y = x^5$ is not a soln.

Ask: $y = x^2$ $\Rightarrow$ $y' = 2x$

Is $2x = 5(x^2)$? No. $y = x^2$ is not a soln.

In fact, if $y = x^n$, $y' = nx^{n-1}$

Is $nx^{n-1} = 5x^4$? No.

My soln $y$ is not $x^n$.

If $y = e^x$, $y' = e^x$ $\Rightarrow$ Is $y' = 5e^x$? No

If $y = e^{5x}$, $y' = 5e^{5x} = y' = 5y$

$5e^{5x} = 5(e^{5x})$

$y = e^{5x}$ is, actually, wrong.

The best answer is $y = Ce^{5x}$.
Solve: \( y' = -xy \)

\[
y = e^{2x}, \quad y' = 2e^{2x} = 2y, \quad y' = 2y
\]

\[
y = e^{6x}, \quad y' = 6e^{6x} = 6y, \quad y' = 6y
\]

\[
y = e^{-2x}, \quad y' = -2e^{-2x} \cdot [-2x] = 2x
\]

\[
y = e^{-x^2}, \quad y' = e^{-x^2} \cdot [-2x] = e^{-x^2}(-2x) = -2xe^{-x^2} = -2xy.
\]

\[
y = e^{-\frac{x^2}{2}}, \quad y' = e^{-\frac{x^2}{2}} \cdot -x = -xe^{-\frac{x^2}{2}} = -xy
\]

So our solution is \( y = Ce^{-\frac{x^2}{2}} \)

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If \( y' = -xy \) AND \( y(0) = 2 \), find \( y \).

First, find \( y \)

\[
y = Ce^{-\frac{x^2}{2}}
\]

Next, plug in initial condition:

\[
y(0) = C \cdot e^{-\frac{0^2}{2}} = 2
\]

\[
= C(1) = 2
\]

\[
\therefore \quad y = 2e^{-\frac{x^2}{2}}.
\]
Which is a family of solutions for \( y' = -4xy^3 \)?

1. \( y = \frac{1}{c + 4x^2} \Rightarrow y' = -\frac{8x}{(c + 4x^2)^2} \)
   \[ -4xy^3 = \frac{-4x}{(c + 4x^2)^2} \]
   \[ y \text{ is not a soln.} \]

2. \( y = \frac{1}{(c + 4x^2)^2} \Rightarrow y' = -2\frac{16x}{(c + 4x^2)^3} \)
   \[ -4xy^3 = \frac{-4x}{(c + 4x^2)^2} \]
   \[ y \text{ is not a soln.} \]

3. \[ \frac{1}{\sqrt{c + 4x^2}} \Rightarrow y' = -\frac{1}{2}\frac{8x}{(c + 4x^2)^{3/2}} \]
   \[ -4xy^3 = \frac{-4x}{(c + 4x^2)^{3/2}} \]
   \[ y \text{ is a soln.} \]

General soln \( y' = -4xy^3 \) is \( y = \frac{1}{\sqrt{c + 4x^2}} \).
Find all $k$ such that $y = e^{kt}$ is a soln to $y'''' - 16y' = 0$

$y = e^{kt}, \ y' = ke^{kt}, \ y'' = k^2e^{kt}, \ y''' = k^3e^{kt}$

$y'''' - 16y' = k^3e^{kt} - 16ke^{kt} = 0$

$= ke^{kt}(k^2 - 16)$

$k = 0$ \hspace{1cm} $k = \pm 4$ \hspace{1cm} $k = 4$

$	herefore y = e^{4t}, e^{4t}, 1$

No $k \ s.t. \ e^{kt} = 0$.

Same Qn, but $y'''' + 16y' = 0$

$k e^{kt}(k^2 + 16) = 0$

$k = \pm 4$ \hspace{1cm} $e^{4it} = \cos 4t + i\sin 4t$

$= \text{only soln is} \ y = e^0$

$y = \sin(4t)$

$y' = 4\cos(4t)$

$y'' = -16\sin(4t)$

$y''' = -64\cos(4t)$

Also, $\cos(4t)$
Show that \( y = \frac{1 + Ce^t}{1 - Ce^t} \) is a solution for \( y' = \frac{1}{2} (y^2 - 1) \)

\[
y' = \frac{\left[ (1 + Ce^t) - [1 - Ce^t] (1 + Ce^t) \right]}{(1 - Ce^t)^2} = \frac{Ce^t (1 - Ce^t) + Ce^t (1 + Ce^t)}{(1 - Ce^t)^2}
\]

\[
= \frac{2Ce^t}{(1 - Ce^t)^2}
\]

\[
\frac{1}{2} (y^2 - 1) = \frac{1}{2} \left[ \left( \frac{1 + Ce^t}{1 - Ce^t} \right)^2 - 1 \right] = \frac{1}{2} \left[ \frac{(1 + Ce^t)^2 - (1 - Ce^t)^2}{(1 - Ce^t)^2} \right] = 2Ce^t / (1 - Ce^t)^2
\]
Let $P(t)$ be the population of rabbits at time $t$.

Claim: $\frac{dP}{dt} = kP$, $k$ a positive constant. \[\text{unrealistic, allows for infinite growth}\]

So suppose we know that a certain area can hold a maximum of $M$ rabbits.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

if $P$ is small relative to $M$, $\frac{P}{M} \approx 0 \Rightarrow \frac{dP}{dt} \approx kP$

if $P = M \Rightarrow \frac{P}{M} = 1 \Rightarrow \frac{dP}{dt} = 0$

if $P > M \Rightarrow \frac{P}{M} > 1$

$\therefore \left(1 - \frac{P}{M}\right) < 0$

$\therefore \frac{dP}{dt}$ is also negative