9/28/17

Last Time: Intro to D.E.
Today: Slope Fields & Euler's Method
Future: How We Meet at 11:30p

Show that \( y = \frac{4}{(c+x^2)^2} \) is a soln to
\( y' = -4xy \). 

\[
y' = 4(c+x^2)^{-2} = 4 \cdot (-2)(c+x^2)^{-3} \cdot 2x
\]

\[
e^{\frac{-16x}{(c+x^2)^3}}
\]

\[
-4xy^{3/2} = -4x \cdot \left(\frac{4}{(c+x^2)^2}\right)^{3/2} = \frac{-4x \cdot 8}{(c+x^2)^3} = \frac{-32x}{(c+x^2)^3}
\]

\( \therefore \) False, \( y \) is not a soln.

A solution would be \( y = \frac{1}{(c+x^2)^2} \)

If \( y(3) = 4 \), find \( y \).
\( y = \frac{1}{(c+9)^2} \)

\( \therefore \frac{1}{4} = (c+9)^2 \Rightarrow \frac{1}{2} = c + 9, \ C = -\frac{17}{2} \)

\( \therefore y = \frac{1}{(-\frac{17}{2}+x^2)^2} \)
What we know about D.E.:

1. Given a D.E. and a function $y$, we can check whether or not $y$ is a solution to D.E.

2. Given a general solution $y$ of a particular point, we can find a particular solution.

Our goal is to find a function $y$ which satisfies a D.E... This turns out to be really, really, REALLY hard (sometimes).

Sometimes we can find $y$,

Sometimes we can't (but others can)

... NO ONE can (yet)

But there are certain things we can do to better understand every D.E. now.
what can we say about \( y' = x - y \)?

It is easy to plug in \((x,y)\) values and get \(y'\) back:

\[
\begin{align*}
(0,2) & \rightarrow y' = -2 \\
(0,1) & \rightarrow y' = -1 \\
(0,0) & \rightarrow y' = 0 \\
(0,-1) & \rightarrow y' = 1 \\
(0,-2) & \rightarrow y' = 2 \\
(1,2) & \rightarrow 1 \\
(1,1) & \rightarrow 0 \\
(1,0) & \rightarrow 2 \\
(1,-1) & \rightarrow 2 \\
(1,-2) & \rightarrow 2 \\
(2,2) & \rightarrow 0 \\
\vdots & \\
(3,-2) & \rightarrow 2
\end{align*}
\]

Slope Field in a Direction Field.

Tells us how solutions behave.

Look at \( y = Ce^x + x - 1 \), \( y' = x - y \).

\[
y' = -Ce^x + 1, \quad x - [Ce^x + x - 1] = -Ce^x + 1 = y'
\]
Let \( y' = y^2 - y \), Find \( \lim_{x \to \infty} y \)

If \( y' = 0 \), \( y = \pm 2 \)

\( y = 2 \), \( y' = 5 \)
\( y = 0 \), \( y' = 0 \)
\( y = -2 \), \( y' = 0 \)
\( y = -2 \), \( y' = 5 \)

\( y = 2 \) is an unstable equilibrium.
\( y = -2 \) is a stable equilibrium.

If \( y(0) > 2 \), \( y \to \infty \)
If \( y(0) = 2 \), \( y \to 2 \)
If \( y(0) < 2 \), \( y \to -2 \)

\( y = 2 \), \( -2 \) are called equilibrium solutions, \( equilibrium \) values that make \( y' = 0 \).
Let's suppose $y' = 1 - x + 2y$, $y(0) = 1$.

Approximate $y(3)$.

Idea:

at $(0, 1)$, $y' = 3$

at $(1, 4)$, $y' = 8$

at $(2, 12)$, $y' = 23$

$y(3) \approx 35$

This is a visual rep of what is called Euler's Method (Euler's Approximation Method).

given an initial value, $(x_0, y_0)$, some desired approximation, $y(x_1)$

we pick a step size $h$

Better approximation, make $h$ very small.
Euler's method is a way to approximate a solution given a DE and initial value \((x_0, y_0)\).

- \(y_{n+1} = y_n + m \cdot h\)
  - \(h\), step-size, \(\Delta x\)
  - \(m\), slope at \((x_n, y_n)\) \([m = y']\)

If \(y' = 1 + x - 2y\), approximate \(y(2)\) when \(y(1) = 3\), \(h = \frac{1}{2}\)

\((x_0, y_0) = (1, 3)\)
- \(x_1 = \frac{3}{2}\)
- \(x_2 = 2\)

\(y_1 = m \left(\frac{1}{2}\right) + y_0 = m \left(\frac{1}{2}\right) + 3 = -4 \left(\frac{1}{2}\right) + 3 = -2 + 3 = 1\)

\((x_1, y_1) = \left(\frac{3}{2}, 1\right)\)
\(y' \text{ at } \left(\frac{3}{2}, 1\right) = 1 + \frac{3}{2} - 2 = \frac{3}{2}\)

\(y_2 = m \left(\frac{1}{2}\right) + y_1 = m \left(\frac{1}{2}\right) + 1 = \left(\frac{3}{2}\right) \cdot \frac{1}{2} + 1 = \frac{3}{4}\)

\((x_2, y_2) = (2, \frac{3}{4}) \iff y(2) \approx \frac{3}{4}\)