Today: Introduction to Differential Equations

Future: HW05 Due Monday
       Exam II on Tue Oct 17th, 3 weeks.

We are going to (briefly) study Differential Equations. A DE. is an equation which includes a derivative.

\[ y' + 3y = x^2 \]
\[ y'' + (y')^2 = \sin(x) \]
\[ xy' = \sin(y^2) \]

Our ultimate goal is to find \( y \).

In a very silly way we have been looking at D.E. all semester.

\[ y' = \frac{x}{x^2 + 1} \Rightarrow y = \frac{1}{2} \ln(x^2 + 1) + C \]

The D.E.s we are studying have both \( y \) and \( y' \) in the same equation.
Given a D.E., we want to find $y$.

Example: $y' = 5y$.

**Guess + Check**

- $y = x^2 \Rightarrow y' = 2x \Rightarrow 2x \neq 5x^2 \quad \text{NO}$.
- $y = x^5 \Rightarrow y' = 5x^4 \Rightarrow 5x^4 = 5(x^5) \quad \text{NO}$.
- $y = x^n \Rightarrow y' = nx^{n-1} \Rightarrow nx^{n-1} = 5x^5 \quad \text{NO, expand on diff.}$

$x^n$ is not a solution, no matter the $n$.

**Try** $y = e^x \Rightarrow y' = e^x \Rightarrow y' = 5y$.

$e^x = 5e^x$, **No**

$y = e^{5x} \Rightarrow y' = 5e^{5x} \Rightarrow y' = 5y$.

$5e^{5x} = 5(e^{5x})$ \quad \text{Yes}.

$y = e^{5x}$ is not the best answer.

$\underline{y = Ce^{5x}}$ is the best answer.
Solve: $y' = -xy$

$y = e^{2x}, \ y' = 2e^{2x}, \ y' = 2y$

$y = e^{3x}, \ y' = 3e^{3x}, \ y' = 3y$

$y = e^{10x}, \ y' = 10e^{10x}, \ y' = 10y$

$y = e^{-x^2}, \ y' = e^{-x^2}[-2x] = -2xe^{-x^2} = -2xy$

$y = e^{-\frac{x^2}{2}}, \ y' = e^{-\frac{x^2}{2}}(-x) = -xy$

$\therefore \ y = Ce^{\frac{-x^2}{2}}$

If $y' = -xy$, find $y$.

$y(3) = 2,$

First, find $y \Rightarrow y = Ce^{\frac{-x^2}{2}}$

Next, plug in the initial condition:

$y(3) = Ce^{\frac{-9}{2}} = 2$

$= \frac{C}{e^{\frac{-9}{2}}} = 2 \Rightarrow C = 2e^{\frac{-9}{2}}$

$\therefore \ y = 2e^{\frac{9}{2}}e^{\frac{-x^2}{2}} = 2e^{\frac{9-x^2}{2}}$
Which of the following is a family of solutions for \( y' = -4xy^3 \):

(a) \( y = \frac{1}{c+4x^2} \Rightarrow y' = -\frac{8x}{(c+4x^2)^2} \neq 4xy^3 \), \( y \) is not a solution.

(b) \( y = \frac{1}{(c+4x^2)^2} \Rightarrow y' = -2\frac{c+4x^2}{(c+4x^2)^3} \neq 4xy^3 \), \( y \) is not a solution.

(c) \( y = \frac{1}{\sqrt{c+4x^2}} \Rightarrow y' = -\frac{1}{c+4x^2} \neq 4xy^3 \), \( y \) works.
Find all $k$ such that $y = Ce^{kt}$ is a solution to $y''' - 16y' = 0$

$y = Ce^{kt}$, $y' = Cke^{kt}$, $y'' = Ck^2e^{kt}$, $y''' = Ck^3e^{kt}$

$\therefore y''' - 16y' = Ck^3e^{kt} - 16Cke^{kt} = 0$

$Ck^3e^{kt}(k^2 - 16) = 0$

Case 1: $k^2 - 16 = 0$

$k = \pm 4$

Case 2: $Ck^3 = 0$

$k = 0$

No $k$

$\therefore y = Ce^t, Ce^{-4t}, C$

Same OA, but $y''' + 16y' = 0$

$Ck^3e^{kt}(k^2 + 16) = 0$

$k = 0 \quad \therefore y = C$.

$y = \sin(4t)$

$y' = 4 \cos(4t)$

$y'' = -16 \sin(4t)$

$y''' = -64 \cos(4t)$

$y''' + 16y' = -64 \cos(4t) + 16(4 \cos(4t)) = 0$

$y = \sin(4t)$

$y = \cos(4t)$
T/F? If \( y' = \frac{1}{2}(y^2 - 1) \), then
\[
y = \frac{1 + Ce^t}{1 - Ce^t}
\]

\[
y' = \frac{[1 + Ce^t]'(1 - Ce^t) - [1 - Ce^t]'(1 + Ce^t)}{(1 - Ce^t)^2} = \frac{Ce^t(1 - Ce^t) - (-Ce^t)(1 + Ce^t)}{(1 - Ce^t)^2}
\]

\[
= \frac{2Ce^t}{(1 - Ce^t)^2}
\]

\[
\frac{1}{2}(y^2 - 1) = \frac{1}{2}\left(\left(\frac{1 + Ce^t}{1 - Ce^t}\right)^2 - 1\right) = \frac{1}{2}\left(\left(\frac{(1 + Ce^t)^2}{(1 - Ce^t)^2}\right) - \left(\frac{(1 - Ce^t)^2}{(1 - Ce^t)^2}\right)\right)
\]

\[
= \frac{1}{2}\left(\frac{(1 + 2Ce^t + C^2e^{2t}) - (1 - 2Ce^t + C^2e^{2t})}{(1 - Ce^t)^2}\right)
\]

\[
= \frac{1}{2}\left(\frac{4Ce^t}{(1 - Ce^t)^2}\right) = \frac{2Ce^t}{(1 - Ce^t)^2}
\]

Yes, if \( y' = \frac{1}{2}(y^2 - 1) \), then \( y = \frac{1 + Ce^t}{1 - Ce^t} \).
One application of Differential Equations is that they model real life phenomena.

Let $P(t)$ be a population of rabbits.

Claim: $\frac{dP}{dt} = kP$, $k$ is a constant.

Unrealistic, allows for infinite # of rabbits.

Suppose we knew that in a certain time, the max number of rabbits is $M$.

$\frac{dP}{dt} = KP(1 - \frac{P}{M})$

- if $P$ is small relative to $M$, $\frac{P}{M} \approx 0$
- $\therefore \frac{dP}{dt} \approx kP$

- if $P \approx M \Rightarrow \frac{P}{M} \approx 1 \Rightarrow 1 - \frac{P}{M} \approx 0 \Rightarrow \frac{dP}{dt} = 0$

- if $P > M \Rightarrow (1 - \frac{P}{M}) < 0$
- $\therefore \frac{dP}{dt} < 0$