Use Euler's Method w/ step size 2 to estimate $y(8)$ when $y(0) = 2$ and $y' = x - y$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>-2</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

$y_{n+1} = m \cdot h + y_n$, $m$ is slope at $(x_n, y_n)$

$y_1 = (-2) \cdot 2 + 2 = -2$

$y_2 = 4 \cdot 2 - 2 = 6$

$y_3 = -2 \cdot 2 + 6 = 2$

$y_4 = 4 \cdot 2 + 2 = 10$

$y(8) \approx 10$
A D.E. is separable if

\[ y' = f(x) \cdot g(y) \iff \frac{dy}{dx} = f(x) \cdot g(y) \iff \frac{dy}{g(y)} = f(x) \, dx \]

Ex: \[ y' = 3e^x \cdot y^2 \iff \frac{dy}{y^2} = 3e^x \, dx \]

\[ y' = (x+3) \cdot \sin(y) \iff \frac{dy}{\sin(y)} = (x+3) \, dx \]

\[ y' = x - y \iff \frac{dy}{dx} = x - y \iff \frac{dy}{y-x} = dx \]

\[ y' = x - xy = x(1-y) \Rightarrow \frac{dy}{1-y} = x \, dx \]

\[ y' = 2y + 5 \Rightarrow \frac{dy}{2y+5} = dx \]
Solve: \( y' = x^2 y', \quad x > 0, \quad y > 0 \)

\[
\frac{dy}{dx} = x^2 y' \implies \frac{dy}{y'} = x^2 \, dx \implies \int \frac{dy}{y'} = \int x^2 \, dx
\]

\[
\ln(y) = \frac{1}{3} x^3 + C \implies y = e^{\frac{1}{3} x^3 + C} = e^{\frac{1}{3} x^3} \cdot e^C
\]

\[
y = Ce^{\frac{1}{3} x^3}
\]

\[
y' = y^2 \sin(x), \quad y(0) = 2
\]

\[
\frac{dy}{y^2} = \sin(x) \, dx \implies \frac{-1}{y} = -\cos(x) + C \implies \frac{-1}{y} = -\cos(x) + C
\]

\[
y = \frac{1}{\cos(x) - C}, \quad 2 = \frac{1}{\cos(0) - C} \implies 2 = \frac{1}{1 - C} = \frac{1}{1 - C} \implies C = \frac{1}{2}
\]

\[
y = \frac{1}{\cos(x) - \frac{1}{2}}
\]

\[
y' = \frac{e^{x+1}}{4y}, \quad y(1) = -1
\]

\[
4y \, dy = e^{x+1} \, dx
\]

\[
2y^2 = e^{x+1} + C \implies y^2 = \frac{e^{x+1}}{2} + C \implies y = \sqrt{\frac{e^{x+1}}{2} + C}
\]

\[
y = -\sqrt{\frac{e^{x+1}}{2} + C} \implies 1 = +\sqrt{\frac{e^2}{2} + C} \implies C = 1 - \frac{e^2}{2}
\]

\[
y = -\sqrt{\frac{e^{x+1}}{2} + [1 - \frac{e^2}{2}]}\]
A large tank contains 20 kg of salt + 5000 L of H₂O. A brine w/ 0.03 kg/L salt concentration enters at a rate of 25 L/min. The solution is mixed and leaves at the same rate. How much salt remains after 30 min?

Our goal is to write down the D.E. that describes this scenario.

\[ A(t) = \text{Amount of salt at time } t \text{ (kg)} \]
\[ A(0) = 20 \text{ kg} \]
\[ A(30) \text{ is what we want.} \]

\[ \frac{dA}{dt} = \text{(rate in)} - \text{(rate out)} \]
\[ = \left( \frac{0.03 \text{ kg}}{L} \cdot 25 \text{ L/min} \right) - \left( \frac{A(t)}{5000 \text{ L}} \cdot 25 \text{ L/min} \right) \]

\[ \Rightarrow \frac{dA}{dt} = 0.75 - \frac{A}{200} = \frac{3}{4} - \frac{A}{200} = \frac{150 - A}{200} \]

\[ \frac{dA}{150 - A} \Rightarrow -\ln(150 - A) = \frac{t}{200} + C \Rightarrow \ln(150 - A) = \frac{-t}{200} + C \]

\[ 150 - A = C \cdot e^{-\frac{t}{200}} \Rightarrow A = 150 - Ce^{-\frac{t}{200}} \]
But, \( A(0) = 20 \)

\[ 20 = 150 - Ce^0 = 150 - C \]

\[ C = 130 \]

\[ A = 150 - 130e^{-\frac{t}{200}} \]

\[ A(30) = 150 - 130e^{-\frac{30}{200}} \]

\[ \approx 28.384 \text{ kg} \]

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Qn: How much salt is in this tank as \( t \to \infty \).

\[ 150 - 130e^{-\frac{t}{200}} = 150 - \frac{130}{e^{\frac{t}{200}}} \to 150 \]

Pretend we start with an empty tank, and we fill it with the 0.03 kg/L solution.

\[(0.03 \text{ kg/L})(5000 \text{ L}) = 150 \text{ kg} \]
You deposit money that earns 2% interest, compounded continuously. Find how much you have at \( t=0 \), when \( A(0) = A_0 \).

\[
\frac{dA}{dt} = .02A \implies \frac{dA}{A} = .02 dt
\]

\[
\ln(A) = .02t + c
\]

\[
A = e^{.02t + c} = Ce^{.02t}, \quad A(0) = A_0 = C \cdot e^0 = C
\]

\[
A(t) = A_0 e^{.02t}, \quad P=P_0 e^{.02t}
\]

If \( P(t) \) is directly proportional to \( \frac{dP}{dt} \implies P(t) = k \cdot \frac{dP}{dt} \), we say \( P \) follows the law of natural growth.

Often times there is a max value of \( P \), call it \( M \). \( P \) follows logistic growth model if

\[
\frac{dP}{dt} = kP(1-\frac{P}{M}), \quad \implies \frac{dP}{P(1-\frac{P}{M})} = k dt
\]
The next type of D.E. we will look at is called a **First-Order Linear D.E.**

\[ A(x) \cdot y' + B(x) \cdot y = C(x) \iff y' + P(x) y = Q(x) \]

\[ x^2 y' + \frac{1}{x} y = \sin(x) \]

\[ \sqrt{x} y' + \frac{1}{e^{x+1}} y = \frac{2}{e^{x+2}} \]

Not \( y'' + y' + y = 3 \)

Not \( y' + (y')^2 = 4 \)

Not \( y' + y y' = 10 \)

**Solve:** \( x^2 y' + x y = 2x^2, x > 0 \)

\[ y' + \frac{1}{x} y = 2, \quad P(x) = \frac{1}{x}, \quad Q(x) = 2 \]

Magic: We will compute the Integration Factor:

\[ I(x) = e^{\int P(x) \, dx} = \exp \left( \int P(x) \, dx \right) \]

\[ I(x) = \exp \left( \int \frac{1}{x} \, dx \right) = \exp \left( \ln(x) \right) = x \]

\[ x y' + y = 2x \]

\[ \int [x y]' = \int 2x \Rightarrow xy = x^2 + c \]

\[ \therefore y = x + \frac{c}{x} \]

\[ x^2 \left[ x + \frac{c}{x} \right]' + x \left[ x + \frac{c}{x} \right] = x^2 \left( 1 - \frac{c}{x^2} \right) + x^2 + c \]

\[ = x^2 - c + x^2 + c = 2x^2 \]