01/28/2020

Last Time: IBP, Trig Integration

Today: Trig Integration, §7.2
Trig Substitution §7.3

Future: LMs
HW02

- $\sin^2(x)+\cos^2(x)=1 \longrightarrow \sin^2(x)=1-\cos^2(x)$
  $\cos^2(x)=1-\sin^2(x)$
  $\tan^2(x)+1=\sec^2(x)$
  $\tan^2(x)=\sec^2(x)-1$

- $\sin^2(x)=\dfrac{1-\cos(2x)}{2}=\dfrac{1}{2}-\dfrac{\cos(2x)}{2}$

- $\cos^2(x)=\dfrac{1+\cos(2x)}{2}=\dfrac{1}{2}+\dfrac{\cos(2x)}{2}$

Trig Integration Strategies

- $\int \cos(x)\left[\text{Bunch of } \sin(x)\right]dx$, $u=\sin(x)$
- $\int \sin(x)\left[\text{Bunch of } \cos(x)\right]dx$, $u=\cos(x)$
- $\int \sec^2(x)\left[\text{Bunch of } \tan(x)\right]dx$, $u=\tan(x)$
- $\int \sec(x)\tan(x)\left[\text{Bunch of } \sec(x)\right]dx$, $u=\sec(x)$

$$\int_0^{\pi/4} \cos^2(x)\,dx = \int_0^{\pi/4} \frac{1}{2} + \frac{\cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4}\Big|_0^{\pi/4}$$

$$= \left[\frac{\pi/4}{2} + \frac{\sin(\pi/2)}{4}\right] - \left[0 + \frac{\cancel{\sin(0)}}{4}\right] = \frac{\pi}{8} + \frac{1}{4} = \frac{1}{4}\left[\frac{\pi}{2} + 1\right]$$

$$= \frac{1}{8}\left[\pi + 2\right]$$

$$\int \cos^2(3x)\,dx = \int \frac{1}{2} + \frac{\cos(6x)}{2}\,dx = \dots$$

$$\int \tan^5(x)\sec^7(x)\,dx = \int \sec^2(x)\left[\tan^5(x)\sec^5(x)\right]dx \qquad u = \tan(x) \text{ STOP}$$

$$\int \sec(x)\tan(x)\left[\tan^4(x)\sec^6(x)\right]dx \qquad u = \sec(x)$$

$$= \int \sec(x)\tan(x)\left[\left[\tan^2(x)\right]^2 \cdot \sec^6(x)\right]dx = \int \sec(x)\tan(x)\left[(\sec^2(x)-1)^2\sec^6(x)\right]dx$$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$$= \int (u^2-1)^2 u^6\,du = \int (u^4 - 2u^2 + 1)u^6\,du = \int u^{10} - 2u^8 + u^6\,du$$

$$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C =$$

$$\boxed{= \frac{1}{11}\sec^{11}(x) - \frac{2}{9}\sec^9(x) + \frac{1}{7}\sec^7(x) + C}$$

Trig Substitution: Ugly Integral $\xrightarrow{\text{Trig sub}}$ Nicer Trig Integral.

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3(\theta)}{\sqrt{1-\sin^2\theta}}\cdot \cos\theta\, d\theta = \int \frac{\sin^3(\theta)}{\cancel{\cos\theta}}\cdot \cancel{\cos\theta}\, d\theta$$

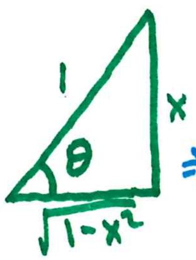
$\left\{ \begin{array}{l} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{array} \right.$

$$= \int \sin^3\theta\, d\theta = \int \sin\theta \cdot [\sin^2\theta]\, d\theta =$$

$\theta = \sin^{-1}(x)$

$$= \int \sin\theta\,[1-\cos^2\theta]\, d\theta = \int u^2 - 1\, du$$

$\sin(\theta) = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$



$\Rightarrow \cos(\theta) = \sqrt{1-x^2}$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$d\theta = \frac{du}{-\sin\theta}$

$$= \frac{u^3}{3} - u + C \cancel{du} =$$

$$\frac{\cos^3(\theta)}{3} - \cos\theta + C = \boxed{\frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} + C}$$

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx = \cdots = \int_0^{\pi/2} \sin^3\theta\, d\theta = \cdots = \int \sin(\theta)\,(1-\cos^2\theta)\, d\theta = \cdots$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$\theta = 0,\ u = 1$

$\theta = \pi/2,\ u = 0$

$$\cdots = \int_1^0 u^2 - 1\, du$$

$$= \int_0^1 1 - u^2\, du$$

$$= u - \tfrac{1}{3}u^3 \Big|_0^1 = \cdots$$

$$\boxed{2/3}$$

HOW TO RECOGNIZE TRIG SUB:

$\sqrt{1-x^2}$ $\qquad (3+x^2)^{3/2}$ $\qquad \dfrac{1}{(4+x^2)^2}$ $\qquad \dfrac{x^2}{(x^2-9)^{3/2}}$

$$\boxed{\begin{aligned} A^2-X^2 &\Leftrightarrow X = A\sin\theta \\ X^2-A^2 &\Leftrightarrow X = A\sec\theta \\ X^2+A^2 &\Leftrightarrow X = A\tan\theta \end{aligned}}$$

Rewrite, but do not solve, using trig sub

$$\int \frac{8\,dx}{\sqrt{x^2+16}} = \int \frac{8\cdot 4\cdot \sec^2\theta\, d\theta}{\sqrt{(4^2\tan^2\theta + 16)}} = \int \frac{8\cdot 4\cdot \sec^2\theta\, d\theta}{\sqrt{16\cdot \sec^2\theta}}$$

$X = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$$= \int \frac{32\sec^2\theta\, d\theta}{4\sec\theta} = 8\int \sec\theta\, d\theta$$

$$\int \frac{x^2}{\sqrt{9x^2-4}}\,dx = \int \frac{x^2}{\sqrt{(3x)^2-2^2}}\,dx = \int \frac{\left(\frac{2}{3}\sec\theta\right)^2}{\sqrt{4\sec^2\theta-4}} \cdot \frac{2}{3}\sec\theta\tan\theta\, d\theta$$

$3x = 2\sec\theta$

$3\,dx = 2\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\frac{8}{27}\cdot \sec^3\theta \cdot \tan\theta\, d\theta}{\sqrt{4\cdot \tan^2\theta}} = \frac{4}{27}\int \sec^3\theta\, d\theta$$

$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{(4-4\sin^2\theta)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{(4\cos^2\theta)^{3/2}} =$$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x=0 \Rightarrow 0 = \sin\theta \Rightarrow \theta = 0$

$x=1 \Rightarrow \frac{1}{2} = \sin\theta \Rightarrow \theta = \pi/6$

$$\int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta\Big|_0^{\pi/6} =$$

$$\frac{1}{4}\left[\frac{\sqrt{3}}{3} - 0\right] = \boxed{\sqrt{3}/12}$$

$$\int_1^2 \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\sqrt{4\cdot\sec^2\theta}}$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\cancel{x=2,\ \tan\theta = 1,\ \theta = \pi/4}$

$\cancel{x=1,\ \tan\theta = \frac{1}{2},\ \theta = ?}$

$$= \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \int \frac{1}{4}\cdot\frac{\sec\theta}{\tan^2\theta}\,d\theta$$

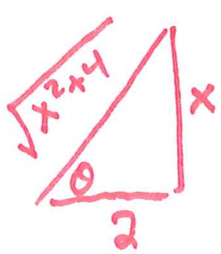
$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{u^2}\,du = -\frac{1}{4u} =$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{-1}{4\sin\theta}\Bigg|_{x=1}^{x=2}$$

$$\tan\theta = \frac{x}{2} = \frac{opp}{adj}$$



$$\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}} \Rightarrow -\frac{\sqrt{x^2+4}}{4x}\Bigg|_1^2$$

$$= -\frac{\sqrt{8}}{8} - -\frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{4} - \frac{\sqrt{8}}{8} = \frac{\sqrt{5}}{4} - \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{1}{4}\left(\sqrt{5}-\sqrt{2}\right)}$$