02/13/2020

Last Time: Exam I

Today: An Introduction to Differential Equations, §9.1

Future: LM, HW

Exam II on Thursday 03/05

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We are going to (briefly) study Differential Equations. A D.E. is an equation w/ a derivative.

- $y' + 3y = x^2$
- $y'' + (y')^2 = \sin(x)$
- $xy' = \sin^3(y)$

Our goal is to find the function $y(x)$ that satisfies the D.E.... $y =$ "bunch of $x$'s"

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In a <u>very</u> silly way, we have been studying DE. all semester:

$$y' = \frac{x}{x^2+1} \Rightarrow y = \frac{1}{2}\ln(x^2+1) + C$$

The D.E. we are looking at will have $y$'s + $y'$'s + maybe some $x$'s.

Solve: $y' = 6y$ ..... Guess + Check.

$$y = x^2 \Rightarrow [x^2]' = 6(x^2)$$
$$2x = 6x^2 \quad \times$$

$$y = x^n \Rightarrow [x^n]' = 6(x^n)$$
$$nx^{n-1} = 6x^{n} \quad \times$$

$$y = e^x \Rightarrow [e^x]' = 6(e^x)$$
$$e^x = 6e^x \quad \times$$

$$y = 6e^x \quad [6e^x]' = 6(6e^x)$$
$$6e^x = 36e^x \quad \times$$

$$y = e^{6x} \quad [e^{6x}]' = 6\cdot(e^{6x})$$
$$6e^{6x} = 6e^{6x} \quad \checkmark$$

$$y = e^{6x} + C \Rightarrow [e^{6x} + c]' = 6(e^{6x} + c)$$
$$6e^{6x} = 6e^{6x} + 6C \quad \times$$

$$\boxed{y = C\cdot e^{6x}} \Rightarrow [Ce^{6x}]' = 6(Ce^{6x})$$
$$6Ce^{6x} = 6Ce^{6x} \quad \checkmark$$

$y' = -xy$

$y = x^2$ $\quad [x^2]' = -x(x^2)$

$2x = -x^3$ $\times$

$y = \sin(x)$ $\quad [\sin(x)]' = -x(\sin(x))$

$\cos(x) = -x\sin(x)$ $\times$

$y = e^x$ $\quad [e^x]' = -xe^x$

$e^x = -xe^x$ $\times$

$y = e^{x^2}$ $\quad [e^{x^2}]' = -xe^{x^2}$

$2xe^{x^2} = -xe^{x^2}$ $\times$

$y = -\frac{1}{2}e^{x^2}$, $\quad [-\frac{1}{2}e^{x^2}]' = -x(-\frac{1}{2}e^{x^2})$

$= -\frac{1}{2}\cdot 2x \cdot e^{x^2} = \frac{x}{2}e^{x^2}$

$-xe^{x^2} = \frac{x}{2}e^{x^2}$ $\times$

$y = e^{-\frac{x^2}{2}}$, $\quad [e^{-x^2/2}]' = -x(e^{-x^2/2})$

$-xe^{-x^2/2} = -xe^{-x^2/2}$ ✓

$y = Ce^{-x^2/2}$, you can check this

If $y' = -xy$ AND $y(3) = 2$, find $y$.

$y = Ce^{-x^2/2}$ $\begin{cases} y(3) = Ce^{-9/2} \\ y(3) = 2 \end{cases}$

$\therefore Ce^{-9/2} = 2$

$\frac{C}{e^{9/2}} = 2$, $C = 2e^{9/2}$

$y = 2e^{9/2}e^{-x^2/2} = 2e^{9/2 - x^2/2}$

Find $y(1) = 2e^{9/2 - 1/2} = 2e^4$

Which of the following is a solution to

$y' = -4xy^3$

(a) $y = \dfrac{2}{C+4x^2}$

$$\left[2(C+4x^2)^{-1}\right]' = -4x\left(\frac{2}{C+4x^2}\right)^3$$

$$-2(C+4x^2)^{-2} \cdot 8x = \frac{-32x}{(C+4x^2)^3}$$

$$\frac{-16x}{(C+4x^2)^2} = \frac{-32x}{(C+4x^2)^3} \quad \times$$

(b) $\underbrace{y = \dfrac{1}{\sqrt{C+4x^2}}}_{\checkmark}$

$$\left[(C+4x^2)^{-1/2}\right]' = -4x\left(\frac{1}{\sqrt{C+4x^2}}\right)^3$$

$$-\frac{1}{2}(C+4x^2)^{-3/2} \cdot 8x = \frac{-4x}{(C+4x^2)^{3/2}}$$

$$\frac{-4x}{(C+4x^2)^{3/2}} = \frac{-4x}{(C+4x^2)^{3/2}} \quad \checkmark$$

(c) $y = \dfrac{2}{\sqrt{C+4x^2}}$

$\times$

$$\frac{-8x}{(C+4x^2)^{3/2}} = \frac{-32x}{(C+4x^2)^{3/2}}$$

Find all numbers $k$ such that $y = e^{kt}$ is a solution to $y''' - 16y' = 0$

$$[e^{kt}]''' - 16[e^{kt}]' = 0$$

$$k^3 e^{kt} - 16 k e^{kt} = 0$$

$$e^{kt} \cdot k(k^2 - 16)$$

$$e^{kt} \cdot k(k+4)(k-4) = 0$$

$$k = 0, -4, 4 \Rightarrow y = e^{4t}, e^{-4t}, e^{0} = 1$$

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Find all numbers $k$ such that $y = e^{kt}$ is a solution to $y'' + 16y = 0$

It works.

$$[e^{kt}]'' + 16[e^{kt}] = 0$$

$$k^2 e^{kt} + 16 e^{kt}$$

$$e^{kt}(k^2 + 16) = 0 \Rightarrow k = \pm 4i \Rightarrow e^{4it}, e^{-4it} \Rightarrow$$

$$y = \left.\begin{matrix} \cos(4t) \\ \sin(4t) \end{matrix}\right\}$$

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Suppose $y' = y^2 - y - 6$, where is $y$ increasing

" " $y' > 0$.

$$y' = (y-3)(y+2)$$

| | | | | | |
|---|---|---|---|---|---|
| $y'$ | + | 0 | − | 0 | + |
| $y$ | -100 | -2 | 0 | 3 | 4 |

$\therefore$ $y$ is increasing $\Longleftrightarrow$ $y < -2$ or $y > 3$.