02/20/2020

Last Time: Slope Fields + Euler's Method } All D.E.

Today: Separable Eqns + Word Problems

Future: LMs, HW, Exam on 03/05 (Thursday)

A differentiable equation is <u>separable</u> if:

$$y' = f(x) \cdot g(y) \iff \frac{dy}{dx} = f(x) \cdot g(y) \iff h(y)\, dy = f(x)\, dx$$

Ex: $y' = 3e^x y^2 \iff \frac{dy}{dx} = 3e^x y^2 \iff \frac{dy}{y^2} = 3e^x dx$

Ex: $y' = (2x+1)\cos(y) \iff \frac{dy}{\cos(y)} = (2x+1)dx$

Non-Ex: $y' = x - y \iff \frac{dy}{dx} = x - y \iff dy = (x-y)\, dx$

Ex: $y' = x - xy \iff y' = x(1-y) \iff \frac{dy}{1-y} = x\, dx$

Ex: $y' = 2y + 5 \iff \frac{dy}{dx} = 2y+5 \iff \frac{dy}{2y+5} = 1\, dx$

$y' = 3x^2 y \qquad x>0,\ y>0$

$$\int \frac{dy}{y} = \int 3x^2\,dx$$

$$\left[Ce^{x^3}\right]' = Ce^{x^3}\cdot\left[x^3\right]' = Ce^{x^3}\cdot 3x^2 = 3x^2\cdot y$$

$$\ln(y) = x^3 + c$$

$$y = e^{x^3+c} = e^{x^3}e^{c} = \boxed{Ce^{x^3}}$$

Find y, $y' = \dfrac{xy^2}{1+x^2} \qquad x>0,\ y>0$

$$\Rightarrow \int \frac{dy}{y^2} = \int \frac{x}{1+x^2}\,dx \qquad u = 1+x^2$$

$$-y^{-1} = \tfrac{1}{2}\ln(1+x^2) + C$$

$$\rightarrow\ y = \frac{-1}{\frac{1}{2}\ln(1+x^2)+C} = \cancel{\frac{-1}{\frac{1}{2}\ln(1+x^2)} + C}$$

Find y when $y' = \dfrac{e^{x+1}}{4y}$, $y(1) = -1$

$$\int 4y\,dy = \int e^{x+1}\,dx$$

$$2y^2 = e^{x+1} + C$$

$$y^2 = \frac{e^{x+1}+C}{2} = \frac{e^{x+1}}{2} + C$$

$$\therefore\ y = -\sqrt{\frac{e^{x+1}}{2} + c}$$

$$y = -\sqrt{\frac{e^{x+1}}{2} + c} = \boxed{-\sqrt{\frac{e^{x+1}}{2} + \frac{2-e^2}{2}}}$$

$$y(1) = -\sqrt{\frac{e^2}{2} + c} = -1$$

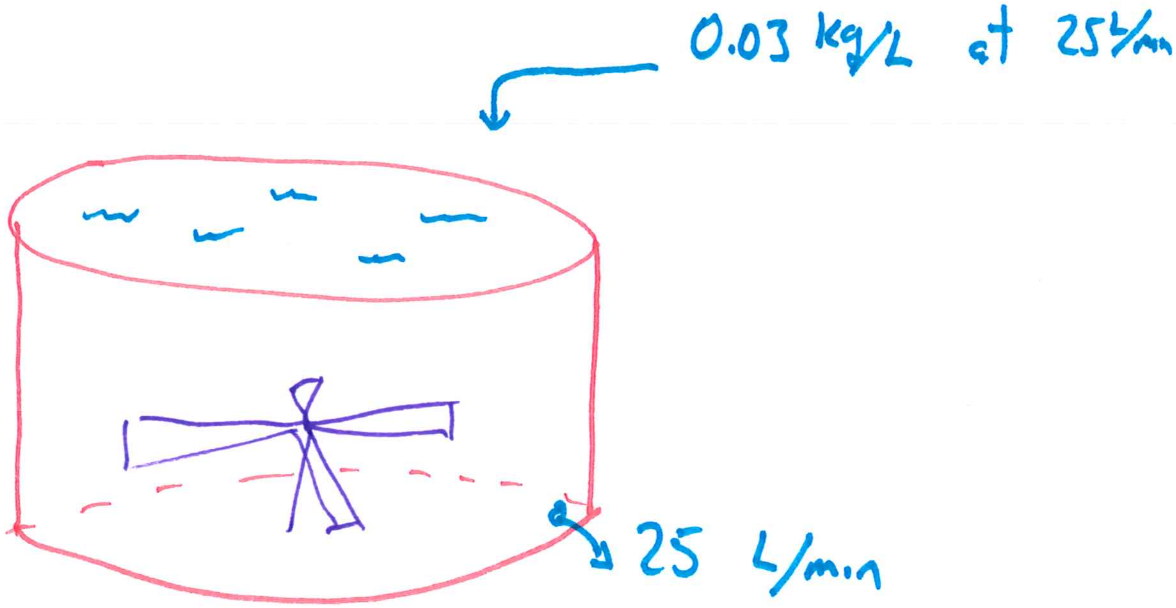
$$\Rightarrow \sqrt{\tfrac{e^2}{2} + c} = 1$$

$$\therefore\ \frac{e^2}{2} + c = 1$$

$$c = 1 - \frac{e^2}{2} = \frac{2-e^2}{2}$$

A large tank contains 30 kg of salt + 5000L of water. A brine with 0.03 kg/L salt concentration enters the tank at a rate of 25 L/min. The solution is mixed and leaves at the same rate.

How much salt is left in the tank after 30 min?



To solve this we need D.E.

Let $A(t)$ be the amount of salt in Kg at time t.

t time in minutes

$A(0) = 30$ kg

$A(30)$ is what I want.

$\frac{kg}{min}$ $\frac{dA}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$

$$= \left(.03\,\frac{kg}{L} \cdot 25\,\frac{L}{min}\right) - \left(25\,\frac{L}{min} \cdot \frac{A(t)\ kg}{5000\ L}\right)$$

$$= \left(.75\ kg/min\right) - \left(\frac{A}{200}\,\frac{kg}{min}\right)$$

$$A' = \frac{3}{4} - \frac{A}{200} = \frac{150-A}{200},\ A(0)=30$$

$$\Rightarrow \frac{dA}{dt} = \frac{150-A}{200}$$

$$\therefore \int \frac{dA}{150-A} = \int \frac{1}{200}\,dt$$

$$-\ln|150-A| = \frac{t}{200} + C$$

$$\ln|150-A| = \frac{-t}{200} + C$$

$$\cancel{e^{\ln|150-A|} = e^{-t/200+C}}$$

$$150 - A = e^{-t/200+C}$$

$$\to 150 - A = Ce^{-t/200}$$

$$\Rightarrow A = 150 - Ce^{-t/200},\ A(0) = 30$$

$$\therefore A(0) = 150 - Ce^0 = 150 - C = 30 \quad \therefore C = 120$$

$$\therefore A = 150 - 120e^{-t/200}$$

$$\therefore A(30) = 150 - 120e^{-30/200} \Rightarrow 46.7\ kg \text{ at } t=30$$

If $t \to \infty$, $A \to 150$

Suppose we started w/ an empty tank: How much salt is in there once we fill it up? $(5000L)(0.03\,K/L) = 150$

Suppose you deposit money into an account that earns 3% interest <u>compounded</u> <u>continuously</u>. How much money do you have at time t if $A(0) = A_0$?

$$\frac{dA}{dt} = .03A$$

$$\int \frac{dA}{A} = \int .03\,dt$$

$$\ln(A) = .03t + C$$

$$A = e^{.03t+c} = e^{.03t} \cdot e^{c} = Ce^{.03t}$$

$$A(0) = Ce^{0} = C = A_0$$

$$\therefore \boxed{A = A_0 e^{.03t}}$$

The Temperature of an object is T. The object is removed from a hot oven. The rate at which the object cools is proportional to the difference between T and the ambient temp T_m.

$$\frac{dT}{dt} = k(T - T_m)$$

Suppose $T_m = 50^\circ$

$T(1) = 250^\circ$

$T(3) = 150^\circ$

$T(0) = ?$

$$\frac{dT}{dt} = k(T-50)$$

$$\frac{dT}{T-50} = k\,dt$$

$$\ln(T-50) = kt + c$$

$$T-50 = Ce^{kt}$$

$$\therefore T = Ce^{kt} + 50$$

$t=3$ $\quad 150 = Ce^{3k} + 50 \Rightarrow 100 = Ce^{3k}$

$t=1$ $\quad 250 = Ce^{k} + 50 \Rightarrow 200 = Ce^{k}$

$$\Rightarrow \begin{matrix} 200 = 2Ce^{3k} \\ 200 = Ce^{k} \end{matrix} \Rightarrow \frac{2\cancel{C}e^{3k}}{e^{k}} = \frac{\cancel{C}e^{k}}{e^{k}}$$

$$2e^{2k} = 1 \Rightarrow e^{2k} = 1/2$$

$$k = \tfrac{1}{2}\ln(1/2)$$

$$\approx -0.346$$

$$\therefore 200 = Ce^{-0.346}$$

$$\Rightarrow C \approx 282.68^\circ$$

$$\therefore T = 282.68e^{-0.346t} + 50$$

$$\therefore T(0) = 282.68\,e^{0} + 50$$

$$= \boxed{332.68^\circ}$$

One day, In Theory, you will get a job + you will have the option to contribute to a 401(k) or 403(b) retirement plan. Suppose you earn, on average, 5% interest. ~~Fso~~ Suppose you make an initial contribution I and a monthly contribution M.

Find $A(t)$. $A(t)$ amount of money at time t

t - time in years

I - A_0

M - monthly payment

$$\frac{dA}{dt} = .05A + 12M$$

$\rightsquigarrow$ 35 yrs, $I=500$, $M=500$

$\rightsquigarrow$ \$573,429

$\rightsquigarrow$ 45 yrs, $I=500$, $M=500$

$\Rightarrow$ less \$60,000

$\rightsquigarrow$ \$1,023,272.17