02/25/2020

Last Time: Separable Equations
Word Problems

Today: First Order Linear

Future: LMs, ~~HW~~

M: HW06 Due

T: Class

W: HW07 Due at 9p

Th: Exam 2

The next type of D.E. we will look at is 1^{st} Order Linear:

$$A(x)\cdot y' + B(x)\cdot y = C(x) \overset{A(x)\neq 0}{\Longleftrightarrow} \underbrace{y' + P(x)y = Q(x)}_{\text{Standard Form}}$$

Ex: $x^2y' - 2e^x y = \cos(x) \quad x>0 \Leftrightarrow y' - \frac{2e^x}{x^2}y = \frac{\cos(x)}{x^2} \quad x>0$

Non-Ex: $xy' + 2xy^2 = 2$

$e^{y'} + y = x^2$

$5\cdot y'\cdot y = 4x$

Ex: $x^2y' + \ln(x)y = 0 \Leftrightarrow x^2\frac{dy}{dx} = -\ln(x)\cdot y \Leftrightarrow \frac{dy}{y} = -\frac{\ln(x)}{x^2}dx$

$$x^2 y' + 3xy = 1, \quad x > 0$$

$$y' + \frac{3}{x} y = x^{-2}$$

Find The Integration Factor:

$$\mu = e^{\int p(x)\,dx} = \exp\left(\int p(x)\,dx\right)$$

$$\mu = e^{\int \frac{3}{x} dx} = e^{3\ln(x)} = \cancel{3x} = e^{\ln(x^3)} = x^3$$

$$y' + P(x) y = Q(x)$$

$$\mu y' + \mu P(x) y = \mu Q(x)$$

$$[y \cdot \mu]' = \mu Q(x)$$

$$y' + \frac{3}{x} y = x^{-2}$$

$$x^3 \cdot y' + 3x^2 y = x$$

$$[x^3 \cdot y]' = x$$

$$\frac{x^3 \cdot y}{x^3} = \int x\,dx = \frac{\frac{1}{2}x^2 + C}{x^3}$$

$$\Rightarrow \quad x^3 \cdot y = \frac{1}{2}x^2 + C$$

$$\therefore \boxed{y = \frac{1}{2x} + Cx^{-3}}$$

$$x^2y' + xy = 1 \quad , \quad y(1) = 4$$

$$y' + \frac{1}{x}y = \frac{1}{x^2} \quad , \quad \mu = \exp\left(\int \frac{1}{x}\,dx\right) = e^{\ln(x)} = x$$

$$\therefore \left[y \cdot x\right]' = \frac{1}{x}$$

$$\Rightarrow y \cdot x = \ln(x) + C \Rightarrow y = \frac{\ln(x) + C}{x}$$

$$y(1) = 4 = \frac{\ln(1) + C}{1} = C$$

$$\therefore \boxed{y = \frac{\ln(x) + 4}{x}}$$

$$y' + 3x^2y = 6x^2 \qquad \mu = \exp\left(\int 3x^2\,dx\right) = e^{x^3}$$

$$\therefore \left[e^{x^3} \cdot y\right]' = 6x^2e^{x^3}$$

$$\therefore e^{x^3} \cdot y = \int 6x^2e^{x^3}\,dx = \int 2e^u\,du = 2e^u + C = 2e^{x^3} + C$$

$$u = x^3$$

$$du = 3x^2\,dx$$

$$\Rightarrow e^{x^3} \cdot y = 2e^{x^3} + C \quad , \quad \boxed{y = 2 + Ce^{-x^3}}$$

$xy' + 2y = \sqrt{x} \qquad x > 0$

$y' + \frac{2}{x} y = x^{-1/2}$, $\mu = \exp\left(\int \frac{2}{x} dx\right) = e^{2\ln(x)} = x^2$

$[x^2 \cdot y]' = x^{3/2}$

$\therefore x^2 \cdot y = \int x^{3/2} dx = \frac{2}{5} x^{5/2} + C$

$\therefore y = \frac{2}{5}\sqrt{x} + Cx^{-2}$

$y' + y = x$, $y(0) = 5$

$\mu = e^{\int 1 dx} = e^x$

$[e^x \cdot y]' = xe^x$

$e^x \cdot y = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$\therefore y = x - 1 + Ce^{-x}$, $y(0) = 0 - 1 + Ce^0 = -1 + C = 5 \quad \therefore C = 6$

$\boxed{y = x - 1 + 6e^{-x}}$

$y' - \frac{2}{x}y = \ln(x), \quad x > 0$

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = e^{-2\ln(x)} = \dots = x^{-2}$$

$\therefore \quad [x^{-2} \cdot y]' = x^{-2} \cdot \ln(x)$

$\therefore \quad x^{-2} \cdot y = \int \frac{\ln(x)}{x^2}\,dx = -\frac{\ln(x)}{x} - \int -\frac{1}{x^2}\,dx = \frac{-\ln(x)}{x} - \frac{1}{x} + C$

$u = \ln(x) \qquad dv = x^{-2}dx$

$du = \frac{1}{x} \qquad v = -x^{-1}$

$$y = -x\ln(x) - x + Cx^2$$

$$y' + \frac{4}{x}y = x^3 \cdot y^2 \qquad x>0,\ y>0$$

$$y^n \Rightarrow v = y^{1-n}$$

$$\Rightarrow \frac{y'}{y^2} + \frac{4}{x}\cdot\frac{1}{y} = x^3$$

$$\Rightarrow y^2 \Rightarrow v = y^{1-2} = y^{-1} = \frac{1}{y}$$

$$v' = -y^{-2}\cdot y' = \frac{-y'}{y^2}$$

$$-v' + \frac{4}{x}v = x^3$$

$$v' - \frac{4}{x}v = -x^3$$

$$\mu = \exp\left(\int -\frac{4}{x}\,dx\right) = e^{-4\ln(x)} = \cdots = x^{-4}$$

$$\left[x^{-4}\cdot v\right]' = -\frac{1}{x}$$

$$x^{-4}\cdot v = -\ln(x) + C$$

$$v = -x^4\ln(x) + Cx^4 = \frac{1}{y}$$

$$\therefore \boxed{y = \frac{1}{-x^4\ln(x) + Cx^4}}$$

$$6y' - 2y = xy^4$$

$$v = y^{1-4} = y^{-3} = \frac{1}{y^3}$$

$$\frac{6y'}{y^4} - 2\cdot\frac{1}{y^3} = x$$

$$v' = -3y^{-4}\cdot y' = \frac{-3y'}{y^4}$$

$$-2v' - 2v = x$$

$$v' + v = -\frac{x}{2} \quad , \quad \mu = e^{\int 1 dx} = e^x$$

$$[e^x \cdot v]' = -\frac{x}{2}e^x$$

$$\therefore e^x \cdot v = -\frac{1}{2}\int xe^x dx = -\frac{1}{2}[xe^x - e^x + c]$$

$$e^x \cdot v = \frac{-xe^x}{2} + \frac{e^x}{2} + C$$

$$v = \frac{-x}{2} + \frac{1}{2} + Ce^{-x} = \frac{-x}{2} + \frac{1}{2} + \frac{c}{e^x} = \frac{-xe^x + e^x + 2C}{2e^x}$$

$$\therefore v = \frac{-xe^x + e^x + C}{2e^x} = \frac{1}{y^3}$$

$$\therefore y = \left(\frac{2e^x}{-xe^x + e^x + C}\right)^{1/3}$$

$$y' + \frac{1}{x}y = \frac{1}{x^2 - 4x + 3} = \frac{1}{x^2 - 4x + 3}$$

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$[x \cdot y]' = \frac{x}{x^2 - 4x + 3}$$

$$x \cdot y = \int \frac{x}{x^2 - 4x + 3}\,dx$$

$$\frac{A}{x-3} + \frac{B}{x-1} = A(x-1) + B(x-3) = x$$

$x = 1 \Rightarrow B = -\frac{1}{2}$

$x = 3 \Rightarrow A = \frac{3}{2}$

$$x \cdot y = \int \frac{3/2}{x-3} + \frac{-1/2}{x-1}\,dx$$

$$x \cdot y = \frac{3}{2}\ln|x-3| + -\frac{1}{2}\ln|x-1| + C$$

$$\therefore\ y = \frac{3\ln|x-3|}{2x} - \frac{\ln|x-1|}{2x} + \frac{C}{x}$$