02/27/2020

Last Time: 1st Order Linear

Today: Parametric Equations

Future: M: HW, #4 11:30p

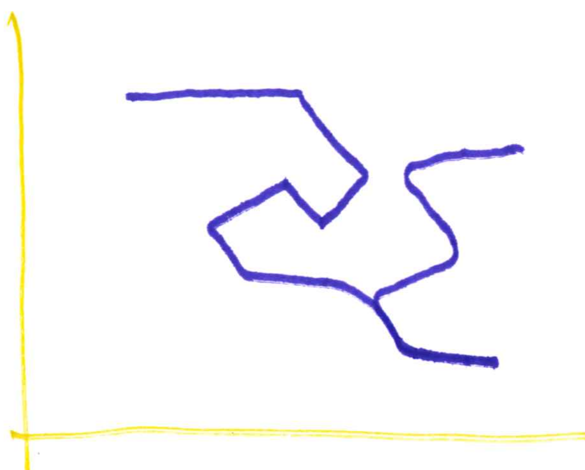
T: Class

W: HW, 9p

Th: Exam

There has been a bias towards functions.

Now we will study things that aren't functions.

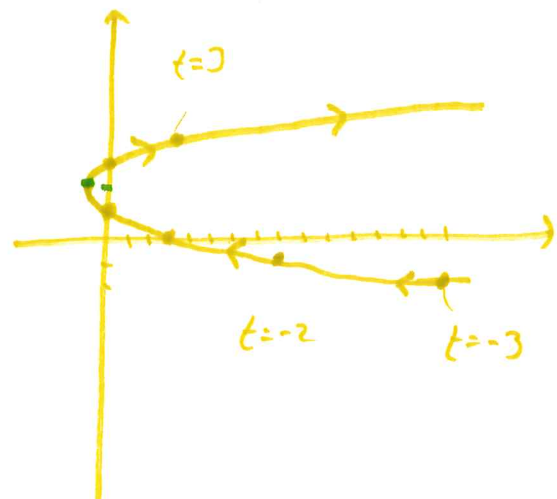


Sketch the following:

$$x(t) = t^2 - 2t$$
$$y(t) = t + 1$$

parameter

t	x	y
-3	15	-2
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4



Ways to visuelize/graph parametric equations

① Plot Points, connect Dots

② Eliminate the parameter

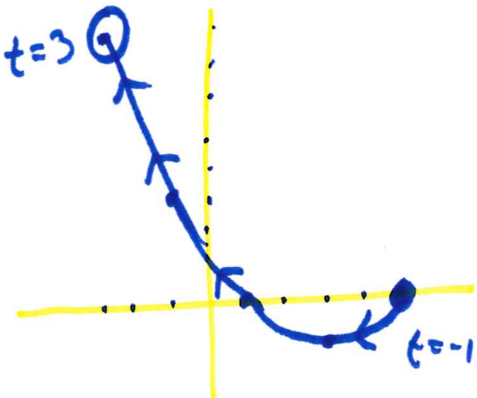
$$x = t^2 - 2t$$
$$y = t + 1 \iff t = y - 1$$

$$\therefore x = (y-1)^2 - 2(y-1)$$
$$x = y^2 - 2y + 1 - 2y + 2$$
$$= y^2 - 4y + 3 \longrightarrow y = \frac{-b}{2a} = \frac{4}{2} = 2$$
$$x = (y-1)(y-3)$$
$$y = 1, 3$$

$$x = (\cancel{y}-2)^2 - 1$$
$$x + 1 = (y-2)^2$$
$$\therefore y - 2 = \pm\sqrt{x+1}$$
$$\therefore y = 2 \pm \sqrt{x+1}$$

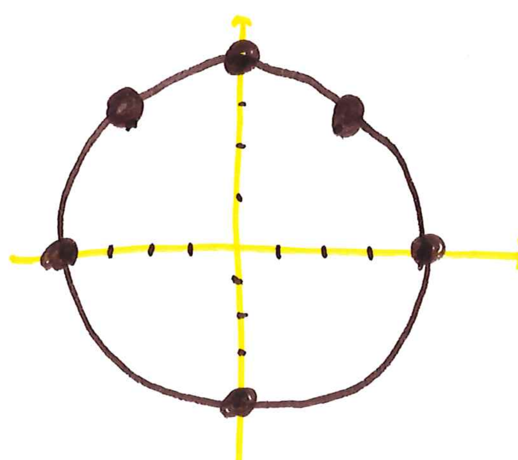
Sketch: $x = 3 - 2t$ $\quad -1 \le t < 3$
$y = t^2 - 1$

t	x	y
-1	5	0
0	3	-1
1	1	0
2	-1	3
3	-3	8



$x = 4\cos(t)$ $\quad 0 \le t \le 2\pi$
$y = 4\sin(t)$

t	x	y
0	4	0 ←
$\pi/4$	$2\sqrt{2}$	$2\sqrt{2}$
$\pi/2$	0	4
$3\pi/4$	$-2\sqrt{2}$	$2\sqrt{2}$
π	-4	0
$3\pi/2$	0	-4
2π	4	0 ←



$x = 4\cos t$
$y = 4\sin t$, Eliminate the parameter.

$x = 4\cos\left(\sin^{-1}\left(\frac{y}{4}\right)\right)$ is wrong, Domain issues + Range issues.

$\sin^2(x) + \cos^2(x) = 1$

$\sin^2(t) + \cos^2(t) = 1$

$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{4}\right)^2 = 1 \Rightarrow \frac{y^2}{4^2} + \frac{x^2}{4^2} = 1 \Rightarrow \boxed{x^2 + y^2 = 4^2}$

Parametric Eqns + Calculus

Given $x(t)$, $y(t)$, find $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Find the eqn of the line tangent to
$x = t^3 - 1$, $y = 3t^2 + 1$ at $t = 1$.

Hint: $y - y_0 = m(x - x_0)$

$(x_0, y_0) = (1^3 - 1, 3 \cdot 1^2 + 1) = (0, 4)$

$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{[3t^2+1]'}{[t^3-1]'} = \frac{6t}{3t^2} \overset{t=1}{=\!=} \boxed{2}$

$\therefore$ $y - 4 = 2(x - 0)$
$y = 2x + 4$

Find where the graph has a H. Tangent + a V. tangent

$$x = 2t^3 - 3t^2, \quad y = 2t^3 - 6t$$

$$\text{H Tangent} \Leftrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0 \Rightarrow \frac{dy}{dt} = 0, \quad 6t^2 - 6 = 6(t^2 - 1) = 0$$

$$\therefore t = 1 \text{ or } -1$$

$$\text{V Tangent} \Leftrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \infty \Rightarrow \frac{dx}{dt} = 0, \quad 6t^2 - 6t = 6t(t-1)$$

$$\therefore t = 0 \text{ or } 1$$

$t = -1$, H. Tangent at $(-5, 4)$

$t = 0$, V. Tangent at $(0, 0)$

$t = 1$, Cusp point at $(-1, -4)$

$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2} \quad \text{Is wrong}$$

$$\frac{d}{dx}(y) = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} \rightsquigarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

Find $\frac{d^2y}{dx^2}$ when $x = t^2$, $y = t^3 - 3t$ at $t = 1$.

First, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$

Next, $$\frac{d^2y}{dx^2} = \frac{\left[\frac{3t^2-3}{2t}\right]'}{[t^2]'} = \frac{\frac{6t \cdot 2t - 2(3t^2 - 3)}{(2t)^2}}{2t} = \frac{6t^2 + 6}{8t^3} \overset{t=1}{=} \boxed{\frac{3}{2}}$$

Arc length

Given a set of parametric equations, the length of the curve from $t=a$ to $t=b$ is:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: $x = 4\cos(t)$ $\quad 0 \le t \le 2\pi$

$y = 4\sin(t)$

$$L = \int_0^{2\pi} \sqrt{\left(\left[4\cos(t)\right]'\right)^2 + \left(\left[4\sin(t)\right]'\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-4\sin(t))^2 + (4\cos(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{16\sin^2 t + 16\cos^2 t}\, dt = \int_0^{2\pi} 4\, dt = 4t\Big|_0^{2\pi} = 8\pi$$

$x = \cancel{R\cos t}\ R\cos t \implies L = 2\pi R.$

$y = R\sin t$

Find the arc length:

$x = 1 + 3t^2$ $\quad 0 \le t \le \sqrt{3}$

$y = 4 + 2t^3$

$$L = \int_0^{\sqrt{3}} \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^{\sqrt{3}} 6t\sqrt{1 + t^2}\, dt = \int_1^4 3u^{1/2}\, du = \frac{3u^{3/2}}{3/2}\Bigg|_1^4$$

$u = 1+t^2$

$du = 2t\,dt$

$dt = \frac{du}{2t}$

$x=0,\ u=1$

$x=\sqrt{3},\ u=4$

$$2\left(4^{3/2} - 1^{3/2}\right)$$

$$= 2(7) = \boxed{14}$$