06/26/17

Last Time: Implicit Differentiation
Derivatives of Logarithms

Today: Exponential Growth/Decay
Related Rates

Future:
HW 6 - T
HW 7 - Th
Exam II - F

---

Find \( \frac{dy}{dx} \) at \((1, -1)\) when \(2y^2 + xy - 1 = 0\).

1. Solve for \(y\), take derivative. We can't do that here.
2. Use Implicit differentiation,

\[
\left[ 2y^2 \right]' + [xy]' - [1]' = [0]'
\]

\[
4y \cdot y' + x \cdot y + x \cdot y' = 0
\]

\[
4yy' + y + xy' = 0 \quad \text{NOTE: } y' = \frac{dy}{dx}
\]

\[
4yy' + xy' = -y
\]

\[
y' (4y + x) = -y \quad \Rightarrow \quad y' = \frac{-y}{4y + x}, \quad y'(0, -1) = \frac{-1}{1}
\]
\[
\frac{4}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 8
\]

\[
[4x^{-\frac{1}{2}} + 3y^{-\frac{1}{2}} = 8]'
\]

\[
= -2x^{-\frac{3}{2}} - \frac{3}{2} y^{-\frac{3}{2}} \cdot y' = 0
\]

\[
\Rightarrow -\frac{3}{2} y^{-\frac{3}{2}} = 2x^{-\frac{3}{2}}
\]

\[
y' = \frac{2x^{-\frac{3}{2}}}{\frac{3}{2} y^{-\frac{3}{2}}} = \frac{\frac{2}{x^{\frac{3}{2}}}}{\frac{3}{2y^{\frac{3}{2}}}} = \frac{2}{x^{\frac{3}{2}}} \cdot \frac{2y^{\frac{3}{2}}}{3} \cdot \frac{4}{3} \cdot \frac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}}
\]

\[
= \frac{4}{3} \left( \frac{y}{x} \right)^{\frac{3}{2}}
\]

\[
y' = -\frac{4}{3} \left( \frac{y}{x} \right)^{\frac{3}{2}}
\]
Exponential Growth and Decay:

There are situations in the world that are dictated by this equation: \( \frac{dy}{dt} = ky \), \( k \) is a constant. Law of natural growth.

If \( y \) obeys the law of natural growth, then
\[
y = y(0) \cdot e^{kt}, \quad P = P_0 e^{rt}, \quad P_0 = P(0)
\]

If \( \frac{dy}{dt} \) is growing, \( P = P_0 e^{rt} \),
If \( \frac{dy}{dt} \) is decaying, \( P = P_0 e^{-rt} \).

Ex: The half-life of radium-226 is 1590 years. This means \( \frac{1}{2} \) of the radium decays in 1590 years.

A sample of radium is 100 mg. Find the formula after \( t \) years:
\[
m(t) = P_0 e^{rt} = 100 \cdot e^{-rt}
\]
done, once we know \( r \), half-life is 1590 years.
\[ m(t) = 1000 \cdot 100 \cdot e^{-r \cdot t}, \text{ half life is } 1590 \]

\[
\frac{50}{100} = \frac{100 \cdot e^{-r \cdot (1590)}}{100}
\]

\[
\frac{1}{2} = e^{-r \cdot (1590)} \quad \ln(a^x) = x \ln(a)
\]

\[
\ln\left(\frac{1}{2}\right) = \ln\left(e^{-r \cdot (1590)}\right) = -r \cdot (1590) \cdot \ln(e) = -r \cdot (1590)
\]

\[
\ln\left(\frac{1}{2}\right) = -r \cdot (1590) \Rightarrow r = \frac{-1}{1590} \ln(2)
\]

\[
\therefore m(t) = 100e^{-\left(\frac{-1}{1590} \ln(2)\right) \cdot t} = 100e^{\frac{\ln(2) \cdot t}{1590}}
\]

---

Compound Interest:

You have \( A_0 \) dollars in a bank earning \( r \) interest, compounded annually.

\[ A = A_0 (1+r)^t \]

\$1000, earns 5\%, how much will you have after 50 years.

\[ A = 1000 \cdot \left(1 + .05\right)^{50} = 1000 \cdot (1.05)^{50} \approx \$11,467.40 \]
<table>
<thead>
<tr>
<th></th>
<th>Money</th>
<th>Interest</th>
<th>Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan 1</td>
<td>1000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>June 30</td>
<td>1000</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Dec 31</td>
<td>1000</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Jan 1</td>
<td>1050</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Dec 31</td>
<td>1050</td>
<td>52.50</td>
<td></td>
</tr>
<tr>
<td>Jan 1</td>
<td>1102.50</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Dec 31</td>
<td>1102.50</td>
<td>55.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1157.63</td>
<td></td>
</tr>
</tbody>
</table>

Compounded semi-annually: \( A = A_0 \left(1 + \frac{r}{2}\right)^{2t} \) \( \rightarrow \) 511,813.72

4 times a year: \( A = A_0 \left(1 + \frac{r}{4}\right)^{4t} \)

\( n \) times: \( A = A_0 \left(1 + \frac{r}{n}\right)^{nt} \)

What happens if we compound the interest every single moment of time. This is compounded continuously (aka, \( n \rightarrow \infty \))

\[ A = A_0 e^{rt} \]
Example:

You pull the string in at a rate of 20%. How fast is the boat moving along the water?

<table>
<thead>
<tr>
<th>t</th>
<th>l</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
</tbody>
</table>

\[ a^2 + b^2 = c^2 \]
\[ 5^2 + l^2 = 100^2 \]

\[ l = 99.8749 \]

\[ \Delta l = 2.0026 \]

\[ t = 1 \Rightarrow 5^2 + l^2 = 96^2 \Rightarrow l = 97.8723 \]

\[ \Delta l = 2.0026 \]

\[ t = 2 \Rightarrow 5^2 + l^2 = 96^2 \Rightarrow l = 95.8697 \]

\[ \Delta l = 2.0028 \]

\[ t = 3 \Rightarrow 5^2 + l^2 = 94^2 \Rightarrow l = 93.8669 \]

\[ \Delta l = 2.0028 \]

\[ t = 4 \Rightarrow 5^2 + l^2 = 60^2 \Rightarrow l = 57.7913 \]

\[ \Delta l = 2.0072 \]

\[ t = 5 \Rightarrow 5^2 + l^2 = 58^2 \Rightarrow l = 57.7941 \]

\[ \Delta l = 2.0041 \]

\[ t = 6 \Rightarrow 5^2 + l^2 = 8^2 \Rightarrow l = 6.2441 \]

\[ \Delta l = 2.927 \]

\[ t = 7 \Rightarrow 5^2 + l^2 = 6^2 \Rightarrow l = 3.316 \]

The idea of related rate.
$5^2 + l^2 = y^2$, take derivative with respect to $t$.

$[25]' + [l^2]' = [y^2]'$

$0 + 2ll' = 2yy'$

$ll' = yy'$

$ll' = 2y$  \[\Rightarrow\]  $l' = \frac{dl}{dt} = \frac{-2y}{l}$

Right triangle:

$a^2 + b^2 = c^2$

Similar triangles:

$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$