Practice Exam Part 1

Old Stuff

1 *Compute the following limits:*
a) $\lim_{n\to\infty} \frac{\cos n}{n^2}$

Since $-1 \geq \cos n \leq 1$, $\frac{-1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}$ *for all* n.
So, $\lim_{n\to\infty} \frac{\cos n}{n^2} \leq \lim_{n\to\infty} \frac{\cos n}{n^2} \leq \lim_{n\to\infty} \frac{1}{n^2}$.

And, $\lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

So, $\lim_{n\to\infty} \frac{\cos n}{n^2} = 0$.

b) $\lim_{x\to 0} \frac{\sin x}{x}$

At 0, *our function is in the indeterminate form* $\frac{0}{0}$, *so we can apply L'Hospital's rule.*

So, $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$
.

c)$\lim_{x\to 1^+} \frac{\ln x^2}{x-1}$

At 1, *our function is in the indeterminate form* $\frac{0}{0}$, *so we can apply L'Hospital's rule.*

So, $\lim_{x\to 1^+} \frac{\ln x^2}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{x^2}2x}{1} = 2$

Notice, b) and c) are both in the indeterminate form $\frac{0}{0}$, *but converge to different limits!*

d) $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n}$

As n approaches ∞, the limit goes to the indeterminate form 1^{∞}. So, it could converge to anything!

Notice $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n} = \lim_{n\to\infty} e^{\ln\left(\frac{n}{n+1}\right)^{2n}}$.

We are concerned with $\lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^{2n}$.

$\lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^{2n} = \lim_{n\to\infty} 2n\ln\left(\frac{n}{n+1}\right)$, *which is now in the indeterminate form* $\infty\cdot 0$.

$\lim_{n\to\infty} 2n\ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{2n}}$, *which is in the indeterminate form* $\frac{0}{0}$. *So, we can apply L'Hospital's rule.*

$\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{2n}} = \lim_{n\to\infty}\frac{\frac{n+1}{n}\frac{(n+1)\cdot 1 - n\cdot 1}{(n+1)^2}}{\frac{1}{2}\cdot\frac{-1}{n^2}}$

$= \lim_{n\to\infty}\frac{\frac{1}{n}\frac{1}{(n+1)}}{\frac{1}{2}\cdot\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{-2n^2}{n(n+1)} = -2$

So, our original limit is $\lim_{n\to\infty} e^{\ln\left(\frac{n}{n+1}\right)^{2n}} = e^{-2}$.

e) $\int_1^{\infty}\frac{1}{x\ln x}dx$

This is integral is actually improper with respect to both limits of integration.

Also, $\int_1^{\infty}\frac{1}{x\ln x}dx = \int_1^{3}\frac{1}{x\ln x}dx + \int_3^{\infty}\frac{1}{x\ln x}dx$.

First look at $\int_1^{3}\frac{1}{x\ln x}dx$.

$\int_1^{3}\frac{1}{x\ln x}dx = \lim_{t\to 1^+}\int_t^{3}\frac{1}{x\ln x}dx$

If $u = \ln x$, *then* $du = \frac{1}{x}$.

So, $\lim_{t\to 1^+}\int_t^3 \frac{1}{x\ln x} = \lim_{t\to 1^+}\int_{\ln t}^{\ln 3}\frac{1}{u}du = \lim_{t\to 1^+} \ln u|_{\ln t}^{\ln 3} = \lim_{t\to 1^+}\ln\ln 3 - \ln\ln t = \infty$.

2 *Which converge? Which diverge? Provide some reasoning and describe convergence if it occurs.*
a) $\sum_{n=4}^{\infty}\frac{1}{(\ln n-3)^5}$

The function $f(x) = (\ln x - 3)^5$ *is concave increasing. This means there is integer* n_0 *such that* $n_0 > (\ln n_0 - 3)^5$ *and any number bigger than* n_0*. So, if* $n > n_0$ $\frac{1}{n} < \frac{1}{(\ln n-3)^5}$.

And the sum $\sum_{n=n_0}^{\infty}\frac{1}{n}$ *diverges. So,* $\sum_{n=n_0}^{\infty}\frac{1}{(\ln n-3)^5}$ *diverges by comparison test. This means that* $\sum_{n=4}^{\infty}\frac{1}{(\ln n-3)^5}$ *diverges as well.*

b) $\sum_{n=1}^{\infty}\frac{n^2-n+1999}{n^4+5!n^3+n-10}$

Converges by limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Notice, the comparison test will not work!

c) $\sum_{n=1}^{\infty}\frac{a}{r^n}$ *where* $r = 1/2, -1, 5$

This is a geometric series where each term of the sum is r^{-1} *times the previous. So, we need* $|r^{-1}| < 1$*, for the series to converge.*

So, when r *is 5, the series converges and when* $r = 1/2, -1$ *the series diverges.*

New Stuff

3 $f(x,y) = -xye^{-x^2-y^2}$ *(Picture on p. 932)*

a) What are the critical points for $f(x, y)$?
$f_x(x,y) = -ye^{-x^2-y^2} - xye^{-x^2-y^2}(-2x) = (2x^2y - y)e^{-x^2-y^2}.f_x(x,y) = 0$, *when* $x = \pm\frac{1}{\sqrt{2}}$ *or* $y = 0$.

$f_y(x,y) = -xe^{-x^2-y^2} - xye^{-x^2-y^2}(-2y) = (2xy^2 - x)e^{-x^2-y^2} f_y(x,y) = 0$, *when* $y = \pm\frac{1}{\sqrt{2}}$ *or* $x = 0$.

So, the critical points are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ *and* $(0, 0)$.

b) What type of critical points are each?
$f_{xx}(x,y) = (2x^2y - y)e^{-x^2-y^2}(-2x) + (4x)e^{-x^2-y^2}$.

$f_{yy}(x,y) = (2xy^2 - x)e^{-x^2-y^2}(-2y) + (4y)e^{-x^2-y^2}$.

$f_{xy}(x,y) = (2x^2y - y)e^{-x^2-y^2}(-2y) + (2x^2 - 1)e^{-x^2-y^2}$
$= (-4x^2y^2 + 2y^2 + 2x^2 - 1)e^{-x^2-y^2}$.

Looking at $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$ *for all of the critical points will tell us what type of critical point each one is.*

$D(0,0) = 0 \cdot 0 - (-1)^2 = -1$ *So, (0,0) is a saddle point.*

$f_{xx}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{2}{e} = f_{xx}(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ $f_{yy}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{2}{e} = f_{yy}(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$f_{xy}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0 = f_{xy}(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

So $D(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = D(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{4}{e^2} > 0$. *Since,* $f_x x$ *is positive in both cases,* $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *and* $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ *are local minima.*

$f_{xx}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{-2}{e} = f_{xx}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ $f_{yy}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{-2}{e} = f_{yy}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$f_{xy}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0 = f_{xy}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

So $D(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = D(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{4}{e^2} > 0$. *Since,* $f_x x$ *is negative in both cases,* $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *and* $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ *are local maxima.*

c) What are the global mins and maxs for this function? The limit of the function as x or y goes to ∞ is 0. Since, $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{-1}{2e}$ and $f(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = f(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2e}$, $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ are global maxima and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ are global mimina.

4 *One leg of a right triangle is increasing at a rate of 1/3 m/s while the other is decreasing at a rate of 1/10 m/s. If the first leg is 10 and the second leg is 20 what rate is the area changing?*

The area of a right triangle is one half the product of the legs. If one leg has length x and the other has length y, the area A is $\frac{1}{2}xy$.

Since x and y are both changing with respect to time we can figure out their formulas. $x = \frac{t}{3}$ and $y = \frac{-t}{10}$.

We want to find a value for $\frac{dA}{dt}$.

By the chain rule, $\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} = \frac{1}{2}y \cdot 1/3 + \frac{1}{2}x \cdot 1/10$.

(Shortcut: It turns out we didn't need to find out what x and y were in terms of t, since we only needed $\frac{dx}{dt}$ and $\frac{dy}{dt}$ which were given.)

5 *A fish is swimming upward along a spiral $f(t) = < -\sin(\frac{\pi t}{10}), \cos(\frac{\pi t}{10}), 2t >$. The temperature is given by $g(x, y, z) = x^2 + xy + z^3 + 10z^2$.*

a) What is the temperature when the fish is at height 10?
We the fish is at height 10, t=5. So, $f(5) = < -1, 0, 10 >$. $g(-1, 0, 10) = 1 + 0 + 1000 + 1000 = 2001$. (Which seems silly for a temperature, but that's why this is a math class.)

b) What is the direction the fish is going in at height 10?
$f'(t) = < -\frac{\pi}{10}\cos(\frac{\pi t}{10}), -\frac{\pi}{10}\sin(\frac{\pi t}{10}), 2 >$. $f'(5) = < 0, -\frac{\pi}{10}, 2 >$.

c) At what rate is the temperature changing at this point in the direction the *fish is going?*

The gradient vector for the temperature is
$\bigtriangledown g(x, y, z) =< 2x + y, x, 3z^2 + 20z >$.

So, $\bigtriangledown g(-1, 0, 10) =< -2, -1, 500 >$

We want the directional derivative in the direction of the tangent vector.
$D_u(x, y) = \bigtriangledown g(-1, 0, 10) \cdot \frac{f'(5)}{|f(5)|} =< -2, -1, 500 > \cdot \frac{<0, -\frac{\pi}{10}, 2>}{\sqrt{4+\frac{\pi^2}{100}}}$
$= \frac{1}{\sqrt{4+\frac{\pi^2}{100}}}(\frac{\pi}{10} + 1000)$

Practice Exam part 2

Old Stuff

6. Consider the power series $g(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{n!}$
a) What is the radius of convergence?

By the ratio test, the radius of convergence is infinite. This means the series converges for any value of x.

b) Estimate g(.01) to 2 places.

Since, $g^{(n+1)}(x) = 2^{n+1}$. From this we get $|R_n(x)| \leq \frac{2^{n+1}}{(n+1)!}|x|^{n+1}$. We want to get $|R_n(x)| \leq .005$.

This happens when $n = 2$. So we just need to add up the 1st and 2nd terms.

$\frac{2(.01)^1}{1!} + \frac{2(.01)^2}{2!} = .02 + .0001 = .0201$

c) What is g'(x)? Estimate g'(.02) to 2 places.

$f(x) = g'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{(2x)^n}{n!} = \sum_{n=1}^{\infty} \frac{(2^n x^{n-1})}{(n-1)!}$

$f^{(n+1)}(x) = g^{(n+2)}(x) = 2^{n+2}$, $|R_n(x)| \leq \frac{2^{n+2}}{(n+1)!}|x|^{n+1}$. We want to get $|R_n(x)| \leq .005$.

This also happens when $n = 2$. So we just need to add up the 1st and 2nd terms.

$\frac{2(.02)^0}{0!} + \frac{2^2(.02)^1}{1!} = 2 + .08 = 2.08$

d) Let $G(x) = \int_{t=0}^{x} g(t)dt$. What is $G(.03)$ to 3 places?

$G(x) = \sum_{n=1}^{\infty} \frac{2^n x^{n+1}}{(n+1)!}$

$G^{(n+1)}(x) = 2^n$. $|R_n(x)| \leq \frac{2^n}{(n+1)!}|x|^{n+1}$. We want to get $|R_n(x)| \leq .0005$.

This also happens when $n = 2$. So we just need to add up the 1st and 2nd terms.

$\frac{2(.03)^2}{2!} + \frac{2^2(.03)^3}{3!} = .009 + .000018 = .009$

(The first component is t^2 not t. Part a was changed from the work sheets handed out as well.)
7. Let $\vec{f}(t) =< t^2, -t^2, t^2 >$ be the position of a particle at time t.
a) What values of t does the particle go through $(4, -4, 4)$?

t=2, t=-2.

b) Compute the velocity, speed, and unit tangent vector at these times.

$\vec{v}(t) =< 2t, -2t, 2t >$ Speed $= |\vec{v}(t)|$. unit tangent $\vec{v}(t)/|\vec{v}(t)|$.

So, $\vec{v}(2) =< 4, -4, 4 >$. Speed $= |\vec{v}(2)| = \sqrt{16 + 16 + 16} = 4\sqrt{3}$. unit tangent at 2= $\vec{v}(2)/|\vec{v}(2)| =< 4, -4, 4 > /4\sqrt{3} =< 1, -1, 1 > /\sqrt{3}$

So, $\vec{v}(-2) =< -4, 4, -4 >$. Speed $= |\vec{v}(-2)| = \sqrt{16 + 16 + 16} = 4\sqrt{3}$. unit tangent at -2 $= \vec{v}(-2)/|\vec{v}(-2)| =< -4, 4, -4 > /4\sqrt{3} =< -1, 1, -1 > /\sqrt{3}$

c) How far does the particle travel from t=-1 to t=1?

We want to use the formula for Arc length. However, we want the total length of the path.

Arc length $= 2\int_{t=0}^{1} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt = 2\int_{t=0}^{1} \sqrt{4t^2 + 4t^2 + 4t^2}dt = 2\int_{t=0}^{1} 2t\sqrt{3}dt = 2t^2\sqrt{3}\Big]_0^1 = 2.$

Notice if we just integrated from -1 to 1 we would have gotten zero as an answer. This would account for the total displacement but not the total distance traveled. The reason why these differ is because on the interval from -1 to 0 the integral contribute negative arc length that is equal and opposite that of the interval from -1 to 0.

8. a) Find the distance from the plane $x+2y+3z-2 = 0$ to the point $(0, 2, 3)$.

From a formula in our book, this distance is

$D = \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}$, where the point is (x, y, z).

So $D = \frac{0+2\cdot 2+3\cdot 3-2}{\sqrt{1+2^2+3^2}} = \frac{11}{\sqrt{14}}$

b) Find the intersection of $x + 2y + 3z - 2 = 0$ and $x - 3y + 2z = 0$

The intersection of two planes is either the whole plane or a line. Since these planes have different normal vectors they can't be the same plane. So, they must intersect in a line.

First we need to find the direction of this line. This line will be in the direction of the cross product of the two normal vectors.

This direction is $< 1, 2, 3 > \times < 1, -3, 2 >= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -3 & 2 \end{vmatrix} =< -12, 5, 7 >$.

Now we want to find a point in both planes. By adding the equations we get $5y + z - 2 = 0$. If $y = 0$, then $z = 2$. Plugging in the values gives us that $x = -4$.

So the symmetric equations for this line are $\frac{x+4}{-12} = \frac{y}{5} = \frac{z-2}{7}$.

c) Find the distance from the line $\frac{x-3}{2} = \frac{y}{-2} = \frac{z+4}{2}$ to the point $(1, 1, 1)$.

First find a point on this line. $(3, 0, -4)$ will do. The vector from this point to $(1, 1, 1)$ is $< 2, -1, -5 >$.

Now we need to find the angle θ between $< 2, -1, -5 >$ and $< 2, -2, 2 >$ the direction vector of our line.

$\cos\theta = \frac{<2,-1,-5>\cdot<2,-2,2>}{|<2,-1,-5>||<2,-2,2>|} = \frac{4+2-10}{(\sqrt{30})(\sqrt{12})} = \frac{-4}{\sqrt{360}}$.

So $\sin\theta = \frac{\sqrt{344}}{\sqrt{360}}$

The distance we want is $| < 2, -1, -5 > | \sin\theta = \sqrt{30}\frac{\sqrt{344}}{\sqrt{360}} = \frac{\sqrt{344}}{\sqrt{12}} = \frac{\sqrt{86}}{\sqrt{3}}$.

New Stuff

9. Evaluate the integral $\int_{x=-1}^{2} \int_{y=x}^{x^2} f(x, y)dydx$, where $f(x, y) = x^3y + y^2$.

$\int_{x=-1}^{2} \int_{y=x}^{x^2} (x^3y + y^2)dydx = \int_{x=-1}^{2} (\frac{1}{2}x^3y^2 + \frac{1}{3}y^3\Big|_{y=x}^{x^2})dx$

$= \int_{x=-1}^{2} (\frac{1}{2}x^3(x^2)^2 + \frac{1}{3}(x^2)^3 - (\frac{1}{2}x^3x^2 + \frac{1}{3}x^3))dx$

$= \int_{x=-1}^{2} (\frac{1}{2}x^7 + \frac{1}{3}x^6 - \frac{1}{2}x^5 - \frac{1}{3}x^3)dx$

$(\frac{1}{16}x^8 + \frac{1}{21}x^7 - \frac{1}{12}x^6 - \frac{1}{12}x^4)\Big|_{x=-1}^{2} = 16 + \frac{127}{21} - \frac{1}{16} - \frac{82}{12} \approx 15.15$

10. Let $f(x, y) = 2\sin x + \cos 3y$.

a) Find the equations for the tangent planes at $x = \pi/2$, $y = 2\pi/3$ and $x = 3\pi$, $y = 0$.

The equation for a tangent plane is
$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

So we need to find $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. At the given points.

First $f_x(x, y) = 2\cos x$ and $f_y(x, y) = -3\sin 3y$.

At $x_0 = \pi/2$, $y_0 = 2\pi/3$,
$z_0 = f(\pi/2, 2\pi/3) = 2 + 1 = 3$
$f_x(\pi/2, 2\pi/3) = 0$
$f_y(\pi/2, 2\pi/3) = 0$
So the equation for the tangent plane is $z - 3 = 0$.

At $x_0 = 3\pi$, $y_0 = 0$,
$z_0 = f(3\pi, 0) = 0 + 1 = 1$
$f_x(3\pi, 0) = 2$
$f_y(3\pi, 0) = 0$
So the equation for the tangent plane is $z - 3 = 2(x - 3\pi)$.

b) Give a vector normal to the curve at both of these points.
The normal vector to our tangent plane will is $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 >$.

So the normal vector to the first plane is $< 0, 0, -1 >$.

And the normal vector to the second plane is $< 2, 0, -1 >$.

c) Which of these points is a critical point?

Since the tangent plane is flat at $x_0 = \pi/2$, $y_0 = 2\pi/3$, it is a critical point.