M341 **Final Exam**, May 11, 2005

1. True or False:

(a) If $\mathcal{V}$ is an n-dimensional vector space with ordered basis B, then $L : \mathcal{V} \to \mathbb{R}^n$ given by $L(v) = [v]_B$ is a linear transformation.

(b) The negation of "A and B" is "not A and not B"

(c) The function $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x, y, z]) = [x + 1, y - 2, z + 3]$ is a linear operator.

(d) Given two square matrices A and B, then $|AB| = |A||B|$ and $|A + B| = |A| + |B|$.

(e) The solution set of any linear system has three possibilities: no solution, or a unique solution, or infinitely many solutions.

(f) Given matrices A and B, if AB is defined, then $AB = 0$ implies that either $A = 0$ or $B = 0$.

2. Prove that if A is a nonsingular symmetric matrix, then A^{-1} is also symmetric.

3. State the contrapositive, converse, and inverse of the following statement: If $\|x + y\| \neq \|x\| + \|y\|$, then y is not parallel to x.

4. Calculate the determinant of $A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix}$ by row reducing A to upper triangular form.

5. Use cofactor expansion along any row or column to find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 5 & 2 \\ 4 & 3 & -1 & 0 \\ -6 & 8 & 0 & 0 \\ 1 & 7 & 0 & -3 \end{bmatrix}.$$

Be sure to use cofactor expansion to find any 3×3 determinants needed *as well.*

6. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Find a nonsingular matrix P having all integer entries, and a diagonal matrix D such that $D = P^{-1}AP$.

7. Prove that the mapping $f : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(A) = BAB^{-1}$, where B is some fixed nonsingular $n \times n$ matrix, is a linear operator.

8. Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, and $L([-2, 5, 2]) = [2, 1, -1]$, $L([0, 1, 1]) = [1, -1, 0]$, and $L([1, -2, -1]) = [0, 3, -1]$. Find $L([2, -3, 1])$.

9. Find a maximal linearly independent subset of

$$S = \left\{ \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -6 & 2 \\ 4 & -8 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} \right\}$$

in $\mathcal{M}_{22}$. What is $\dim(\text{span}(S))$?

10. (a) Find the transition matrix from B-coordinates to C-coordinates if $B = (\ [10,-17,8],\ [-4,10,-5],\ [29,-36,16]\ )$ and $C = (\ [12,-12,5],\ [-5,3,-1],\ [1,-2,1]\ )$ are ordered bases for $\mathbb{R}^3$.

(b) Given $\mathbf{v} = [-109, 155, -71] \in \mathbb{R}^3$, find $[\mathbf{v}]_B$, and use your answer to part (a) to find $[\mathbf{v}]_C$.