Research Statement

Erxiao Wang

My research is in the area of **differential geometry**, centering around various applications of **integrable systems** to submanifold geometries. I will first explain the overall goal of this very interdisciplinary and active field, then summarize my contributions and some ongoing and future projects.

1 Introduction

Following the important work of Zakharov-Shabat [48] and Ablowitz-Kaup-Newell-Segur [1] in $1970's$, there are systematic constructions of hierarchies of integrable differential equations associated to a complex simple Lie algebra with various reality conditions given by finite order automorphisms (see, e.g., [38, 35]). Drinfeld-Sokolov [13] constructed generalized KdV and mKdV hierarchies for any affine Kac-Moody Lie algebra using this ZS-AKNS scheme. In particular, Sine-Gordon equation and Sinh-Gordon equation are two real forms of the -1-flow or Toda-type equation in the mKdV-hierarchy for the simplest affine algebra: some extension of the loop algebra of $\mathrm{sl}(2,\mathbb{C})$. It is amazing that these two equations have already appeared in classical differential geometry for constant negative Gauss curvature surfaces (or pseudo-spheres) and constant mean curvature surfaces (or soap films). For example, Bäcklund [2] constructed his famous transformation for pseudo-spheres around 1883, which produced many explicit solutions of Sine-Gordon equation $\omega_{xy} = \sin\omega$. This transformation and the higher flows in the hierarchy can be regarded as hidden symmetries of such submanifolds. It has ever since become a central problem how to find special submanifolds in higher dimensions which admit similar geometric transformations or have a lot of hidden symmetries. It is now natural to expect the answer to lie in integrable systems, as we will illustrate further using the next rank 2 affine algebra, some extension of the twisted loop algebra of $\mathrm{sl}(3,\mathbb{C})$. The following submanifolds and their structure equations are known:

(1) Indefinite affine spheres in $\mathbb{R}^3$ are given by solutions of $\omega_{xy} = e^{\omega} - e^{-2\omega}$;

(2) Definite affine spheres in $\mathbb{R}^3$ are given by solutions of $\omega_{z\bar{z}} = -e^{\omega} - e^{-2\omega}$;

(3) Special Lagrangian cones in $\mathbb{C}^3$ are given by solutions of $\omega_{z\bar{z}} = -e^{\omega} + e^{-2\omega}$.

The above three equations are three real forms of the -1-flow in the corresponding mKdV-hierarchy, and the real groups are often the groups preserving the corresponding geometries. This gives rise to a very interesting question: *what geometries correspond to other integrable systems?* Research projects related to this question have been proposed in many places and the reader may refer to Terng's survey [35] for more examples and references.

The equation in (1) is called the Tzitzéica equation. It has also been studied in the context of gas dynamics [19] and pseudo-hyper-complex structures on $\mathbb{R}^2 \times \mathbb{RP}^2$ [17]. It is a striking universal feature of integrable systems that the same equation often arises from many unrelated sources. To further convince the reader of the great varieties here, we mention that minimal surfaces and Hamiltonian stationary Lagrangian surfaces in $\mathbb{CP}^2$ also correspond to solutions of integrable systems associated to $\mathrm{sl}(3, \mathbb{C})$, but with different automorphisms [23].

The systematic construction above is just the starting point. It naturally gives rise to a loop group factorization, which in turn provides a method for constructing explicit solutions and symmetries of the equations. One way to construct explicit solutions is through the dressing action by loop group factorization. The original idea goes back to Zakharov-Shabat [48]. In general, the theorems proving the existence of loop group factorizations do not provide any concrete insight into the precise form of the factors and how they depend on the original matrix. In contrast with the general situation, the factorization can be carried out explicitly via residue calculus when one factor is a rational loop group element. This is the work of Terng-Uhlenbeck [38], in which classical Bäcklund transformations were identified explicitly for pseudo-spheres as dressing action of rational loop group elements with a simple pole. Another way of constructing solutions to such a nonlinear PDE is by reducing the PDE to a finite dimensional Hamiltonian system of Lax form. This special class of solutions is called the finite type or finite gap solutions (see, e.g., [32, 21]). This approach has a beautiful link to geometries of algebraic curves (see, e.g., [31, 20, 21, 16]). In particular, periodic solutions fall into this class.

In summary, the primary goal of my research is:

- *to search for interesting geometric objects corresponding to various integrable systems;*
- *to find all transformations/symmetries for such objects and identify their group structure.*
- *to construct different classes of explicit solutions and study their properties such as periodicity.*

2 Previous Results

With my collaborators, I have made several interesting contributions [45, 46, 41] after Terng-Uhlenbeck's work [38], have explored new directions [40] relating to exterior differential systems[1], and have employed algebraic geometry techniques to study calibrated geometry and harmonic maps [36, 47, 11]. I have also been actively involved and given presentations in nearly all conferences and AMS meetings relating to integrable system and calibration. The following serves as a brief sketch of our previous results, *with the main theorems emphasized.*

Transformations of flat Lagrangian immersions and Egoroff nets

Ribaucour constructed his transformation [33] for orthogonal nets in 1872. The iteration of these transformations was studied and applied to Egoroff nets by Liu-Mañas in [27]. Dajczer-Tojeiro generalized sphere congruence and Ribaucour transformations to submanifolds with any dimension and co-dimension in space-forms in a series of papers [14, 15], and applied to flat Lagrangian submanifolds in $\mathbb{C}^n$ with non-degenerate normal bundle.

[1] I would like to thank Robert Bryant for inspiring lectures and discussions in MSRI.

As part of my thesis research [45], a natural λ-family ($\lambda \in \mathbb{R} \setminus \{0\}$) is associated to any such flat Lagrangian submanifold, and a new idea has been introduced to integrate the position function. The main result is that *Ribaucour transformation in [14] is given exactly by the dressing action of some rational loop group element defined via an imaginary pole and a real Hermitian projection onto a line.* A new and interesting observation is that *the family degenerates to an orthogonal-Egoroff net (or E-type orthogonal curvilinear coordinate system) on $\mathbb{R}^n$ when* $\lambda \to 0$. Terng and I have submitted another paper [41] which unifies these two geometries and generalizes both Liu-Mañas' and Dajczer-Tojeiro's work on Ribaucour transformations. *We have not only discovered new types of transformations in closed algebraic formulas, but also described the group structure of these transformations by identifying the generators.* When such a submanifold lies in a hypersphere, it projects onto a $(n-1)$-dimensional flat Lagrangian submanifold in $\mathbb{C}P^{n-1}$ via Hopf projection and *we have simplified the transformations in this important case.* As a by-product, *we have also obtained transformations of Egoroff metrics and spherical (or ∂-invariant) Egoroff metrics*, which is important in Frobenius manifold and quantum cohomology. The beautiful pictures of these orthogonal nets on $\mathbb{R}^2$ and $\mathbb{R}^3$ can be drawn in computer for education purposes.

Tzitzéica transformation is a dressing action

In 1910, Tzitzéica published a classical paper [42] on hyperbolic surfaces in $\mathbb{R}^3$ whose Gauss curvature at any point is proportional to the fourth power of the distance from a fixed point to the tangent plane there. He also constructed his transformation of such surfaces similar to Bäcklund transformation for pseudo-spheres. These surfaces are invariant under affine transformations, and they are now known as affine spheres since all affine normal directions intersect at the fixed point. The classical Tzitzéica equation in the introduction is the structure equation. It is natural to ask whether his transformation is also a dressing action of some loop group element, whether there are new transformations of affine spheres, and what is the group structure of these transformations.

I have given an affirmative answer to the first question[2] in [45] and *have computed the permutability formula* and discussed the other questions in my paper [46]. The theoretical importance of [46] is that *it has classified, for the first time, the simplest rational loop group elements satisfying higher order (> 2) reality conditions.* It is a brute-force computation and the element has 3 poles. One new phenomena is that it is no longer constructed by projections as [38] but by general linear maps of fixed rank. Thus its dressing action and permutability formula are much harder to compute. The good news is that *the answers nevertheless have the same patterns as projection case.* It turns out that *one class of dressing action provides exactly the Tzitzéica transformation and the other provides the dual*, an important relation among affine spheres. Then *I have constructed complex Tzitzéica transformations to produce breather-type solutions* (periodic in certain direction). In particular, when we apply these dressing actions to the vacuum solution $\omega \equiv 0$ and the vacuum affine sphere $x^3 + y^3 + z^3 - 3xyz = 1$, we obtain explicit solutions to draw in computer.

PS: Please read the paper, and you will agree with the referee of this paper that it is *excellent*, regardless of the journal ranking.

[2]Thank Chuu-Lian Terng, Ferus Pedit, and Josef Dorfmeister for valuable discussions.

A new transformation of indefinite affine spheres

Later, *Karen Uhlenbeck and I found a special twisted loop group element with* 6 *poles which do not come from the product of two simple ones above. The dressing action of such elements produces new but complicated transformations.* Hence the group structure is very different from the compact case and we can at best show 'generic' such rational elements are generated by the simple ones and the structure of all rational elements remains too complicated to see by now. We are writing it up as a short paper "*A New Transformation of Indefinite Affine Spheres*" and encourage algebraist to study the delicate structures of rational loop groups in non-compact case.

Explicit conservation laws for an involutive EDS

During my stay at MSRI in Berkeley, I have learned a lot from Bryant [6] on exterior differential systems (EDS). It was easy for him to show that r-dimensional system associated to a rank r symmetric space U/U_0 (see [38]) is also an involutive EDS naturally associated to the symmetric space. It is then an interesting problem in EDS to find all conservation laws. In a joint work [40] with Chuu-Lian Terng, *we have not only constructed infinitely many integral submanifolds and conservation laws, but also generalized Bryant's results to* U/U_0 *defined by a higher order* $k \geq 2$ *automorphism* σ *of* U. In other words, we have constructed an EDS $\mathcal{I}_\sigma$ associated to (G, σ, τ) corresponding to the collection of first flows in (G, σ, τ)-hierarchy, and proved that $\mathcal{I}_\sigma$ *is involutive.* Note that here U is the real form of G fixed by τ and we require $\tau\sigma = \sigma^{-1}\tau^{-1}$. Thus it is different from k-symmetric space (which is defined by commuting σ and τ) when $k \geq 3$. There are many geometries associated to such r-dimensional system. For example, the notion of curved flats in symmetric spaces introduced by Ferus-Pedit [18] is generalized naturally to $k \geq 3$ case. Moreover, Cartan-Kähler theory solves the the Cauchy problem for such EDS in real analytic category, but the inverse scattering technique used by Terng [34] can be generalized easily to solve the case of smooth rapidly decaying initial data for $k \geq 3$.

Associative cones and the first elliptic G_2/T^2-system

The octonions $\mathbb{O}$ are the largest of the four normed division algebras. They are nonassociative. Aut($\mathbb{O}$) is the exceptional Lie group G_2. As a subgroup of $\mathrm{SO}(7, \mathbb{R})$, G_2 still acts transitively on S^6 with the isotropy group isomorphic to SU(3). So there is a nearly Kähler structure on S^6 with the almost complex structure J given naturally by the octonion product, which is however not integrable. Associative cones are cones over almost complex curves in nearly Kähler S^6. Bryant studied a special class of such curves of null-torsion or superminimal type and has given a Weierstrass representation for them to show that there are many such curves of any genus [5]. Later Bolton-Vrancken-Woodward classified such curves into 4 types in [8], using the advanced tool of harmonic sequence. My joint work with Chuu-Lian Terng and Shengli Kong [36] was motivated when we run a seminar on calibrated geometry and special holonomy. We unified their 4 types into one standard integrable system formulation: *the Gauss-Codazzi equations for associative cones in* $\mathbb{R}^7$ *is the first elliptic* G_2/T^2*-system.* We have also constructed a natural moving frame and *provided elementary proof of their classification theorem of* 4 *types.* One type is equivalent to special Lagrangian cones in $\mathbb{C}^3$, and has been well studied. The only mysterious type is actually the general type locally and there is still no compact examples yet.

The Spectral Data of Finite Type Associative Cones

I was then motivated to find almost complex tori in S^6 of general type. It is known (See [4])that such tori, if they exist, must be of finite type. Thus we need to study their spectral data, which consists of a hexagonal algebraic curve (called spectral curve) and a planar flow of line bundles in its Jacobian. Inspired by Hitchin's slides[3] about his recent work on spectral curves and Langland duality, I have been able to prove in my preprint [47] (*submitted*) *the generic smoothness of such spectral curve and have computed its genus and the dimension of the moduli of such curves.* Then we identify a Prym-Tjurin subtorus of $\mathcal{J}(S)$, in which the direction of the line bundle flow must lie, and compute its dimension. Finally we characterize finite type special Lagrangian cones in $\mathbb{C}^3$ as a subclass of these associative cones in terms of the spectral data.

Reconstructing an associative cone from such spectral data is a harder problem. Nevertheless I have done a parameter count for the double-periodicity equations, which tells roughly that 'how many' such tori are expected. The answer is **surprising**: *only rotationally symmetric cones pass the parameter count, which has the smallest possible spectral genus* 5. Thus it is very different from special Lagrangian case, where Carberry and McIntosh [10] showed the existence of many special Lagrangian cones over tori for any even spectral genus. This is the second paper in this direction that I am writing up some part right now.

3 Future Plans

New approaches that I have developed with my collaborators and learned from experts open up many exciting possibilities for future research. I am exploring and will continue to research the following related directions.

Spectral curves and periodic solutions [4] As a continuation of the project searching for general type almost complex tori in S^6, I have a joint project with Emma Carberry[5] to give definite answer to the existence of such tori through a regularity check of certain generalized Abel-Jacobi map, similar to her work with McIntosh [10] for special Lagrangian cones over tori. This method only works for the cases that pass the parameter count.

With the above experience, I plan to study in general the finite type or finite gap solutions to integrable systems and the periodicity problems. Finite type solutions are obtained by solving a Lax equation with a rational parameter λ:

$$\frac{\mathrm{d}A_\lambda}{\mathrm{d}t} = [A_\lambda, B_\lambda],$$

where A and B are polynomials of λ with matrix coefficients. This is an isospectral flow and has a beautiful correspondence to the spectral data (Σ, λ, L_t). Mumford and van Moerbeke [31] used the word 'dictionary' for this correspondence since it touches most aspects of curve theory. While Σ defined by characteristic polynomial of A is independent of t, the eigenvector line bundle L_t^* flows linearly in the Jacobian in most examples. But we cannot say that this dictionary is

[3] I would like to thank Tamas Hausel for giving me Hitchin's slides.

[4] I would like to thank David Ben-Zvi, Tamas Hausel, Gavril Farkas, and Xiaobo Liu for valuable discussions.

[5] She is partially supported by NSF grant

well understood if we cannot tell if there are periodic solutions. This test problem is notoriously hard in the sense that it involves very delicate geometries of curves and their Jacobians under certain Weyl group action. My motivation is to construct tori in some geometries, such as almost complex tori in S^6 Carberry and I are studying right now. After this project, I plan to investigate the following general problems:

1. Inverse Problem: For any simple Lie algebra $\mathcal{G}$, What intrinsic characterization of the spectral data guarantees its origin from the Lax equation for matrices valued in $\mathcal{G}$?

2. Parameter Count: What is the dimension of the moduli of $\mathcal{G}$-spectral curves for a particular genus? How many periodicity equations do we need to solve to obtain (doubly) periodic solutions?

3. Regularity Check: Can we assure the existence of (doubly) periodic solutions, for example, by doing a regularity check for the map from the available parameters to the space of flow directions in the Jacobian?

Constructing KdV hierarchy by restriction of ZS-AKNS This has been a joint project with Chuu-Lian Terng and Karen Uhlenbeck for more than a year and we have made a big progress since spring 2006. Recall that Drinfeld-Sokolov [13] used a quotient to construct generalized KdV hierarchy associated to any affine Kac-Moody Lie algebra of type $X_N^{(r)}$ and a fixed vertex α_k of its Dynkin diagram [24]. We would like to construct them as restrictions of ZS-AKNS hierarchy, at least in some cases. The motivation is to understand both tau function following Wilson [44] and Virasora algebra geometrically for restrictions of ZS-AKNS flows, which may have application in quantum cohomology. Terng and Uhlenbeck have succeeded in $A_n^{(1)}$-KdV case and constructed new flows gauge equivalent to classical KdV ($n = 1$) or Gel'fand-Dickiĭ ($n \geq 2$) hierarchy. This construction used an unusual splitting of the loop algebra. We have applied this new method with partial success in some cases and failed in some other cases, which leads us to think that we can generalize this new construction to all cases $(X_N^{(r)}, \alpha_k)$ where the Dynkin diagram of the affine algebra becomes simply-laced after removing the vertex α_k. This is a large project but we are in steady progress right now.

Geometries of generalized harmonic maps [6] Let U/U_0 be any k-symmetric space. The first elliptic U/U_0-system correponds to primitive harmonic map (harmonic map when $k = 2$) of a Riemann surface into U/U_0. Similarly we say the j-th elliptic U/U_0-system [35] for $j \geq 2$ correponds to generalized harmonic map into U/U_0. While the geometries of the first elliptic systems are well understood (they give rise to harmonic surfaces in symmetric spaces or Lie groups with certain isotropy order [3]), the only known surface geometry associated to the j-th elliptic U/U_0-system for $j \geq 2$ is on Hamiltonian stationary Lagrangian surfaces in 4-dimensional Hermitian symmetric spaces given by Hélein and Romon [23]. It has become very interesting to find other examples and the corresponding variational problems. Dorfmeister-Pedit-Wu's Weierstrass-type representation [12] can also be adapted to construct solutions of them. I know another special example and would like to explore this direction in depth.

[6] I would like to thank Chuu-Lian Terng and Josef Dorfmeister for suggesting this project.

Lagrangian isometric immersions into complex space forms Recall that in our study [41], flat Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$ can also be viewed as Lagrangian isometric immersions of flat real space form into flat complex space form and the complex space form of constant positive holomorphic sectional curvature. It is natural to study further flat Lagrangian submanifolds in $\mathbb{C}H^n$, the complex space form of constant negative holomorphic sectional curvature. As a comparison with Chen-Dillen-Verstraelen-Vrancken's work [9], we would also like to investigate Lagrangian isometric immersions of real space forms of constant curvature $c \neq 0$ into complex space forms. It is also interesting to apply our method of integrating the position function developed in [41] to other special submanifolds in some vector space to find new transformations.

Deformations of flat Lagrangian immersions and Egoroff nets via Higher Order Flows This is the continuation of our study on algebraic transformations of flat Lagrangian immersions and Egoroff nets [41] with C.-L. Terng. The structure equations of these geometric objects are the first order $\mathrm{U}(n)/O(n)$-system. But there is a whole hierarchy of commuting flows with higher and higher order upon this first order flow, and they thus form the hidden symmetries of these geometric objects. In this work in progress, we are trying to identify these flows as deformations of the geometric objects and compute their effects on various curvatures.

New projects in Singapore During my first month in Singapore, I have already discussed with Professor Xingwang Xu many times about new research projects. He is very interested in my work in affine differential geometry and we are collaborating now on another problem in this geometry. Moreover, we are also trying to understand Lawson's famous work on minimal surfaces in S^3 [26] from a different point of view. We have done some computation and know what we expect. This would be an interesting short paper.

In conclusion, I have been not only the best following Terng-Uhlenbeck's direction on integrable systems, but also have learned new techniques from other directions to solve an open problem: the existence of associative cones over tori of general type. I will explore further and deeper along these directions. At the same time, I will keep on learning new skills and also anticipate new collaborations from other fields in exploring this very interdisciplinary wonderland.

References

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C. and Segur, H., *The inverse scattering transform - Fourier analysis for nonlinear problems*, Stud. Appl. Math., **53** (1974), no. 4, 249-315.

[2] Bäcklund, A.V., *Concerning Surfaces with constant negative curvature*, New Era Printing Co., Lancaster, PA, 1905. (original 1883)

[3] Burstall, F.E., *Harmonic tori in spheres and complex projective spaces*, J. Reine Angew. Math. **469** (1995), 149-177.

[4] Bolton, J., Pedit, F., and Woodward, L.M., *Minimal surfaces and the affine field model*, J. Reine Angew. Math. **459** (1995), 119-150.

[5] Bryant, R., *Submanifolds and special structures on the octonians*, J. Differential Geom. **17** (1982), 185-232.

[6] Bryant, R., *Lectures given at MSRI "Integrable system seminar"*, 2003, unpublished notes.

[7] Bobenko, A.I. and Schief, W.K., *Affine spheres: discretization via duality relations*, Experiment. Math. **8** (1999), no. 3, 261-280.

[8] Bolton, J., Vrancken, L., and Woodward, L.M., *On Almost complex curves in the nearly Kähler 6-sphere*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 407-427.

[9] Chen, B.-Y., Dillen, F., Verstraelen, L. and Vrancken, L., *Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\widetilde{M}^n(4c)$*, Math. Proc. Cambridge Philos. Soc. **124** (1998), no. 1, 107-125.

[10] Carberry, E. and McIntosh, I., *Minimal Lagrangian 2-tori in $\mathbb{C}P^2$ come in real families of every dimension*, J. London Math. Soc. (2) **69** (2004), no. 2, 531-544.

[11] Carberry, E. and Wang, E., *Existence of almost complex tori in S^6*, in preparation, will be available at http://www.math.utexas.edu/~ewang.

[12] Dorfmeister, J., Pedit, F. and Wu, H., *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), no. 4, 633-668.

[13] Drinfel'd, V.G. and Sokolov, V.V., *Lie algebras and equations of Korteweg-de Vries type*, Itoginauki **24** (1984), 81-180.

[14] Dajczer, M. and Tojeiro, R., *The Ribaucour transformation for flat Lagrangian submanifolds*, J. Geom. Anal. **10** (2000), 269-280.

[15] Dajczer, M. and Tojeiro, R., *An extension of the classical Ribaucour transformation*, Proc. London Math. Soc. **85** (2002), 211-232.

[16] Donagi, Ron; Markman, Eyal, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, Integrable systems and quantum groups (Montecatini Terme, 1993), 1-119, Lecture Notes in Math., **1620**, Springer, Berlin, 1996.

[17] Dunajski, M., *Hyper-complex four-manifolds from the Tzitzéica equation*, nlin.SI/0108017.

[18] Ferus, D. and Pedit, F., *Curved flats in symmetric spaces*, Manuscripta Math., **91** (1996), 445–454

[19] Gaffet, B., *A class of 1-d gas flows soluble by the inverse scattering transform*, Physica **26**, (1984), 123-131.

[20] Griffiths, Phillip A., *Linearizing flows and a cohomological interpretation of Lax equations*, Amer. J. Math. **107** (1985), no. 6, 1445-1484 (1986).

[21] Hitchin, N., *Harmonic maps from a 2-torus to the 3-sphere*, J. Differential Geom., **31** (1990), no. 3, 627-710.

[22] Harvey, R. and Lawson, B., *Calibrated geometies*, Acta Math. **148** (1982), 47-157.

[23] Hélein, Frédéric and Romon, Pascal, *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces*, Differential geometry and integrable systems (Tokyo, 2000), 161-178, Contemp. Math., **308**, Amer. Math. Soc., Providence, RI, 2002.

[24] Kac, V.G., *Infinite dimensional Lie algebras*, third edition Cambridge University Press, 1990.

[25] Katzarkov, L. and Pantev, T., *Stable G_2 bundles and algebraically completely integrable systems*, Compositio Math. **92** (1994), no. 1, 43-60.

[26] Lawson, H. Blaine, Jr., *Complete minimal surfaces in S^3*, Ann. of Math. (2) **92** (1970), 335–374.

[27] Liu, Q.P. and Mañas, M., *Symmetric reduction of the vectorial fundamental transformation: application to the Darboux-Egorov equations*, J. Phys. A **32** (1999), 5921-5927.

[28] McIntosh, I., *Special Lagrangian cones in $\mathbb{C}^3$ and primitive harmonic maps*, J. Lond. Math. Soc. (2) **67** (2003), 769-789.

[29] McIntosh, I., *A construction of all non-isotropic harmonic tori in complex projective space*, Internat. J. Math. **6** (1995), no. 6, 831-879.

[30] McIntosh, I., *Two remarks on the construction of harmonic tori in $\mathbb{C}P^n$*, Internat. J. Math. **7** (1996), no. 4, 515-520.

[31] van Moerbeke, Pierre and Mumford, David, *The spectrum of difference operators and algebraic curves*, Acta Math. **143** (1979), no. 1-2, 93-154.

[32] Pinkall, U. and Sterling, I., *On the classification of constant mean curvature tori*, Ann. of Math., **130** (1989), no. 2, 407-451.

[33] Ribaucour, A., Comp. Rend. Acad. Sci. Paris **74** (1872) 1489.

[34] Terng, C.L., *Soliton equations and differential geometry*, J. Differential Geom. **45** (1997), no. 2, 407-445.

[35] Terng, C.L., *Geometries and symmetries of soliton equations and integrable elliptic equations*, preprint (2002)

[36] Terng, C.L., Kong, S. and Wang, E., *Associative cones and integrable systems*, Chin. Ann. Math 27B(2), 2006, 153-168.

[37] Terng, C.L. and Uhlenbeck, K., *Poisson actions and scattering theory for integrable systems*, Surveys in Differential Geometry: Integrable systems [integrable systems], 315–402, Surv. Differ. Geom., IV, Int. Press, Boston, MA.

[38] Terng, C.L. and Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure. Appl. Math. **53** (2000), no. 1, 1-75.

[39] Terng, C.L. and Uhlenbeck, K., *Geometry of solitons*, Notices Amer. Math. Soc. **47** (2000), no. 1, 17-25.

[40] Terng, C.L. and Wang, E., *Curved flats, exterior differential systems, and conservation laws*, Complex, contact and symmetric manifolds, 235–254, Progr. Math., 234, Birkhauser Boston, Boston, MA, 2005.

[41] Terng, C.L. and Wang, E., *Transformations of flat Lagrangian immersions and Egoroff nets*, **accepted**, to appear in Asian Journal of Mathematics.

[42] Tzitzéica, G., *Sur une nouvelle classe des surfaces*, C. R. Acad. Sci. Paris, **150** (1910), 955–956.

[43] Uhlenbeck, K., *Harmonic maps into Lie group: classical solutions of the Chiral model*, J. Differential Geom. **30** (1989), no. 1, 1-50.

[44] Wilson, G., *The τ-functions of the $\mathcal{G}$AKNS equations*, Integrable systems (Luminy, 1991), 131-145, Progr. Math., **115**, Birkhäuser Boston, Boston, MA, 1993.

[45] Wang, E., *Submanifold geometries and integrable systems*, Ph.D. Thesis, Northeastern University, available online http://www.math.utexas.edu/˜ewang.

[46] Wang, E., *Tzitzéica transformation is a dressing action*, J. Math. Phys. 47, 053502 (2006).

[47] Wang, E., *The Spectral Data of Finite Type Associative Cones*, preprint, submitted.

[48] Zakharov, V.E. and Shabat, A.B., *Integration of non-linear equations of mathematical physics by the inverse scattering method, II*, Funct. Anal. Appl. **13** (1979), 166-174.