Sums of Divisors and Egyptian Fractions

Charles N. Friedman
Department of Mathematics
The University of Texas
Austin, Texas 78712

Proposed running head: **Sums of Divisors**

Mailing address:
Charles N Friedman
Math. Dept. RLM 8.100
University of Texas
Austin Texas, 78712

Abstract

We discuss the problem of representing a natural number n as a sum of certain of its distinct positive proper ($\neq n$) divisors. If this is possible n is called *semiperfect*. We present a method which leads in certain cases to a verification that all abundant numbers with prime divisors limited to a fixed set of primes are semiperfect.

We then discuss the *Egyptian fraction* representation of positive rational numbers (*i.e.* the representation as a sum of reciprocals of distinct integers > 1.) In particular we obtain results relating the primes occurring in such representations to the primes occurring in the denominator of the rational being represented. This leads to further results concerning the existence of semiperfect numbers with certain prime factors.

1. Introduction

In this article, we consider the question, "When is a given natural number n equal to the sum of certain of its distinct positive proper ($\neq n$) divisors?" We call numbers satisfying this condition *semiperfect*, and the results to be discussed are all closely related to or motivated by this notion. For example, suppose that

$$D = d_1 + \cdots + d_k \tag{1.1}$$

where the d_i are distinct proper divisors of D. If we set $X_i = D/d_i$, then eq (1.1) becomes

$$1 = \frac{1}{X_1} + \cdots + \frac{1}{X_k} \, . \tag{1.2}$$

Now the reciprocal of a natural number $X > 1$ is called an *Egyptian fraction*, and we will say a rational number has an *Egyptian fraction representation* if it can be written as a sum of *distinct* Egyptian fractions. Thus, (1.2) is an Egyptian fraction representation of the number 1, and clearly such a representation gives rise to a semiperfect number, $X_1 \cdots X_k$. We consider Egyptian fraction representations later, and actually consider the representations of rational numbers other than 1, partly because it seems no more difficult to do so, but also because the theory seems interesting in itself. In particular we address the question, "Which X_i are necessary in representing some fraction whose denominator involves a given set of prime factors?"

A closely related notion is the following: A natural number n the sum of whose proper divisors is $\geq n$ but which is not semiperfect is called *weird*. All known weird numbers appear to be even; e.g. a computer search by M. Mossinghoff at University of Texas - Austin reveals no odd weirds less than 2^{32}. It seems unlikely that odd weird numbers are nonexistent, but we shall prove the nonexistence of odd weirds with certain allowed prime factors, and related results.

We introduce some notation and further definitions which will be useful. The symbol $\mathbf{p}$ will always denote a vector of increasing (but not necessarily successive) primes, i.e.

$$\mathbf{p} = (p_1, p_2, \cdots, p_k), \qquad p_1 < p_2 < \cdots < p_k, \qquad p_i \ \text{ primes.} \tag{1.3}$$

$\mathbf{p}$ is called *odd* if all the p_i are odd. (Otherwise we call $\mathbf{p}$ *even.*)
Other bold letters will denote vectors of arbitrary non-negative integers, e.g. $\mathbf{a} = (a_1, \cdots, a_k)$, and we will then use the notation

$$\mathbf{p}^{\mathbf{a}} = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_k}^{a_k} \tag{1.4}$$

(assuming, of course, that $\mathbf{p}$ and $\mathbf{a}$ are of equal length.)
Let $\sigma(n)$ denote the sum of all the positive divisors of n. Then if $\mathbf{p}^{\mathbf{a}}$ is as above,

$$\sigma(\mathbf{p}^{\mathbf{a}}) = \prod_{i=1}^{k} (1 + p_i + \cdots + p_i^{a_i}) \qquad (1.5).$$

Recall that n is called *abundant* if $\sigma(n) \geq 2n$. We define a vector $\mathbf{p}$ as above to be abundant provided *some* number of form $\mathbf{p}^{\mathbf{a}}$ is abundant. By dividing $\sigma(\mathbf{p}^{\mathbf{a}})$ by $\mathbf{p}^{\mathbf{a}}$ in eq (1.5) and letting $a_i \to \infty$, we see that a *sufficient* condition for $\mathbf{p}$ to be abundant is that

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \geq 2. \tag{1.6}$$

We define a vector $\mathbf{p}$ to be weird if $\mathbf{p}$ is abundant and every abundant number of form $\mathbf{p}^{\mathbf{a}}$ is weird. It seems unlikely that odd weird $\mathbf{p}$ exist; we discuss this later, and deduce from our results on Egyptian fractions that odd weird $\mathbf{p}$ with certain allowed p_i do not exist.

Several of our results on representation by Egyptian fractions are improvements of previous results of other authors on the subject (see e.g. the applications to Egyptian fractions in [**3**]).

2. Minimal Abundants and Non-existence of Weirds

In this section we describe how it may be shown for certain $\mathbf{p}$ that no weird numbers of form $\mathbf{p}^{\mathbf{a}}$ exist, i.e. that all abundant such numbers are semiperfect. For example, it will be seen that all abundant numbers of form $3^a 5^b 7^c$ are semiperfect. The idea is rather simple. First we define a *minimal abundant* to be an abundant number none of whose proper divisors is abundant. Then every abundant is a multiple of a minimal abundant; in particular, every weird number is a multiple of a minimal abundant. If we show that every minimal abundant of form $\mathbf{p}^{\mathbf{a}}$ for some fixed $\mathbf{p}$ is semiperfect, then there can be no weirds of this form, since it is trivial that a multiple of a semiperfect number is again semiperfect. The crucial relevant circumstance is that the number of minimal abundants with fixed $\mathbf{p}$ is always *finite.* This is a consequence of the following theorem.

Theorem 2.1 Let $\mathbb{N}$ denote the natural numbers, and consider the lattice $\mathbb{L} = \mathbb{N}^k$ with the partial ordering $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i, i = 1 \cdots k$. Then any subset of $\mathbb{L}$ contains at most finitely many minimal elements.

Remarks: It is easy to see that even when $k = 2$ there is no *a priori* bound on the number of minimal elements which is independent of the subset in question. The application to minimal abundants with fixed $\mathbf{p}$ involves the subset of $\mathbb{L}$ consisting of those $\mathbf{a}$ for which $\mathbf{p}^{\mathbf{a}}$ is abundant; in this situation one might look for a bound on the number of minimal $\mathbf{a}$ depending on $\mathbf{p}$ in some understandable way, but we know of no such result.

Proof of the theorem: Suppose that we had an infinite sequence of minimal elements of some subset of $\mathbb{L}$. We could then find a subsequence with the property that some fixed component of the elements of the subsequence tends to ∞. The set of all remaining components of the elements of our subsequence cannot be bounded (hence finite), otherwise our elements could not all be minimal. Hence we may extract a subsequence of our subsequence for which two fixed components both tend to ∞. Continuing in this way we eventually get a subsequence of minimal elements of our subset for which *all* components tend to ∞. This is evidently inconsistent with the minimality of these elements. ∎

Examples; Some examples of abundant $\mathbf{p}$ are $(3, 5, 7)$, $(3, 7, 11, 13)$, and $(5, 7, 11, 13, 17, 19, 23)$, etc. Consider first the case of $(3, 5, 7)$. A simple computation shows that the minimal abundants involving $3, 5, 7$ are $3^3 \cdot 5 \cdot 7$, $3^2 \cdot 5^2 \cdot 7$, and $3^2 \cdot 5 \cdot 7^2$. It is easy to check that these are all semiperfect. Consider, for example, $3^3 \cdot 5 \cdot 7 = 945$. It is not difficult to see that 945 can be written as a sum of certain of its proper divisors; note that $\sigma(945) - 945 = 1920 - 945 = 975$ can be written as a sum of proper divisors of 945 as a consequence of the definition of σ, and $975 - 945 = 30 = 3 \cdot 7 + 5 + 3 + 1$ is also a sum of proper divisors of 945, and therefore 945 can also be represented in this way. The same technique can be applied to the remaining two minimal abundants involving $3, 5, 7$. One could also invoke the non-existence of odd weirds $\leq 2^{32}$ mentioned in the previous section. One may proceed in a similar way with numbers involving exactly the prime factors $3, 7, 11, 13$. In this case the minimal abundants are $3^4 \cdot 7 \cdot 11 \cdot 13$, $3^3 \cdot 7^2 \cdot 11 \cdot 13$, $3^3 \cdot 7 \cdot 11^2 \cdot 13$, $3^3 \cdot 7 \cdot 11 \cdot 13^2$, $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^3$, $3^2 \cdot 7^2 \cdot 11^3 \cdot 13^2$. As before one checks easily that these are all semiperfect.

Clearly the method we have used is of rather limited usefulness. The major drawback is the difficulty in finding the minimal abundants involving a fixed set of primes; this is elementary, but laborious. In addition, it seems unlikely that any information concerning general $\mathbf{p}$ or even infinitely many $\mathbf{p}$ may be deduced. Unfortunately, we know of no other way to obtain results concerning *all* numbers of form $\mathbf{p}^{\mathbf{a}}$ for some fixed $\mathbf{p}$. For this reason, we turn in the next section to results which are weaker, but more general in a sense. In particular, we ask, "Given fixed $\mathbf{p}$, when are *some* numbers of form $\mathbf{p}^{\mathbf{a}}$ semiperfect?"

2. Semiperfect Numbers and Egyption Fractions

It appears plausible that if $\mathbf{p}$ is abundant, then there are always semiperfect numbers of form $\mathbf{p}^{\mathbf{a}}$. We remark that this is certainly true if $\mathbf{p}$ is even, albeit due to a rather special circumstance. Namely, one sees easily that if p is an odd prime, then $2^k p$ is semiperfect provided $2^{k+1} \geq p + 1$. (This is also exactly the condition for $2^k p$ to be abundant.) To verify this, we simply note that $2^k p = (1 + 2 + \cdots + 2^{k-1})\, p + p$; the terms excluding the last p are a sum of distinct proper divisors of $2^k p$, and p can be written (binary expansion) as a sum of some of the divisors $1, 2, \cdots, 2^k$ provided the cited condition on k holds. Since a positive multiple of a semiperfect number is again obviously semiperfect, the possibility of even weird $\mathbf{p}$ is ruled out.

We remark also that there are many even weird numbers, the first of which is 70. To get further examples, one will, according to the previous remark, have to use other odd prime factors, but it is easy to see directly that there are infinitely many even weirds. In this regard, suppose that n is not semiperfect, and p is a prime $> \sigma(n)$. A sum of distinct proper divisors of $p \cdot n$ is of form $p\Sigma_1 + \Sigma_2$ where the Σ_i are (possibly empty) sums of divisors (proper in Σ_1) of n. But $p \cdot n - (p\Sigma_1 + \Sigma_2) = p\,(n - \Sigma_1) - \Sigma_2$; since the right hand side of this equation evidently cannot vanish if n and p have the properties stated, it follows that $n \cdot p$ is not semiperfect either. In particular, if n is weird, $p \cdot n$ will also be weird if p is a sufficiently large prime. (One might attempt to construct an odd weird by multiplying an odd n which is not abundant by some large p, as above, but the result, although not semiperfect will also fail to be abundant.)

Leaving the case of even $\mathbf{p}$, suppose that we have an odd, abundant $\mathbf{p}$. Why might we expect some $\mathbf{p^a}$ to be semiperfect? First we note that if $\mathbf{p} = (p_1, \cdots, p_k)$, then

$$\sum\nolimits_{\mathbf{a}\geq \mathbf{0}} \frac{1}{\mathbf{p^a}} = \prod\nolimits_{i=1}^{k} (1 + \frac{1}{p_i} + \cdots + \frac{1}{{p_i}^l} + \cdots) = \prod\nolimits_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{3.1}$$

(This much is true whether $\mathbf{p}$ is abundant or not.) If we denote by $1 < X_1 < X_2 < \cdots < X_i < \cdots$ the numbers > 1 of form $\mathbf{p^a}$,taken in increasing order, then the finite sums of the form $\sum' \frac{1}{X_i}$ (the $'$ indicates that the sum is over some finite set of distinct indices i) lie in the interval $[0, R]$ where $R = \prod_i \frac{p_i}{p_i - 1} - 1$. If $\mathbf{p}$ is abundant, then $R \geq 1$ and these finite sums $\sum' \frac{1}{X_i}$ are *dense* in the interval $[0, R]$. To verify this, we first note that the condition $R \geq 1$ follows from the fact that

$$2 \leq \frac{\sigma(\mathbf{p^a})}{\mathbf{p^a}} \leq \prod\nolimits_{i=1}^{k} \frac{p_i}{p_i - 1} \qquad \text{(for } \mathbf{p^a} \text{ abundant).} \tag{3.2}$$

Hence, for each $i = 1, \cdots$, we have

$$\frac{1}{X_i} \leq \frac{1}{X_i} \cdot \sum\nolimits_{j\geq 1} \frac{1}{X_j} = \sum\nolimits_{j\geq 1} \frac{1}{X_i X_j} \leq \sum\nolimits_{j\geq i+1} \frac{1}{X_j}. \tag{3.3}$$

Then if we wish to approximate some $\xi \in [0, R]$ by sums $\sum' \frac{1}{X_i}$, we simply add terms $\frac{1}{X_i}$ which are as large as possible so as to never exceed ξ; the inequality just noted ensures that whenever we must skip some term(s) $\frac{1}{X_i}$ which would make our sum exceed ξ, the remaining terms have a sufficiently large sum that we may continue the approximation until further term(s) must be skipped; since these skipped terms decrease to 0, we may form sums $\sum' \frac{1}{X_i}$ which are arbitrarily close to ξ. This density result lends credence to the possibility of writing $1 = \sum' \frac{1}{X_i}$, and then the product of the X_i appearing as denominators in the sum is a semiperfect number of form $\mathbf{p^a}$.

In fact, we will show that much more than this is true for many abundant $\mathbf{p}$. Namely, it will often be the case that *every* rational of form $\frac{A}{\mathbf{p^a}} \in [0, R)$ (where A is a natural number) can be written as such a finite sum. We remark first that in trying to construct such an Egyptian Fraction representation, we may assume our rational is arbitrarily small in magnitude since we may arrive at this case after subtraction of some appropriate terms. For example, we may find an index j such that

$$0 \le \frac{B}{\mathbf{p^b}} = \frac{A}{\mathbf{p^a}} - \sum_{i=1}^{j-1} \frac{1}{X_i} < \frac{1}{X_j}. \tag{3.4}$$

(In (3.4), j is the first index for which $\sum_{i=1}^{j} \frac{1}{X_i} > \frac{A}{\mathbf{p^a}}$.) Then we only need to find a representation of $\frac{B}{\mathbf{p^b}}$, and it follows from the last inequality in (3.4) that the terms $\frac{1}{X_i}$ that we use in representing $\frac{B}{\mathbf{p^b}}$ will necessarily be distinct from those already used in (3.4). Evidently, we may subtract several sums of successive $\frac{1}{X_i}$ from $\frac{A}{\mathbf{p^a}}$ and produce a non-negative $\frac{B}{\mathbf{p^b}}$ which is $< \frac{1}{X_j}$ for some index j not yet used. Hence we may assume that the original rational $\frac{A}{\mathbf{p^a}}$ that we wish to represent is as small as we wish; in fact we shall need a condition of the form

$$0 \le \frac{A}{\mathbf{p^a}} < \frac{2}{p_1^{r-1}} \tag{3.5}$$

where r is a positive integer to be determined later. We remark that since we can multiply (3.5) by $\frac{\mathbf{p^b}}{\mathbf{p^b}}$, we may assume that the exponont $\mathbf{a}$ in (3.5) is as large as we wish (in the ordering defined in Theorem 2.1.)

Assuming now that $\frac{A}{\mathbf{p^a}}$ satisfies a condition of the form in (3.5) , we describe an algorithm which will in certain cases lead to a an Egyptian Fraction representation

$$\frac{A}{\mathbf{p^a}} = \frac{1}{D_1} + \cdots + \frac{1}{D_n}, \qquad D_i \text{ distinct} \tag{3.6}$$

where the D_i are in fact each of the form $\mathbf{p}^{\mathbf{b}_i}$ ($\mathbf{p}$ being identical for all terms of (3.6), of course.) Our basic strategy is as follows. First we look for a divisor D of $\mathbf{p^a}$ which satisfies

$$\frac{A}{2} < D \le A. \tag{3.7}$$

If such a D exists, then we have

$$\frac{A}{\mathbf{p^a}} - \frac{1}{\mathbf{p^a}/D} = \frac{A'}{\mathbf{p^a}}, \qquad 0 \le A' = A - D < \frac{A}{2}. \tag{3.8}$$

We thus have decreased the numerator A of our fraction to A', and we then repeat this process as long as possible. Notice that because $A' < A/2$, the successive reciprocals that we subtract from our original fraction are *distinct.* When can our procedure fail? Suppose, for example that there exists no divisor D of $\mathbf{p}^{\mathbf{a}}$ satisfying (3.7). Then for every positive divisor d of $\mathbf{p}^{\mathbf{a}}$ we have $d \leq \frac{A}{2}$ or $d > A$, and so if we denote the positive divisors of $\mathbf{p}^{\mathbf{a}}$ by $d_1 < d_2 < \cdots < d_j$, we must have

$$d_1 < \cdots < d_i \leq \frac{A}{2} < A < d_{i+1} < \cdots < d_j. \tag{3.9}$$

(Note that A is less than some divisor of $\mathbf{p}^{\mathbf{a}}$ by (3.5).) In particular, we have then

$$2d_i \ \leq A \ < \ d_{i+1}. \tag{3.10}$$

Of course, (3.10) can hold if $2d_i = 2d_1 = 2 \leq A < p_1 = d_2$; we now state a condition on $\mathbf{p}$ which forces this to be the only situation for which (3.10) and (3.5) may hold.

Lemma 3.1: Suppose that $p_1 < p_2 < \cdots < p_k$ are *successive* (but $p_1 \neq 2$ necessarily) primes satisfying the condition

$$p_k \ < \ p_1^r \ < 2p_k \qquad \text{, some integer } r. \tag{3.11}$$

Suppose as above that $d_1 < \cdots < d_j$ are the positive divisors of $\mathbf{p}^{\mathbf{a}}$, where all $a_i > 0$ and $a_1 \geq r$. Then if A is a positive integer satisfying

$$2d_m \ \leq A \ < d_{m+1}\ , \quad \text{some } m$$

we have

$$\text{either} \ \ A \ < \ p_1 \qquad \text{or} \qquad \frac{A}{\mathbf{p}^{\mathbf{a}}} \ \geq \ \frac{2}{p_1^{r-1}}.$$

Remark: If the successive p_i don't satisfy (3.11), then we can always enlarge to a set of successive primes for which (3.11) does hold. In fact, let r_0 be the first integer for which $p_k < p_1^{r_0}$. If p_l is the largest prime $< p_1^{r_0}$, then $p_l < p_1^{r_0} < 2p_l$ by Bertrand's theorem ([**2**]). In this case, the successive primes $p_1 < \cdots < p_l$ satisfy the hypotheses of the lemma; we note that $p_l < p_1^{r_0} < p_1 p_1^{r_0 - 1} < p_1 p_k$.

Proof of the lemma: Put $d = d_m = \mathbf{p}^{\mathbf{b}} = p_1^{b_1} \cdots p_k^{b_k}$. We may assume that no pair b_i, b_{i+1} satisfies $b_i > 0$ and $b_{i+1} < a_{i+1}$, otherwise we may produce another divisor d' of $\mathbf{p}^{\mathbf{a}}$ with $d < d' < 2d$, by putting $d' = p_{i+1}p_i^{-1}d$ (using Bertrand's theorem again.) So d must be of the form

$$d = p_1^0 \cdots p_i^0 p_{i+1}^{b_{i+1}} p_{i+2}^{a_{i+2}} \cdots p_k^{a_k}. \tag{3.12}$$

If $i = k$, so $d = 1$, then $A < d_2 = p_1$. Otherwise, p_k must appear to a positive power. We may then produce another divisor d'' of $\mathbf{p}^{\mathbf{a}}$ with $d < d'' < 2d$ by putting $d'' = p_1^r p_k^{-1} d$ *unless* $i + 1 = 1$ and $b_1 + r > a_1$. In the latter case, $b_1 - a_1 > -r$, so $\frac{A}{\mathbf{p}^{\mathbf{a}}} \geq \frac{2d}{\mathbf{p}^{\mathbf{a}}} \geq \frac{2}{p_1^{r-1}}$. ∎

From the above results we may deduce the following:

Theorem 3.1: Suppose that $\mathbf{p} = (p_1, p_2, \cdots, p_k)$ is a fixed vector of successive primes which is abundant and satisfies (3.11) for some r. Let $R = \prod_i \frac{p_i}{p_i - 1} - 1$. Suppose that for each integer ξ satisfying $1 < \xi < p_1$, an equation of the following form holds:

$$\xi \mathbf{p}^{\mathbf{b}} = \mathbf{p}^{\mathbf{c}_1} + \cdots + \mathbf{p}^{\mathbf{c}_j} \tag{3.13}$$

where $\mathbf{p}^{\mathbf{b}} > 1$ and the $\mathbf{c}_i$ are distinct. ($\mathbf{b}$, the $\mathbf{c}_i$ and j may depend on ξ.) Then every rational number X of the form $\frac{A}{\mathbf{p}^{\mathbf{a}}}$ with $X \in [0, R)$ has an Egyptian fraction representation

$$X = \frac{1}{X_1} + \cdots + \frac{1}{X_n}, \qquad X_i \text{ distinct, of form } \mathbf{p}^{\mathbf{a}_i}. \tag{3.14}$$

Remarks: In particular cases, especially if p_1 is not too large, it is often quite easy to check that equations of form (3.13) hold. The theorem then gives information concerning *infinitely* many rational X. It would be nice to be able to remove the conditions (3.13) from the statement of the theorem, but it seems unlikely that they hold automatically as a consequence of the other hypotheses on $\mathbf{p}$ (and as will be seen in the proof of the theorem, these conditions are certainly necessary.)

We note also that if $p_1 = 2$, then there are no conditions of form (3.13), and furthermore (3.11) is clearly satisfied if r is the smallest integer satisfying $p_k < 2^r$.

Proof of the Theorem: As we have remarked earlier, by subtracting appropriate terms from X, we may assume that (3.5) is satisfied; we may also assume that $\mathbf{a}$ is as large as we wish, in particular that it has positive components and $a_1 \geq r$. Lemma 3.1 then implies that we may repeatedly apply the algorithm described immediately preceeding that lemma until we are left with the problem of representing a rational of form $\frac{A'}{\mathbf{p^a}}$ where $A' < p_1$. But the equations (3.13) imply that $\frac{A'}{\mathbf{p^d}}$ has an Egyptian fraction representation provided $\mathbf{d}$ is sufficiently large (divide (3.13) by $\mathbf{p^{b+c_1+\cdots+c_j}}$.) Since we may assume that $\mathbf{a}$ was initially as large as we wish, the proof is complete. ∎

We remark that in applications of the theorem to specific cases, when we subtract terms from X in order to ensure that (3.5) holds, it often occurs that the resulting rational is actually *much* smaller than indicated by (3.5). (See (3.4) and the following discussion.)

We also mention that R.L.Graham ([**1**]) has proved a result which would imply the result of theorem 3.1 for any $\mathbf{p}$ satisfying the hypotheses stated in the theorem and *also* the following "completeness" condition: every sufficiently large natural number can be written as a sum of distinct terms of the form $\mathbf{p^a}$. However, it is not easy to see when this condition holds; for the cases $\mathbf{p} = (3, 5, 7)$ and $\mathbf{p} = (5, 7, 11, 13, 17, 19, 23)$, completeness does hold, but the proof seems to require an analysis similar to (although more tedious than) the one we have followed to prove theorem 3.1. (We give this reasoning in the appendix.) In general, the completeness is unclear. Even if completeness were known to hold for all $\mathbf{p}$ satisfying the hypotheses of theorem 3.1, Graham's argument is considerably more complicated than the one we have given. (It does give some very interesting results in other situations, however.)

Examples: We give some examples of $\mathbf{p}$ satisfying the hypotheses of the theorem.

(i.) Any $\mathbf{p}$ consisting of at least two successive primes, and with $p_1 = 2$. (This already implies that any positive rational has an Egyptian fraction representation, since we may always subtract distinct reciprocals of successive integers ≥ 2 from this rational until the resulting rational is positive but less than the last reciprocal subtracted. Then we may multiply the numerator and denominator by some appropriate factor which produces a denominator of form $\mathbf{p^a}$ where $\mathbf{p}$ is of the type just stated. The existence of a representation can also be proved directly by an elementary argument.)

(ii.) $\mathbf{p} = (3, 5, 7)$. (3.11) clearly holds (with $r = 2$), and for (3.13) we observe that

$2 \cdot 3 = 1 + 5$. (This gives the representation $2/5 = 1/3 + 1/15$.) As an example, suppose we wish to find a representation of 1 using only the primes 3, 5, 7. First we subtract appropriate reciprocals from 1: We find that $1 - 1/3 - 1/5 - 1/7 - 1/9 - 1/15 - 1/21 - 1/27 - 1/35 - 1/45 = 10/945 < 1/63$. (This is exactly (3.4) for the present case.) Then we write $10/945 = 9/945 + 1/945 = 1/105 + 1/945$. Hence,

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{45} + \frac{1}{105} + \frac{1}{945} \tag{3.15}$$

If we multiply (3.15) by 945, we get a representation of 945 as a sum of distinct divisors. It is easy to see that this is the smallest odd, semiperfect number.

(iii.) Any abundant $\mathbf{p}$ with $p_1 = 3$, consisting of successive primes and satisfying (3.11). ((3.13) has already been verified in (ii.).) From this fact, and an argument similar to the one sketched in (i.) above, we may conclude that any positive rational with an *odd* denominator has an Egyptian fraction representation consisting of reciprocals of *odd* integers ≥ 2. (This particular result is also deduced in [**3**], however as a corollary of an argument which is more complicated than that which we have used to prove theorem 3.1.)

(iv.) $\mathbf{p} = (5, 7, 11, 13, 17, 19, 23)$. (3.11) holds with $r = 2$, and for (3.13) we observe that $2 \cdot 7 = 1 + 13$, $3 \cdot 7 = 1 + 7 + 13$, $4 \cdot 7 = 5 + 23$. Clearly the same argument applies to $\mathbf{p} = (5, 7, 11, 13, 17, 19, 23, ..., p_k)$ provided the primes in $\mathbf{p}$ are successive and (3.11) holds. (We have seen that we can always enlarge $\mathbf{p}$ so that (3.11) holds for the enlarged $\mathbf{p}$.)

Theorem 3.1 gives information concerning the Egyptian fraction representations of an arbitrary positive rational number X. For we may always multiply the numerator and denominator of X by appropriate factors so as to have $X = \frac{A}{\mathbf{p}^{\mathbf{a}}}$, and such that the hypotheses of the theorem are satisfied. (This follows from the above examples. Of course, if we wish to add as few new primes as possible to the primes initially in the denominator of X, then it is useful to know many $\mathbf{p}$ satisfying the hypotheses of the theorem.)

We may also deduce from theorem 3.1 the following

Corollary 3.1: Suppose $\mathbf{p}$ satisfies the hypotheses of theorem 3.1. Then $\mathbf{p}$ is not weird; *i.e.*, there are semiperfect numbers of form $\mathbf{p}^{\mathbf{a}}$. Equivalently, there is a positive integer e depending only on $\mathbf{p}$, such that if $n = \mathbf{p}^{\mathbf{a}}$ with $a_i > 0$ for all i, then n^e is semiperfect.

Proof: The existence of semiperfect numbers of form $\mathbf{p^a}$ follows immediately from the fact that according to the theorem, 1 has an Egyptian fraction representation (3.14). The equivalence of the final statement is a consequence of the facts that the exponents occuring in the prime power factorization of n^e may be made as large as desired by taking e large and that a multiple of a semiperfect number is again semiperfect. ∎.

The corollary doesn't rule out the possibility that there are odd weird numbers of form $\mathbf{p^a}$,with $\mathbf{p}$ satisfying the hypotheses of theorem 3.1, and there seems to be no strong evidence that none exist. On the other hand, the density result discussed following (3.1) (and also theorem 3.1) indicate that weird $\mathbf{p}$ are likely to be much harder to find. We note that the non-existence of weird $\mathbf{p}$ is easily seen to be equivalent to the following condition: For each abundant n there is a positive integer e, depending on n, such that n^e is semiperfect. (We have already seen that this is true if n is even.)

4. Appendix

In this section we discuss briefly the the "completeness" question discussed following the proof of theorem 3.1. Suppose $\mathbf{p}$ satisfies the hypotheses of the theorem, and let

$$1 < d_1 < \cdots < d_j < \cdots \tag{4.1}$$

be the increasing sequence of *all* numbers of form $\mathbf{p^a}$ with $a_i \geq 0$. It follows from the argument given in lemma 3.1 that

$$d_{j+1} < 2d_j \ , \qquad j \geq 1. \tag{4.2}$$

From this fact it follows that if $n \geq p_1$, we may find some d_j with $n/2 < d_j \leq n$. If $n' = n - d_j \geq p_1$, we do the same for n', and by repeating this process sufficiently many times (this is essentially just the algorithm described in (3.7), (3.8)), we see that we may write any non- negative n in the form

$$n = \xi + {\sum}' \mathbf{p^a} \ , \quad \xi = 0 \text{ or } 2 \leq \xi < p_1 \tag{4.3}$$

where the notation $\sum{}'$ indicates that the terms in the sum are distinct. Now pick some fixed D of form $\mathbf{p}^{\mathbf{a}}$. If N is a positive integer we can write $N = Dk+r, 0 \le r < D$, and if N is sufficiently large we we may write this last equation in the form $N = D(k-j)+Dj+r,\ 0 \le r < D,\ k-j \ge 0$. We may then write $k-j$ in the form given in (4.3). This yields

$$N = Dj + D\xi + r + D \cdot \sum{}' \mathbf{p}^{\mathbf{a}} \quad , 0 \le r < D, \quad \xi = 0 \text{ or } 2 \le \xi < p_1. \tag{4.4}$$

It follows that if for each r satisfying $0 \le r < D$, there is a j (depending on r only), such that for all ξ with $\xi = 0$ or $2 \le \xi < p_1$, $Dj + D\xi + r$ can be written as a sum of distinct terms of form $\mathbf{p}^{\mathbf{a}}$, *none of which are divisible by* D, then every sufficiently large N can be written as a sum of distinct terms of form $\mathbf{p}^{\mathbf{a}}$. (The condition that the terms used to represent $Dj + D\xi + r$ are not divisible by D is necessary to ensure that these terms do not duplicate any of those occurring in the term $D \cdot \sum{}' \mathbf{p}^{\mathbf{a}}$ in (4.4).)

We can apply this result to the case $\mathbf{p} = (3, 5, 7)$. We take $D = 3$, and then for each of $r = 0, 1, 2$, we must exhibit a j such that for $\xi = 0, 2$, each $Dj + D\xi + r$ can be written as sum of distinct terms of form $5^a 7^b$. For $r = 0$, we take $j = 2$; then $Dj+D\xi = 6+3\xi = 6$ or 12, and $6 = 1+5$, $12 = 5+7$. For $r = 1$, we take $j = 2$ again; then $Dj+D\xi+r = 7+3\xi = 7$ or 13, and $13 = 1+5+7$. Finally, for $r = 2$, we take $j = 10$; then $Dj+D\xi+r = 32+3\xi = 32$ or 38, and $32 = 5^2 + 7$, $38 = 1 + 5 + 5^2 + 7$. This proves that any sufficiently large natural number N can be written as a sum of distinct terms of the form $3^a 5^b 7^c$, and some easy checking shows that this is actually true for all $N > 2$.

An analogous calculation may be successfully applied to the case $\mathbf{p} = (5, 7, 11, 13, 17, 19, 23)$. In further cases, the calculations become very tedious. We intend to take up this and related questions in a future article.

References

1. R.L.Graham, 'On finite sums of unit fractions', *Proc. London Math. Soc.* 14 (1964) 193-207.

2. G.H.Hardy, E.M.Wright, *The Theory of Numbers*, 4th ed. (New York: Oxford University Press, 1960).

3. B.M.Stewart, 'Sums of distinct divisors', *Am. J. Math.* 76 (1954) 779-785.

List of Symbols

pg 1:

$\mathbf{p}$ bold-face p

pg 2:

$\mathbf{p^a}$ bold-face p with bold-face a as exponent

$\prod_{i=1}^{k}$ product symbol with sub and superscripts

σ Greek “sigma”

pg 3:

$\mathbb{N}$, $\mathbb{N}^k$, $\mathbb{L}$ bold-face outlined letters

$\mathbf{a}$, $\mathbf{b}$ bold-face letters

pg 5:

$\prod$ product symbol

$\sum_1$, $\sum_2$ sum symbols with subscripts

$\sum'$ sum symbol with prime

pg 8:

ξ greek “xi”