An Existence and Uniqueness Result of a Nonlinear Two-Dimensional Elliptic Boundary Value Problem

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Abstract

We consider a boundary value problem for the generalized two-dimensional flow equation $\Delta\varphi = \nabla\varphi \cdot \vec{h}$ for $\vec{h}$ a C^α vector field, where the speed is prescribed on a part of the boundary. By using Bers theory combined with elliptic operator theory in nonsmooth domains, we show existence and uniqueness of a $C^{2,\alpha}$ solution with nonvanishing gradient, and we find positive lower and upper bounds for $|\nabla\varphi|$ along with $C^{2,\alpha}$ estimates of φ, in terms of the C^α and L^∞ norms of $\vec{h}$. © 1995 John Wiley & Sons, Inc.

1. Introduction

When studying compressible potential flow, the following equation arises naturally:

Given a strictly positive $C^{1,\alpha}$ density ρ in a flow region Ω, find the flow potential function φ with nonvanishing gradient that satisfies the equation

$$\operatorname{div}(\rho\nabla\varphi) = 0$$

with boundary conditions given by $\nabla\varphi \cdot n = 0$, n the outer unit normal to the boundary, along the lateral walls, φ constant at the inflow boundary, and the speed $|\nabla\varphi|$ prescribed on the remaining boundary section.

We are able to solve uniquely this problem in two dimensions by using the Bers theory for the flow equation.

In fact, for our application in compressible hydrodynamics (see [6]), we need to consider a general C^α, $0 < \alpha < 1$, vector field $\vec{h}$, instead of $\nabla \ell n\rho$, and we must obtain existence of a unique $C^{2,\alpha}$ solution φ with positive lower bound for the speed $|\nabla\varphi|$, along with a $C^{2,\alpha}$ estimate for φ that has linear growth in the C^α norm of h.

Let Ω be a domain in two dimensions with a $C^{2,\alpha}$ boundary except for four points called w_i, $i = 1, \dots, 4$.

Let $\partial_i\Omega$ be the $C^{2,\alpha}$ component of $\partial\Omega$ and let Ω satisfy the condition that $\overline{\partial_i\Omega} \cap \overline{\partial_{i+1}\Omega}$ meet at points w_i forming a $\pi/2$ angle. Thus Ω is a "smooth rectangular" domain. See Figure 1.

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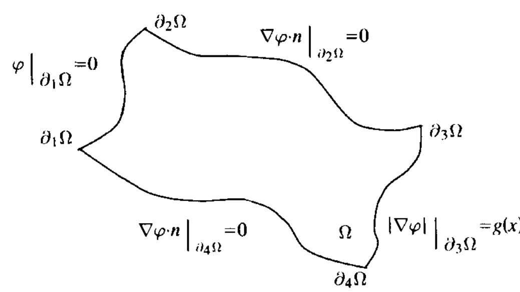

Figure 1. Domain Ω and boundary conditions on φ.

We consider the following elliptic problem: given $\vec{h}=(h_1,h_2)$ a $C^\alpha(\bar\Omega)$ function, $0<\alpha<1$,

$$(1.1)\qquad \begin{cases} \Delta\varphi = \nabla\varphi\cdot\vec{h} & \text{in } \Omega \\ \varphi\Big|_{\partial_1\Omega}=0 \text{ and } \nabla\varphi\cdot n\Big|_{\partial_1\Omega}(w_1)<0 & \text{inflow conditions} \\ \dfrac{\partial\varphi}{\partial n}\Big|_{\partial_2\Omega\cup\partial_4\Omega}=0 & \text{streamline walls} \\ |\nabla\varphi|\Big|_{\partial_3\Omega}=g(x) & \text{prescribed speed} \end{cases}$$

Here n denotes the exterior normal to $\partial\Omega$ and $g(x)$ is a $C^{1,\alpha}(\partial_3\Omega)$ strictly positive function.

We show the following theorem.

THEOREM 1. *There exists a unique solution $\varphi\in C^{2,\alpha}(\bar\Omega)$ of the boundary value problem* (1.1) *with nonvanishing gradient. In addition, the solution φ satisfies*

$$(1.2)\qquad \begin{gathered} \|\varphi\|_{C^{1,\alpha}(\bar\Omega)} \leqq K e^{K\|h\|_{\infty,\bar\Omega}}\left(1+\|h\|_{\infty,\bar\Omega}\right) \\ \textit{and} \\ \|\varphi\|_{C^{2,\alpha}(\bar\Omega)} \leqq K e^{K\|h\|_{\infty,\bar\Omega}}\left(1+\|h\|_{C^\alpha(\bar\Omega)}\|h\|_{\infty,\bar\Omega}+\|h\|^2_{\infty,\bar\Omega}\right)\left(1+\|h\|_{\infty,\bar\Omega}\right) \end{gathered}$$

where K depends on Ω and on the $C^{1,\alpha}$-norm of g; and the following estimates hold,

$$0<ke^{-KH}\leqq|\nabla\varphi|\leqq Ke^{KH}, \quad \textit{and} \quad -\widetilde{K}\leqq\varphi\leqq\widetilde{K} \quad \textit{on } \bar\Omega\,,$$

where k and K depend on Ω and $g(x)$, H depends on the upper bound for $|\vec{h}|$. $\widetilde{K}$ depends on the diameter of Ω and on the upper bound for $|\nabla\varphi|$.

Moreover, $\partial_1\Omega$ is an inflow boundary since

$$\nabla\varphi \cdot n < 0 \qquad on \quad \partial_1\Omega$$

We show in the Appendix that $\bar{\Omega}$ can be uniquely conformally transformed into a fixed rectangle R such that the four boundary sections $\partial_i\Omega, i = 1,4$ are transformed into the sides of R, in the same order.

Thus, we prove Theorem 1 using the two-dimensional complex representation of solutions of the equation $\Delta\varphi = \nabla\varphi \cdot \vec{h}$ (Bers-Nirenberg representation), and the following *key lemma* in the transformed domain R:

LEMMA 1.2. *There exists a* unique *weak solution b in $C^{1,\alpha}(\bar{R})$ for the problem*

$$(1.3) \qquad \begin{cases} \operatorname{div}(\nabla b - f(\mathbf{x}, b)) = 0 & in\ R \\ b = 0 & in\ \Gamma \\ b_n = f(\mathbf{x}, b) \cdot n & in\ \Gamma_1 \end{cases} .$$

where n denotes the outer normal to ∂R, and, the function $f(\mathbf{x}, b) = (f_1, f_2)(\mathbf{x}, b)$ is bounded, $C^\alpha(\bar{R})$ in $\mathbf{x}$ and Lipschitz as function of b.

The domain R is a rectangle where $\partial R = \Gamma \cup \Gamma_1$ with Γ_1 one side of R and $f(\mathbf{x}, 0) \cdot n = 0$ in $\overline{\partial_2 R} \cup \overline{\partial_4 R}$, where $\partial_2 R$ and $\partial_4 R$ are the sides of R adjacent to Γ_1 (compatibility condition for regularity up to the boundary). See Figure 2.

Note: b is a $C^{1,\alpha}(\bar{R})$ weak solution if

$$(1.4) \qquad \int_R \nabla\varphi(\nabla b - f(\mathbf{x}, b)) = 0 \qquad for\ all\ \ \varphi \in H_0^1(\mathbb{R}^2 \setminus \Gamma) ,$$

where $H_0^1(\mathbb{R}^2 \setminus \Gamma)$ is the closure of $C_0^1(\mathbb{R}^2 \setminus \Gamma)$. That means the boundary condition on Γ_1 is a natural boundary condition for weak solutions in the sense (1.4).

In addition, if we denote $\|f\|_{C^\alpha,\mathbf{x},\bar{R}}$ the $C^\alpha(\bar{R})$-norm of f with respect to the first variable and $\|f\|_{\mathrm{Lip},b}$ denotes the Lipschitz norm of f with respect to the second variable, the following estimates hold:

$$(1.5) \qquad \|b\|_{C^{1,\alpha}(\bar{R})} \leqq C\left\{\|f\|_{C^\alpha,\mathbf{x},\bar{R}} + \|f\|_{\mathrm{Lip},b}\right\} \|f\|_{\infty,\bar{R}}$$

and

$$(1.6) \qquad \|b\|_{C^\alpha(\bar{R})} \leqq C\|f\|_{\infty,\bar{R}} ,$$

where C depends on R.

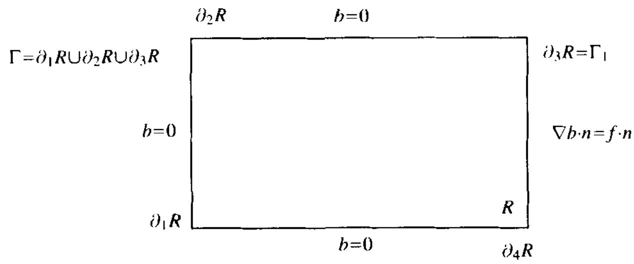

Figure 2. Domain F and boundary conditions on b.

Section 3 deals with the proof of Lemma 1.2.

Remark 1. The nature of estimate (1.2) is due to the boundary conditions from (1.1) and (1.3). Otherwise, the right-hand side of (1.2) might not have linear growth in $\|h\|_{C^\alpha(\bar{\Omega})}$.

Remark 2. In case speed is prescribed all over the boundary for the boundary value problem (1.1), even for $\vec{h} \equiv 0$, uniqueness is lost for some special geometries as shown by G. E. Backus in [2].

2. Proof of Theorem 1

The proof of this theorem makes use of three lemmas. The first one (see Appendix) is a simple proof, included for completeness of our arguments, of the existence of a unique conformal transformation of the domain Ω under consideration into a rectangle R with one fixed given side. This transformation is as regular as the given regularity for each portion $\partial_i\Omega$ of the boundary $\partial\Omega$.

Next, after showing that the equation (1.1) conserves the same form in the new variables, Lemma 2.1 will show how to solve the new equation in a rectangular domain with prescribed data as in (1.1). There, we shall see that the solution has a representation (Bers-Nirenberg representation), whose two components solve a linear homogeneous elliptic problem and a nonlinear elliptic boundary value problem, (1.4), respectively, (Lemma 1.2).

Thus, we reconstruct the solution φ by taking the inverse conformal transformation of the solution given by the Bers-Nirenberg representation (see [4]), which is constructed uniquely by solving the equations for their components in the rectangle with the corresponding boundary data.

From now on we use the complex representation of the flow equation. Following Bers (see [3]), we set

$$w = 2\varphi_z = \varphi_x - i\varphi_y\,, \tag{2.1}$$

and obtain the "complex flow equation"

$$w_{\bar z} = \beta(z)w + \alpha(z)\bar w \tag{2.2}$$

with

$$\beta(z) = (h_1 + ih_2)(z)\,, \qquad \alpha(z) = \overline{\beta(z)}\,. \tag{2.3}$$

Thus, Lemma 3 of [3] states that if w is a solution of (2.2)–(2.3) in a simply connected domain, then we have a solution φ of (1.1) with the form

$$\varphi(z) = \mathrm{Re}\left\{\int_{z_0}^{z} w\,dz\right\}\,. \tag{2.4}$$

We first show that equation (2.2) preserves its form under any conformal transformation. Indeed, let $z' = F(z)$ be a conformal transformation from Ω into Ω', where $w = \varphi_x - i\varphi_y$ is the complex velocity in $\Omega(z)$. In $\Omega'(z')$ let

$$u = \varphi_{z'} = \varphi_z \frac{dz}{dz'} + \varphi_{\bar z}\frac{d\bar z}{dz'} = \varphi_z \frac{dz}{dz'}\,, \tag{2.5}$$

A short computation yields from (2.2)

$$\begin{aligned} u_{\bar z'} &= (\beta w + \bar\beta\,\bar w)\frac{d\bar z}{d\bar z'}\frac{dz}{dz'} = \left(\beta u \frac{d\bar z}{d\bar z'} + \bar\beta \bar u \frac{dz}{dz'}\right) \\ &= \beta\frac{d\bar z}{d\bar z'}u + \bar\beta\frac{dz}{dz'}\bar u \end{aligned}$$

or

$$u_{\bar z'} = \beta' u + \alpha' \bar u \tag{2.6}$$

defined in Ω', with

$$\beta' = \left(\beta\frac{dz}{dz'}\right)(z') \quad \text{and} \quad \beta' = \overline{\alpha'}\,. \tag{2.7}$$

In particular, if F is the conformal transformation (see Appendix) that takes Ω into $\Omega' = R$, the corresponding rectangle, we solve equations (2.6), (2.7) in R so that, using (2.5), the function $w = u\frac{dz'}{dz} = uF_z$ solves (2.1) and the solution φ is then given by (2.4). The boundary data from (1.1) is equivalent to $\Im u = 0$ on $\partial_1 R \cup \partial_2 R \cup \partial_4 R$ and $|u| = |w|\left|\frac{dz}{dz'}\right| = g(x)|F_z| = \tilde g(0,y) > 0$ in $\partial_3 R$.

In addition, the condition $\nabla\varphi \cdot n|_{\partial_1\Omega} < 0$ yields that $w > 0$ on $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega}$. Since $F_z|_{\overline{\partial_1\Omega}\cap\overline{\partial_2\Omega}} > 0$ (see Appendix), then $u > 0$ on $\overline{\partial_1 R} \cap \overline{\partial_2 R}$. This condition will be sufficient to see that $\partial_1\Omega$ is an inflow boundary.

Therefore, we want to prove the following lemma.

LEMMA 2.1. *Let β' be a $C^\alpha(\bar{R})$ function. The boundary value problem*

$$u_{\bar{z}} = \beta' u + \overline{\beta'}\bar{u} \tag{2.8}$$

on a rectangle $R = [C_1, C_2] \times [0,1]$ with boundary data

$$\begin{cases} |u| = \tilde{g}(y) & \text{on } \partial R_3 = \{C_2 + iy,\ 0 \leqq y \leqq 1\} \\ \Im u = 0 & \partial R \setminus \partial R_3 \quad \text{and} \quad u > 0 \quad \text{on } \overline{\partial_1 R} \cap \overline{\partial_2 R} \end{cases} \tag{2.9}$$

has a unique nonvanishing solution that takes the form

$$u = \mathrm{f}(z)e^{s(z)} = \mathrm{f}(z)e^{\Re s(z) + i\Im s(z)}$$

where $\Re s(z)$ and $\Im s(z)$ are $C^{1,\alpha}(\bar{R})$ functions and $\mathrm{f}(z)$ is analytic in R, nonvanishing in $\bar{R}$ and satisfies the same boundary data as u.

In addition, the following estimates hold:

$$\begin{aligned} &\|u\|_{C^\alpha(\bar{R})} \leqq K\|\mathrm{f}\|_{C^\alpha(\bar{R})}(\inf|\mathrm{f}|)^{-2}\, e^{K\|\beta'\,\mathrm{f}\|_{\infty,\bar{R}}}\left(1 + \|\beta'\,\mathrm{f}\|_{\infty,\bar{R}}\right) \quad \text{and} \\ &\|u\|_{C^{1,\alpha}(\bar{R})} \leqq K\|\mathrm{f}\|_{C^{1,\alpha}(\bar{R})}(\inf|\mathrm{f}|)^{-2}\, e^{K\|\beta'\,\mathrm{f}\|_{\infty,\bar{R}}} \\ &\quad\cdot\left(1 + \|\beta'\,\mathrm{f}\|_{\infty,\bar{R}}\right)\left(1 + \|\beta'\,\mathrm{f}\|_{C^\alpha(\bar{R})}\|\beta'\,\mathrm{f}\|_{\infty,\bar{R}} + \|\beta'\,\mathrm{f}\|^2_{\infty,\bar{R}}\right) \end{aligned} \tag{2.10}$$

where K depends on R, and, $\|\mathrm{f}\|_{C^{1,\alpha}(\bar{R})}$ and $\inf|\mathrm{f}|$ depend on $C^{1,\alpha}$-norm and the lower bound for $\tilde{g}$ respectively.

Proof: We first reduce the problem for u to a problem for $a = \Re s$ and $b = \Im s$. The solution u of equation (2.8) admits a representation; see Bers and Nirenberg representation in [4], Chapter II, Section 6.

$$u(z) = \mathrm{f}(z)e^{s(z)}\,. \tag{2.11}$$

Here, f is an analytic function satisfying

$$\begin{aligned} \mathrm{f}_{\bar{z}} &= 0 && \text{in } \bar{R}\,, \quad \text{along with} \\ \Im\mathrm{f} &= 0 && \text{on } \partial R \setminus \partial_3 R\,, \\ \mathrm{f} &> 0 && \text{on } \overline{\partial_1 R} \cap \overline{\partial_2 R}\,, \quad \text{and} \\ |\mathrm{f}| &= \tilde{g}(x) && \text{on } \partial_3 R\,. \end{aligned} \tag{2.12}$$

The function $s(z)$ is complex valued and satisfies

$$\mathrm{f}(z)e^{s(z)}s_{\bar{z}} = u_{\bar{z}} = \beta'\,\mathrm{f}e^{s(z)} + \overline{\beta'\,\mathrm{f}e^{s(z)}}\,,$$

so that

$$s_{\bar{z}} = \beta' + \overline{\beta'}\,\frac{\bar{\mathrm{f}}}{\mathrm{f}}\,e^{-2i\,\Im s(z)} \tag{2.13}$$

with the boundary conditions

$$\Im s = 0 \quad \text{on } \partial R \setminus \partial_3 R \quad \text{and} \quad \Re s = 0 \quad \text{on } \partial_3 R\,. \tag{2.14}$$

We shall show that the boundary value problem (2.12) has a unique nonvanishing solution. Moreover, the boundary value problem (2.13)–(2.14) has a unique solution as well.

In order to show uniqueness of a nonvanishing u solution of the boundary value problem (2.8)–(2.9), let $\tilde{u}$ be another nonvanishing solution of the same problem and set $\tilde{s} = \ell n(\frac{\tilde{u}}{\mathrm{f}})$, with f the unique nonvanishing solution of problem (2.12).

Since u and $\tilde{u}$ have the same data on ∂R, then it is easy to see that $\Im\tilde{s} = 0$ on $\partial R \setminus \partial_3 R$ and $\Re\tilde{s} = 0$ on $\partial_3 R$. Differentiating $\tilde{s}$ with respect to $\bar{z}$ it is easy to see that $\tilde{s}$ solves equation (2.13) since $\mathrm{f}_{\bar{z}} = 0$. Indeed,

$$\tilde{s}_{\bar{z}} = \left(\ell n\left(\frac{\tilde{u}}{\mathrm{f}}\right)\right)_{\bar{z}} = \frac{\tilde{u}_{\bar{z}}}{\tilde{u}} = \beta' + \bar{\beta}'\frac{\bar{\tilde{u}}}{\tilde{u}} = \beta' + \bar{\beta}'\frac{\bar{\mathrm{f}}}{\mathrm{f}}e^{-2i\Im\tilde{s}}\,.$$

Therefore, by the uniqueness of problem (2.13)–(2.14), $\tilde{s} = s$. Then, by the representation (2.11), $\tilde{u}(z) = \mathrm{f}(z)e^{s(z)} = u(z)$ on $\bar{R}$. Thus, the uniqueness of nonvanishing solutions follows.

We first solve problem (2.12). We set $\ell n\,\mathrm{f}(z) = \ell n\,|\mathrm{f}(z)| + i \arg \mathrm{f}(z)$. Thus, if $|\mathrm{f}(z)| \neq 0$, then $\mathrm{f}_{\bar{z}} = 0$ if and only if $(\ell n\,\mathrm{f})_{\bar{z}} = 0$.

Thus, we set $h = \ell n\,\mathrm{f}(z)$ and solve

$$h_{\bar{z}} = 0\,, \qquad \begin{array}{ll} \Im h = 0 & \text{on } \partial R \setminus \partial_3 R\,, \\ \Re h = \ell n\,\tilde{g} & \text{on } \partial_3 R\,. \end{array} \tag{2.15}$$

Problem (2.15) is solved using standard methods. Indeed, take the Cauchy-Riemann equations and solve the corresponding harmonic equation for $\Re h$, with $\frac{\partial \Re h}{\partial n} = 0$ on $\partial R \setminus \partial_3 R$, and find $\Im h$ as the conjugate harmonic function. In particular, $\Delta\,\Im h = 0$, $\Im h = 0$ on $\partial R \setminus \partial_3 R$ and $(\Im h)_n = -\frac{\tilde{g}'}{\tilde{g}}$ on $\partial_3 R$, where $\frac{\tilde{g}'}{\tilde{g}}$ is $C^\alpha(\partial_3 R)$ and $\frac{|\tilde{g}'|}{\tilde{g}} \leqq C$, C depending on the lower and upper bounds of $\tilde{g}(0, y)$ and $\tilde{g}'(0, y)$.

Both, $\Re h$ and $\Im h$ are $C^\infty(R)$ functions.

We use reflection techniques in order to obtain the $C^{1,\alpha}$ regularity up to the boundary of R.

Reflect the obtained harmonic function $\Re h$ (which satisfies $\frac{\partial \Re h}{\partial n} = 0$ on $\partial_1 R \cup \partial_2 R \cup \partial_4 R$ and $\Re h = \ell n\, \tilde{g}$ on $\partial_3 R$) evenly across ∂R_2 and $\partial_4 R$ to a larger reflected rectangular domain $\tilde{R}$ whose sides $\partial_i \tilde{R} \supset\supset \partial_i R$ for $i = 1, 3$. Let $\widetilde{\Re h}$ be the reflected function of $\Re h$. Also, let G be the even reflection of $\ell n\, \tilde{g}$ across the endpoints of $\partial_3 R$ defined on $\partial_3 \tilde{R}$.

Then $\widetilde{\Re h}$ is a harmonic function in $\tilde{R}$ that satisfies $\frac{\partial \widetilde{\Re h}}{\partial n} = 0$ on $\partial_1 \tilde{R} \cup \partial_2 \tilde{R} \cup \partial_4 \tilde{R}$ and $\widetilde{\Re h} = G$ on $\partial_3 \tilde{R}$.

In addition, $\widetilde{\Re h}$ is $C^\infty(\tilde{R}) \cup C^{1,\alpha}(\tilde{R} \cup T)$ where T is any regular portion of the boundary of $\tilde{R}$. Then $\Re h = \widetilde{\Re h}|_{\bar{R}}$ is in $C^\infty(R) \cup C^{1,\alpha}(\bar{R})$.

Therefore, since the harmonic conjugate of $\Re h$ inherits its regularity, $\Im h$ is $C^\infty(R) \cup C^{1,\alpha}(\bar{R})$ and the following estimates hold

$$\|\Re h\ ,\ \Im h\|_{C^{1,\alpha}(\bar{R})} \leqq C \|\tilde{g}\|_{C^{1,\alpha}(\bar{R})}\ ,$$

where C depends on R.

Moreover, by the Hopf maximum principle,

$$\sup_{\partial_3 R} ln\, \tilde{g} \geqq \sup_{\partial R} \Re h \geqq \sup_{\bar{R}} \Re h \geqq \inf_{\bar{R}} \Re h \geqq \inf_{\partial R} \Re h \geqq \inf_{\partial_3 R} ln\, \tilde{g}\ .$$

Thus h is analytic in R with its real part bounded by a constant that depends only on the bounds for $\ell n\, \tilde{g}$ on $\partial_3 R$.

In particular, we obtain $\mathfrak{f}(z) = \exp(h(z))$ has its absolute value bounded below away from zero by a positive constant that depends on the lower bound of $\tilde{g}(y)$ in $\partial_3 R$. In addition, the solution of (2.12) satisfies

$$\|\mathfrak{f}\|_{C^{1,\alpha}(\bar{R})} \leqq \|\tilde{g}\|_{C^{1,\alpha}} \quad \text{and} \quad 0 < k \leqq |\mathfrak{f}(z)| \leqq K\ , \tag{2.16}$$

where C depends on R, $k = \inf \tilde{g}$ and $K = \sup \tilde{g}$, respectively.

Next, we solve (2.13)–(2.14). For this problem we set $s = a + ib$, and we write the corresponding equations and boundary conditions.

We recall

$$s_{\bar{z}} = \tfrac{1}{2}(s_x + i s_y) = \tfrac{1}{2}\{(a_x - b_y) + i(b_x + a_y)\}\ . \tag{2.17}$$

Hence, equations (2.17) and (2.13) yield the following equations

$$\begin{aligned} (a_x - b_y) &= \Re\left(\beta' + \overline{\beta'}\frac{\bar{\mathfrak{f}}}{\mathfrak{f}} e^{-2ib}\right) = p_1(\mathbf{x}, b)\ ,\\ (b_x - a_y) &= \Im\left(\beta' + \overline{\beta'}\frac{\bar{\mathfrak{f}}}{\mathfrak{f}} e^{-2ib}\right) = p_2(\mathbf{x}, b)\ . \end{aligned} \tag{2.18}$$

On the other hand, the right-hand side of equation (2.13) can be written as

$$\frac{\bar{\mathfrak{f}}}{|\mathfrak{f}|^2}(\beta' \mathfrak{f} + \bar{\beta}' \bar{\mathfrak{f}} e^{-2ib}) = |\mathfrak{f}|^{-2}(\mathfrak{f}_R - i\mathfrak{f}_I)((A + iB) + (A - iB)(\cos 2b - i \sin 2b)\ ,$$

where $A + iB = \beta' \mathtt{f}$, and $\mathtt{f} = \mathtt{f}_I + i\mathtt{f}_R$, then the terms p_1 and p_2 from (2.18) can be written as

$$\begin{aligned} p_1(\mathbf{x}, b) &= 2|\mathtt{f}|^{-2}[\mathtt{f}_R A(1 + \cos 2b) \\ &\quad + \mathtt{f}_I B(1 - \cos 2b) - (\mathtt{f}_R B + \mathtt{f}_I A)\sin 2b] \\ p_2(\mathbf{x}, b) &= 2|\mathtt{f}|^{-2}[-\mathtt{f}_I A(1 + \cos 2b) \\ &\quad + \mathtt{f}_R B(1 - \cos 2b) - (\mathtt{f}_R A + \mathtt{f}_I B)\sin 2b] \,. \end{aligned} \tag{2.19}$$

Clearly,

$$p_i(\mathbf{x}, b) = \sum_{k=1}^{3} A_{i_k}(\mathbf{x}) \ell_{i_k}(b) \,, \qquad i = 1, 2 \tag{2.20}$$

where $A_{i_k}(\mathbf{x})$ are real C^α functions in $\bar{R}$ and $\ell_{i_k}(b)$ bounded and Lipschitz.

Since p_i does not depend on $a = \Re s(z)$, system (2.19) yields an elliptic equation for $b(\mathbf{x})$ alone, namely

$$\Delta b = p_{2x} - p_{1y} \,. \tag{2.21}$$

The boundary conditions for b are obtained from (2.14) and the relationship (2.19).

Since $\Re s = a = 0$ on $\partial_3 R$, then $a_y = 0$ on $\partial_3 R$, and by (2.19), $b_x = p_2(\mathbf{x}, b)$ on $\partial_3 R$.

Also, $\Im s = b = 0$ on $\partial R \setminus \partial_3 R$. In particular, since $\mathtt{f}$ is real at $\overline{\partial_3 R} \cup \overline{\partial_4 R}$ and at $\overline{\partial_3 R} \cup \overline{\partial_2 R}$, we have the compatibility condition

$$p_2(\mathbf{x}, 0) = 0 \quad \text{if} \quad \mathbf{x} \in \overline{\partial_3 R} \cap \overline{\partial_4 R} \quad \text{or} \quad \mathbf{x} \in \overline{\partial_3 R} \cup \overline{\partial_2 R} \,. \tag{2.22}$$

Therefore, we must show that there exists a unique solution of (2.21), along with the data

$$\begin{cases} b = 0 & \text{on } \partial R \setminus \partial_3 R \\ b_x = p_2((C_2, y), b) & \text{on } \partial_3 R \end{cases} \,. \tag{2.23}$$

In the next section, we prove Lemma 1.2 which ensures the existence of a unique weak solution $b(\mathbf{x})$ in the class $C^{1,\alpha}(\bar{R})$ of the nonlinear boundary value problem (2.21)–(2.23) that satisfies the estimates

$$\|b\|_{C^{1,\alpha}(\bar{R})} \leqq C \left(\|p\|_{C^\alpha, \mathbf{x}, \bar{R}} + \|p\|_{\mathrm{Lip}, b} \right) \|p\|_{\infty, \bar{R}}, \tag{2.24}$$

and

$$\|b\|_{C^\alpha(\bar{R})} \leqq C \|p\|_{\infty, \bar{R}} \tag{2.25}$$

where C depends on R.

Once we have the unique function $b(\mathbf{x}) = \Im s(z)$, then using equation (2.20) we find a unique $a(\mathbf{x}) = \Re s(z)$ by integrating the field

$$\nabla a(\mathbf{x}) = (b_y + p_1(\mathbf{x}, b), -b_x + p_2(\mathbf{x}, b)) \tag{2.26}$$

where $a(\mathbf{x}) = 0$ on $\partial_3 R$. In particular, $\Delta a = \operatorname{div} p(\mathbf{x}, b)$ with $a_n = 0$ on $\partial_2 R \cup \partial_4 R$, $a_n = p(\mathbf{x}, 0) \cdot n$ on $\partial_1 R$, and $a = 0$ on $\partial_3 R$.

Therefore, a similar result to lemma 3.1 yields that $a(\mathbf{x})$ is a $C^{1,\alpha}(\bar{R})$ function, and by (2.24) and (2.25), satisfies the estimates

$$\|a\|_{C^{1,\alpha}(\bar{R})} \leqq C \left(\|p\|_{C^{\alpha},\mathbf{x},\bar{R}} + \|p\|_{\mathrm{Lip},b} \right) \|p\|_{\infty,\bar{R}} \tag{2.27}$$

and

$$\|a\|_{C^{\alpha}(\bar{R})} \leqq C \|p\|_{\infty,\bar{R}} \,, \tag{2.28}$$

where C depends on R.

We may now finish the proof of Lemma 2.1.

End of Proof of Lemma (2.1): Let the complex function $s(z) = a + ib$ be given, with a and b real $C^{1,\alpha}(\bar{R})$ functions which are the unique solutions of the boundary value problem (2.19), (2.21), and (2.23).

Then $s(z)$ solves the boundary value problem (2.13)–(2.14) uniquely. Therefore, by (2.11), (2.17), and (2.13), $u = \mathfrak{f}(z)e^{a+ib}$ solves the boundary value problem (2.8), (2.9) in the rectangle R uniquely.

Moreover, using estimates (2.24), (2.25), (2.27), and (2.28), along with the form of p given by (2.20), yields the estimates

$$\begin{aligned}
\|u\|_{C^{\alpha}(R)} &\leqq C \|\mathfrak{f}\|_{C^{\alpha}(\bar{R})} e^{\sup_R |\Re s(z)|} \left(1 + \|s\|_{C^{\alpha}(\bar{R})}\right) \\
&\leqq C \|\mathfrak{f}\|_{C^{\alpha}(\bar{R})} (\inf |\mathfrak{f}|)^{-2} e^{C\|\beta' \mathfrak{f}\|_{\infty,R}} \left(1 + \|\beta' \mathfrak{f}\|_{\infty,\bar{R}}\right) \quad \text{and} \\
\|u\|_{C^{1,\alpha}(\bar{R})} &\leqq C \|\mathfrak{f}\|_{C^{1,\alpha}(\bar{R})} e^{\sup_R |\Re s(z)|} \left(1 + \|s\|_{C^{1,\alpha}(\bar{R})}\right) \left(1 + \|s\|_{C^{\alpha}(\bar{R})}\right) \\
&\leqq C \|\mathfrak{f}\|_{C^{1,\alpha}(\bar{R})} (\inf |\mathfrak{f}|)^{-2} e^{C\|A_{i_k}\|_{\infty,\bar{R}}} \\
&\quad \cdot \left(1 + \|A_{i_k}\|_{C^{\alpha}(\bar{R})} \|A_{i_k}\|_{\infty,\bar{R}} + \|A_{i_k}\|^2_{\infty,\bar{R}}\right) \left(1 + \|A_{i_k}\|_{\infty,R}\right) \\
&\leqq C \|\mathfrak{f}\|_{C^{1,\alpha}(\bar{R})} (\inf |\mathfrak{f}|)^{-2} e^{C\|\beta' \mathfrak{f}\|_{\infty,R}} \\
&\quad \cdot \left(1 + \|\beta' \mathfrak{f}\|_{C^{\alpha}(\bar{R})} \|\beta' \mathfrak{f}\|_{\infty,\bar{R}} + \|\beta' \mathfrak{f}\|^2_{\infty,\bar{R}}\right) \left(1 + \|\beta' \mathfrak{f}\|_{\infty,\bar{R}}\right)
\end{aligned} \tag{2.29}$$

where C depends only on R. Hence, (2.10) holds.

Thus, the proof of Lemma 2.1 is now completed.

Proof of Theorem 1: Let F be the conformal transformation and R the resulting rectangle given in the Appendix; then $F: \Omega \to R$ is the unique conformal transformation of the domain Ω onto $R = [C_1, C_2]\times[0,1]$. Recalling w the complex velocity representation in Ω, we have by (2.1), (2.2), and (2.3) that

$$
\begin{aligned}
&w_{\bar{z}} = \beta(z)w + \bar{\beta}(z)\bar{w}\,, \qquad w = \varphi_x - i\varphi_y \quad \text{and} \\
&\beta(z) = (h_1 + ih_2)(z)\,.
\end{aligned} \tag{2.30}
$$

Now, by (2.5), (2.6), and (2.7),

$$
w = u\frac{dF}{dz} = uF_z \tag{2.31}
$$

where u is the unique solution of the flow equation in the rectangle $F(\Omega) = R$ of the boundary value problem given by Lemma 3.1, where

$$
\beta' = (h_1 + ih_2)\left(\overline{\frac{dF}{dz}}\right)^{-1}.
$$

Hence, by Lemma 2.1,

$$
w = \mathfrak{f}(z)e^{s(z)}\cdot F_z\,, \qquad |\mathfrak{f}(z)| \neq 0\,.
$$

Since F is an analytic function in Ω, which is $C^{\infty}(\bar{\Omega})$ (due to the special geometrical properties of Ω) and $F_z \neq 0$ in Ω and $|F_z|$ is uniformly bounded away from zero. See lemma in the Appendix.

Then $\Re w$ and $\Im w$ are $C^{1,\alpha}(\bar{\Omega})$ and by (2.10) the following estimates hold,

$$
\begin{aligned}
&\|w\|_{C^{\alpha}(\bar{\Omega})} \leqq C\|\mathfrak{f}\|_{C^{\alpha}(\bar{\Omega})}(\inf|\mathfrak{f}|)^{-2}e^{C\|\beta'\mathfrak{f}\|_{\infty,\bar{\Omega}}}\left(1 + \|\beta'\|_{\infty,\bar{\Omega}}\|\mathfrak{f}\|_{\infty,\bar{\Omega}}\right), \quad \text{and} \\
&\|w\|_{C^{1,\alpha}(\bar{\Omega})} \leqq C\|\mathfrak{f}\|_{C^{1,\alpha}(\bar{\Omega})}(\inf|\mathfrak{f}|)^{-2}e^{C\|\beta'\mathfrak{f}\|_{\infty,\bar{\Omega}}} \\
&\cdot\left(1 + \left(\|\beta'\|_{C^{\alpha}(\bar{\Omega})}\|\beta'\|_{\infty,\bar{\Omega}} + \|\beta'\|^2_{\infty,\bar{\Omega}}\right)\|\mathfrak{f}\|^2_{\infty,\bar{\Omega}}\right)\left(1 + \|\beta'\|_{\infty,\bar{\Omega}}\|\mathfrak{f}\|_{\infty,\bar{\Omega}}\right),
\end{aligned} \tag{2.32}
$$

and the estimate for the absolute value of w

$$
0 < C\inf\left\{|\mathfrak{f}|\,|F_z|e^{\inf \Re s(z)}\right\} \leqq |w| \leqq C\sup\left\{|\mathfrak{f}|\,|F_z|e^{\sup \Re s(z)}\right\}. \tag{2.33}
$$

The constant C and $|F_z|$ depend only on Ω, the $\inf|\mathfrak{f}|$ is positive and depends on $g(\mathbf{x})$ on $\partial_3 R$, and sup and inf of $\Re s(z) = a(x)$ are bounded and, by (2.26), are bounded from above and below by the upper and lower bounds of $|\mathfrak{f}(z)|$ from (2.16), from $|F_z^{-1}|$, and from $\vec{h} = (h_1, h_2)$ the given coefficients of equation (1.1). Estimate (2.33) thus ensures that $\nabla\varphi$ is positive throughout $\bar{\Omega}$.

Furthermore, by estimate (2.16), the right-hand sides of (2.32) can be estimated by

$$
\begin{aligned}
&C\|\mathfrak{f}\|_{C^{\alpha}(\bar{\Omega})}(\inf|\mathfrak{f}|)^{-2}e^{C\|\beta'\|_{\infty,\bar{\Omega}}}\left(1 + C\|\beta'\|_{\infty,\bar{\Omega}}\right), \quad \text{and} \\
&C\|\mathfrak{f}\|_{C^{1,\alpha}(\bar{\Omega})}(\inf|\mathfrak{f}|)^{-2}e^{C\|\beta'\|_{\infty,\bar{\Omega}}} \\
&\cdot\left(1 + C\|\beta'\|_{C^{\alpha}(\bar{\Omega})}\|\beta'\|_{\infty,\bar{\Omega}} + C\|\beta'\|^2_{\infty,\bar{\Omega}}\right)\left(1 + C\|\beta'\|_{\infty,\bar{\Omega}}\right),
\end{aligned} \tag{2.34}
$$

where C depends on $\|g\|_{C^{1,\alpha}(\partial_3\Omega)}$ and on $|F_z|_{L^\infty}$, i.e., C depends on the data $g(\mathbf{x})$ and on the domain Ω.

Finally, by (2.4), we take

$$\varphi = \operatorname{Re}\left\{\int_{z_0}^{z} w\,dz\right\}, \quad \varphi = 0 \quad \text{on } \partial_1\Omega\,, \tag{2.35}$$

then φ is a real function that solves the boundary value problem (1.1) and, since w has real and imaginary parts in the class of $C^{1,\alpha}(\bar{\Omega})$ functions, then $\varphi \in C^{2,\alpha}(\bar{\Omega})$ and by estimates (2.32) and (2.34)

$$\begin{gathered} \|\varphi\|_{C^{1,\alpha}(\bar{\Omega})} \leqq Ce^{C\|h\|_{\infty,\bar{\Omega}}}\left(1 + \|h\|_{\infty,\bar{\Omega}}\right) \quad \text{and} \\ \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})} \leqq Ce^{C\|h\|_{\infty,\bar{\Omega}}} \\ \cdot\left(1 + \|h\|_{C^{\alpha}(\bar{\Omega})}\|h\|_{\infty,\bar{\Omega}} + \|h\|^2_{\infty,\bar{\Omega}}\right)\left(1 + \|h\|_{\infty,\bar{\Omega}}\right) \end{gathered} \tag{2.36}$$

where C depends on the $C^{1,\alpha}$-norm of $g(\mathbf{x})$ and on the domain Ω, so that estimates in (1.2) hold.

In addition, by (2.33)

$$0 < k\,e^{-C_{\sup}H}C_{\inf} < |\nabla\varphi| < Ke^{C_{\sup}H}C_{\sup} \tag{2.37}$$

where H is the upper bound of $|h|$, $C_{\sup}$ and $C_{\inf}$ bounds depending on $g(\mathbf{x})$ and Ω, and k, K depends only on Ω. We stress once more that $C_{\inf} > 0$, because of the assumption $\inf g(\mathbf{x}) > 0$ on $\partial_3\Omega$; see estimate 2.16.

Furthermore, since the given data $\varphi = 0$ on $\partial_1\Omega$ then $\nabla\varphi \cdot \tau = 0$ along $\partial_1\Omega$. Since $|\nabla\varphi|$ never vanishes in $\bar{\Omega}$ then $\nabla\varphi$ cannot change sign in $\partial_1\Omega$. Thus, $\nabla\varphi \cdot n|_{\partial_1\Omega}(w_1) < 0$, implies $\nabla\varphi \cdot n < 0$ on $\partial_1\Omega$. Therefore φ increases along the paths orthogonal to the level surfaces that start at the boundary $\partial_1\Omega$, and, in particular,

$$0 \leqq \varphi \leqq \widetilde{K} \qquad \text{near } \partial_1\Omega\,. \tag{2.38}$$

In general,

$$-\widetilde{K} \leqq \varphi \leqq \widetilde{K} \qquad \text{in } \bar{\Omega}\,, \tag{2.39}$$

where $\widetilde{K}$, both in (2.38) and (2.39), depends on the diameter of Ω and on the upper bound of $|\nabla\varphi|$.

Now the proof of Theorem 1 is complete.

In the last section, we prove our "key lemma" (1.2) to solve the boundary value problem (2.21), (2.23).

3. Proof of Lemma 1.2

Without loss of generality, we assume that Γ_1 is contained in the axis $x = 0$ and that its lower end is the origin.

(a) Existence

We first show the existence of a weak solution to the boundary value problem (1.3) by constructing a precompact continuous map T from an appropriate Banach space $\mathscr{B}$ into itself. Then we use the Leray-Shauder fixed-point theorem (see Gilbarg-Trudinger, [7], Chapter 11), where $\mathscr{B}$ is $C^\alpha(\bar{R})$.

Thus, we let $v \in C^\alpha(\bar{R})$ such that $v \equiv 0$ on Γ.

We define the map T by letting $b = Tv$ be the unique weak solution in $C^{1,\alpha}(\bar{R})$ of the linear boundary value problem

$$(3.1)\qquad \begin{aligned} \Delta b &= \operatorname{div} f(\mathbf{x}, v(\mathbf{x})) && \text{on } R \\ b &= 0 && \text{on } \Gamma \quad \text{and} \\ b_x &= f_1((0,y), v(0,y)) && \text{on } \Gamma_1 \\ f_1(\mathbf{x}, 0) &= 0 && \text{for any } \mathbf{x} \in \Gamma \cap (\partial_2 R \cup \partial_4 R)\,. \end{aligned}$$

By weak solution we mean that

$$(3.1.\text{w})\qquad \int_R \nabla\varphi \cdot \nabla b = \int_R \nabla\varphi \cdot f(x,v) \qquad \text{for all } \varphi \in H_0^1(\mathbb{R}^2 \setminus \Gamma)\,.$$

Remark. Note that

$$\int_{\partial R} \varphi(\nabla b - f(x,v)) \cdot n\,ds = \int_{\Gamma_1} \varphi\big(b_x - f_1((0,y), v(0,y))\big)\,dy = 0$$

for all $\varphi \in H_0^1(\mathbb{R}^2 \setminus \Gamma)$.

LEMMA 3.1. *The problem* (3.1) *has a unique weak solution b in $C^{1,\alpha}(\bar{R})$ in the sense of* (1.4) *or* (3.1.w) *satisfying the following estimates:*

$$(3.2)\qquad \|b\|_{C^\alpha(\bar{R})} \leqq C\|f\|_{\infty,\bar{R}}$$

$$(3.3)\qquad \|b\|_{C^{1,\alpha}(\bar{R})} \leqq C\left(\|f\|_{C^\alpha,\mathbf{x},\bar{R}} + \|f\|_{\mathrm{Lip},v}\right)\|f\|_{\infty,\bar{R}}$$

where C depends only on $\bar{R}$.

Proof: The proof of this lemma is an application of existence and regularity theorems for elliptic equations in general smooth domain. Although our domain is a rectangle, the nature of the boundary data permits reflecting the boundary appropriately such that we deal with a Dirichlet boundary value transmission problem, and so keeps the interior regularity up to the boundary. Methods discussed by Grisvard in [8] would also be adequate to obtained the same regularity.

Let $\tilde{R}$ be the rectangle given by the union of R, its reflection with respect to Γ_1, and Γ_1; and define $\tilde{v}$ on $\tilde{R}$ as the even reflection of v with respect to Γ_1.

We consider the following boundary value problem

$$\begin{aligned} \Delta\tilde{b} &= \operatorname{div} \tilde{f}(\mathbf{x}, \tilde{v}(\mathbf{x})) \quad \text{in } \tilde{R} \\ \tilde{b} &= 0 \qquad\qquad\qquad\quad \text{on } \partial\tilde{R}\,, \end{aligned} \tag{3.4.1}$$

where $\tilde{f}(\mathbf{x}, \tilde{v}(\mathbf{x}))$ is the extension of $f(\mathbf{x}, v(\mathbf{x}))$ to $\tilde{R}$ given by

$$\begin{cases} f(\mathbf{x}, \tilde{v}(\mathbf{x})) & \text{if } \mathbf{x} \in \bar{R} \\ (-f_1(\mathbf{x}^*, \tilde{v}(\mathbf{x}^*)), f_2(\mathbf{x}^*, \tilde{v}(\mathbf{x}^*))\,) & \text{if } \mathbf{x} \in \bar{\tilde{R}} \setminus \bar{R} \end{cases} \tag{3.4.2}$$

where $\mathbf{x}^* \in R$ is the reflection of $\mathbf{x} \in \bar{\tilde{R}} \setminus \bar{R}$ with respect to Γ_1, that is, f_1 is reflected oddly and f_2 is reflected evenly with respect to Γ_1, respectively.

Now, since $v \in C^\alpha(\bar{R})$ and $\tilde{v}$ is the even reflection of v with respect to Γ_1, then $\tilde{v} \in C^\alpha(\bar{\tilde{R}})$, so that $\tilde{f}$ is Lipchitz in $\tilde{v}$. However, since f_1 does not vanish on Γ_1, then $\tilde{f}$ is discontinuous across Γ_1 but $\tilde{f}$ remains bounded and measurable in $\bar{\tilde{R}}$.

The existence of an H_0^1 solution of problem (3.4) follows from standard results. In fact, a weak solution of problem (3.4.1) must satisfy

$$\int_{\tilde{R}} \nabla\varphi\nabla\tilde{b} = -\int_{\tilde{R}} \nabla\varphi\, f(\mathbf{x}, v) \qquad \text{for all } \varphi \in H_0^1(\tilde{R})\,, \tag{3.5}$$

Since $f \in L^\infty(\tilde{R})$, then

$$\ell(\varphi) = \int_{\tilde{R}} \nabla\varphi\, f(v)$$

is a continuous linear form in H_0^1 satisfying $\|\ell(\varphi)\|_{H_0^1(\tilde{R})} \leqq C\|\varphi\|_{H_0^1}\|f\|_{\infty,\tilde{R}}$ where C depends on the domain.

By the Riesz representation theorem, there exists a unique solution $\tilde{b}$ in H_0^1 of problem (3.4.1) that satisfies

$$\|\varphi\|_{H_0^1(\tilde{R})} \leqq C\|f\|_{\infty,\tilde{R}}\,.$$

Next, the solution $\tilde{b}$ can be reflected oddly across all boundary sections $\partial\tilde{R}_i$, and it defines a $\mathbf{b}$ that solves a similar problem for the enlarged domain $\mathscr{R}$ in the space $H_0^1(\mathscr{R})$ with a right-hand side $\mathbf{f}$. Now this right-hand side $\mathbf{f}$ is measurable and bounded in $\mathscr{R}$ and C^α on each side of $\mathscr{R}$ up to, but not across, $\tilde{\Gamma}_1$ and $\partial_1\tilde{R}$, the extension to $\mathscr{R}$ of Γ_1 and $\partial_1 R$, respectively. See Figure 3.

Recalling Agmon-Douglis-Nirenberg (see [1]) interior regularity results for elliptic problems, it follows that the solution $\mathbf{b}$ of problem (3.4.1) in the domain $\mathcal{R}$ for the subdomain $\bar{\bar{R}} \subset\subset \mathcal{R}$ satisfies the following estimate

$$\|\mathbf{b}\|_{W^{1,p}(\bar{\bar{R}})} \leqq C\|\mathbf{f}\|_{L^p(\mathcal{R})} \leqq C\|f\|_{L^\infty(R)} , \tag{3.6}$$

where C depends only on R, since the definition of $\mathbf{f}$ only involves even or odd reflections of the components of f.

Since $\tilde{b} = \mathbf{b}|_{\tilde{R}}$, the Embedding Theorem yields $\tilde{b} \in C^\alpha(\bar{\bar{R}})$ along with the estimate

$$\|\tilde{b}\|_{C^\alpha(\bar{\bar{R}})} \leqq C\|f\|_{L^\infty(R)} , \tag{3.7}$$

where C depends only on R.

Next, we show first that $\tilde{b}$ is an even function in $\tilde{R}$ and later that any even, and hence any solution of the problem (3.4.1)–(3.4.2) yields a weak solution b of problem (3.1) in the sense (3.1.w).

Due to the uniqueness of the boundary value problem (3.4.1)–(3.4.2), it is enough to show that $\tilde{b}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the reflection of $\mathbf{x}$ with respect to Γ_1, is also a solution. Consequently, $\tilde{b}(\mathbf{x}^*) = \tilde{b}(\mathbf{x})$, i.e., $\tilde{b}$ is an even function.

Note that if $\Gamma_1 \subset \{x = 0\}$, then $\mathbf{x}^* = (x, y)^* = (-x, y)$. Thus $\Delta_\mathbf{x} = \Delta_{\mathbf{x}^*}$ and $\nabla_{\mathbf{x}^*} \cdot = -\partial_x + \partial_y$ as differential operators, hence

$$\Delta_\mathbf{x}\tilde{b}(\mathbf{x}^*) = \Delta_{\mathbf{x}^*}\tilde{b}(\mathbf{x}^*)$$

and, by definition of $\tilde{f}$,

$$\nabla_\mathbf{x} \cdot \tilde{f}(\mathbf{x}, \tilde{v}(\mathbf{x})) = \nabla_\mathbf{x} \cdot \tilde{f}(\mathbf{x}, \tilde{v}(\mathbf{x}^*)) = \nabla_{\mathbf{x}^*} \cdot \tilde{f}(\mathbf{x}^*, \tilde{v}(\mathbf{x}^*)) .$$

Therefore if $\tilde{b}$ is the solution of (3.4.1)–(3.4.2) then

$$\Delta_{\mathbf{x}^*}\tilde{b}(\mathbf{x}^*) = \nabla_{\mathbf{x}^*} \cdot \tilde{f}(\mathbf{x}^*, \tilde{v}(\mathbf{x}^*)) ,$$

so that

$$\Delta_\mathbf{x}\tilde{b}(\mathbf{x}^*(\mathbf{x})) = \nabla_\mathbf{x} \cdot \tilde{f}(\mathbf{x}, \tilde{v}(\mathbf{x})) .$$

In particular, since $\tilde{b}(\mathbf{x}^*) = 0$ on $\partial\tilde{R}$, then $\tilde{b}(\mathbf{x}^*) = \tilde{b}(\mathbf{x})$, so that $\tilde{b}$ is an even function in $\tilde{R}$.

Now, let φ be a test function in $H_0^1(\mathbb{R}^2 \setminus \Gamma)$. Let $\tilde{\varphi}$ be the even reflection of φ to $\tilde{R}$, then $\tilde{\varphi} \in H_0^1(\tilde{R})$, and, for the solution $\tilde{b}$ of problem (3.4)

$$\begin{aligned} 0 &= \int_{\tilde{R}} \nabla\tilde{\varphi}\,(\nabla\tilde{b} - \tilde{f})\,d\mathbf{x} \\ &= \int_{\tilde{R}} \tilde{\varphi}_x\tilde{b}_x + \tilde{\varphi}_y\tilde{b}_y - (\tilde{\varphi}_x\tilde{f}_1 + \tilde{\varphi}_x\tilde{f}_2)\,dxdy \\ &= 2\int_R \nabla\varphi\,(\nabla\tilde{b} - f)\,d\mathbf{x} , \end{aligned}$$

since $\tilde{\varphi}_x, \tilde{b}_x$, and $\tilde{f}_1$ are odd functions and $\tilde{\varphi}_y, \tilde{b}_y$, and $\tilde{f}_2$ are even ones.

Hence (3.1.w) holds for $b = \tilde{b}|_R$, making this b the unique weak solution of the boundary value problem (3.1). In addition, by definition of $\tilde{f}$,

$$\|\tilde{f}\|_{\infty,\bar{\tilde{R}}} = \|f\|_{\infty,\bar{R}} .$$

Then by estimate (3.7)

$$\|b\|_{C^\alpha(\bar{\tilde{R}})} \leqq C\|f\|_{L^\infty(\bar{R})} , \tag{3.8}$$

where C depends only on R, so that estimate (3.2) holds.

In order to obtain the $C^{1,\alpha}$ regularity and estimate (3.3), note that $\tilde{f}|_{\bar{R}} \in C^\alpha(\bar{R})$ and $\tilde{f}\,|_{\overline{\tilde{R}\setminus R}} \in C^\alpha(\tilde{R}\setminus R)$ ($\tilde{f}$ is not C^α across Γ_1) and, by the boundary conditions given on problem (3.1), $\tilde{f}_1 = 0$ on $\partial_2\tilde{R} \cup \partial_4\tilde{R}$.

The $C^{1,\alpha}$ estimate for b in $\bar{R}$ is obtained first for $\mathbf{b}$ by differentiating $C^{2,\alpha}$ solutions of elliptic diffraction problems defined in $\mathscr{R}$ where $\tilde{\Gamma}_1$ is the diffraction boundary and the given right-hand sides are C^α on each side of $\tilde{\Gamma}_1$; see Ladyženskaja and Ural'ceva, [9].

In fact, if $B = (B_1, B_2)$ solves uniquely $\Delta B_i = \mathbf{f}_i(\mathbf{x})$ in $\mathscr{R}$, $B_i = 0$ on $\partial\mathscr{R}$, with $\mathbf{f}_i \in C^\alpha(\bar{\mathscr{R}}_1) \cap C^\alpha(\bar{\mathscr{R}}_2)$, and $\tilde{\Gamma}_1 = \bar{\mathscr{R}}_1 \cap \bar{\mathscr{R}}_2$ for $i = 1, 2$.

By [1], each $B_i \in W^{2,p}(\bar{\mathscr{R}}')$ for $1 < p < \infty$, and, in particular, in $C^{1,\alpha}(\mathscr{R}')$ for $0 < \alpha < 1$, for any regular subdomain $\mathscr{R}'$ of $\mathscr{R}$.

Hence, $[B_i] = 0 = [(B_i)_x]$ on $\tilde{\Gamma}_1$. This implies that B_i is the unique solution of the elliptic diffraction problem defined above. By [9], each B_i is $C^{2,\alpha}(\Lambda_k), k = 1, 2$, where $\Lambda_k = (\mathscr{R}' \cap \mathscr{R}_k) \cup S$ for S any part of $\tilde{\Gamma}_1$ satisfying $\Gamma_1 \subset\subset S \subset\subset \tilde{\Gamma}_1$. Moreover, the following estimate holds

$$\|B_i\|_{C^{2,\alpha}(\Lambda_k)} \leqq C\|\mathbf{f}_i\|_{C^\alpha(\bar{\mathscr{R}})} \leqq C\|f_i\|_{C^\alpha(\bar{R})} , \qquad i = 1,2\ k = 1,2$$

where C depends only on R.

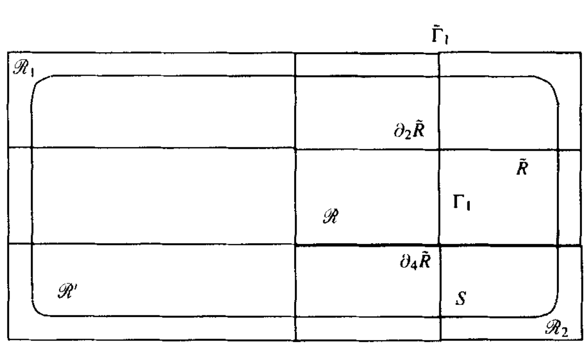

Figure 3. The reflected domain $\mathscr{R}$.

Therefore, since $h = \operatorname{div} B_i - \mathbf{b}$ solves $\Delta h = 0$ in $\mathcal{R}'$ with $h = \operatorname{div} B_i$ on $\partial\mathcal{R}'$, then h is harmonic in $\mathcal{R}'$. Hence, h is C^∞ in the interior of $\mathcal{R}'$, so that $\mathbf{b}$ is as good as div B_i, and estimating $\mathbf{b}$ in $\bar{R} \subset \mathcal{R}'$

$$\|\mathbf{b}\|_{C^{1,\alpha}(\bar{R})} \leqq C\|\operatorname{div} B_i\|_{C^{1,\alpha}(\bar{R})} \leqq C\|B_i\|_{C^{2,\alpha}(\bar{R})} \leqq C\|f\|_{C^\alpha(\bar{R})} .$$

Therefore,

$$\|b\|_{C^{1,\alpha}(\bar{R})} \leqq C \ \|f_1, f_2\|_{C^\alpha(\bar{R})} \tag{3.9}$$

where

$$\begin{aligned} \|f_1, f_2\|_{C^\alpha(\bar{R})} &= \|f\|_{C^\alpha(\bar{R});\mathrm{Lip},v} \\ &\leqq C\left(\|f\|_{C^\alpha,\mathbf{x},\bar{R}}\|f\|_{\infty,\bar{R}} + \|f\|_{\mathrm{Lip},v}\|v\|_{C^\alpha(\bar{R})}\right) \end{aligned} \tag{3.10}$$

where C depends on R. Therefore (3.2) holds. The proof of Lemma (3.1) is now complete.

In order to complete the proof of the existence part of Lemma 1.2, we make use of the Leray-Schauder fixed-point theorem; see [7], Chapter 11.

Let the map $T(v) = b$ be given by the solution of problem (3.1) which can be estimated by (3.2) and (3.3).

Indeed, estimate (3.3) provides the needed a priori estimate to show the operator T from the Banah space $\mathcal{B} = C^\alpha(\bar{R})$ into itself is bounded. Moreover, $T(C^\alpha(R)) \subset C^{1,\alpha}(\bar{R})$ is compactly embbeded in $C^\alpha(\bar{R})$, then T is a compact operator, and, for any b in $\mathcal{B} = C^{1,\alpha}(\bar{R})$ such that $b = \sigma T b$, for some $0 < \sigma < 1$, from (3.3) we obtain the a priori estimate

$$\|b\|_{C^{1,\alpha}(\bar{R})} \leqq \sigma C\left\{\|f\|_{C^\alpha,\mathbf{x},\bar{R}}\|f\|_{\infty,\bar{R}} + \|f\|_{\mathrm{Lip},b}\|f\|_{\infty,\bar{R}}\right\} \leqq M \tag{3.11}$$

where C is independent of b and σ, and so M is independent of σ and b.

Finally, since f is bounded and Lipschitz in v, then the map T is Lipschitz continuous. Indeed, $T(v_1) - T(v_2) = b_1 - b_2$ satisfies the equation $\Delta(b_1 - b_2) = \operatorname{div}(A(\mathbf{x})(v_1 - v_2))$, with $A(\mathbf{x}) = \frac{\partial f}{\partial v}(\mathbf{x}, (\lambda v_1 + (1-\lambda)v_2)(\mathbf{x}))$. Since $A(\mathbf{x}) \in C^\alpha(\bar{R})$ using estimates (3.9) and (3.10) for this equation the following estimates hold:

$$\begin{aligned} \|b_1 - b_2\|_{C^{1,\alpha}(\bar{R})} &\leqq C\left[\|A\|_{C^\alpha(\bar{R})}\|v_1 - v_2\|_{\infty,\bar{R}} + \|A\|_{\infty,\bar{R}}\|v_1 - v_2\|_{C^\alpha(\bar{R})}\right] \\ &\leqq C\|A\|_{C^\alpha(\bar{R})}\|v_1 - v_2\|_{C^\alpha(\bar{R})} . \end{aligned}$$

Thus, T is a continuous map.

Therefore T has a fixed point b which is the solution of the boundary value problem (1.3), and, by the estimate (3.11), (1.5) holds. Similarly from (3.2), (1.6) holds.

(b) Uniqueness of the Solution of Problem (1.3)

THEOREM 3.2. *Let b_1 and b_2 be two solutions in $W^{1,p}(\bar{R})$, $p > 2$, of the boundary value problem* (1.3), *then $b_1 = b_2$ on $\bar{R}$.*

Proof: We prove uniqueness in the reflected domain for the corresponding Dirichlet boundary value problem in a similar way as we worked out Lemma 3.1.

Let $\omega = b_1 - b_2$ on R. Then ω satisfies

$$\Delta\omega = \operatorname{div}(f(\mathbf{x}, b_1) - f(\mathbf{x}, b_2)) = \operatorname{div}(A(\mathbf{x})\omega)$$

where

$$A(\mathbf{x}) = \int_0^1 f_b(\mathbf{x}, t\,b_1 + (1-t)b_2)\,dt\,. \tag{3.12}$$

Since f is Lipschitz in b and C^α in $\mathbf{x}$ in $\bar{R}$, then $A(\mathbf{x})$ is $L^\infty(\bar{R})$.

Then, ω is a $C^{1,\alpha}(\bar{R})$ solution of linear elliptic operator in divergence form

$$\begin{cases} L\omega = \Delta\omega - \operatorname{div}(A(\mathbf{x})\omega) = 0 & \text{on } R \\ \omega = 0 & \text{on } \Gamma \\ \omega_{\mathbf{x}} = A_1(\mathbf{x})\omega & \text{on } \Gamma_1 \end{cases}\,. \tag{3.13}$$

Now if $\tilde{R}$ defined as above is the rectangle that contains R, its reflection with respect to Γ_1 and Γ_1, then, taking $\tilde{b}_1$ and $\tilde{b}_2$ the even reflection of b_1 and b_2 respectively with respect to Γ_1. Their difference $\tilde{\omega}$ is an even function that solves

$$\begin{aligned} L\tilde{\omega} = \Delta\tilde{\omega} - \operatorname{div}(\tilde{A}(x)\tilde{\omega}) = 0 \quad & \text{in } \tilde{R} \\ \omega = 0 \quad & \text{on } \partial\tilde{R} \end{aligned} \tag{3.14}$$

where

$$\tilde{A}(x) = \begin{cases} A(\mathbf{x}) & \text{if } \mathbf{x} \in \bar{R} \\ (-A_1(\mathbf{x}^*), A_2(\mathbf{x}^*)) & \text{if } \mathbf{x} \in \bar{\tilde{R}} \setminus \bar{R} \end{cases}$$

and $x^* \in R$ is the reflection of $x \in \bar{\tilde{R}} \setminus \bar{R}$ with respect to Γ_1.

In particular, for any test function $\varphi \in H_0^1(\tilde{R})$

$$\int_{\tilde{R}} \nabla\varphi(\nabla\tilde{\omega} - \tilde{A}(x)\tilde{\omega}) = 0\,. \tag{3.15}$$

We prove that the boundary value problem (3.14) has a unique solution $\tilde{\omega} = 0$.

In fact, it is enough to see that the adjoint operator to L is solvable in $H_0^1(\tilde{R})$ for any given bounded function in $\tilde{R}$ as a right-hand side. Indeed, by estimate (3.8) $\tilde{\omega} = \tilde{b}_1 - \tilde{b}_2$ is $C^\alpha(\tilde{R})$.

We solve for ψ in $H_0^1(\tilde{\tilde{R}})$ the adjoint problem

$$
\begin{aligned}
L^*\psi = \Delta\psi + \tilde{A}(x)\nabla\psi &= \tilde{\omega} \quad \text{in } \tilde{R} \\
\psi &= 0 \quad \text{on } \partial\tilde{R}\,.
\end{aligned}
\tag{3.16}
$$

Since $\psi \in H_0^1$ is an admissible test function, it satisfies (3.15). Then

$$
0 = \int_{\tilde{R}} \nabla\psi(\nabla\tilde{\omega} - \tilde{A}(x)\tilde{\omega}) = -\int_{\tilde{R}} (\Delta\psi + \tilde{A}(x)\nabla\psi)\tilde{\omega}
\tag{3.17}
$$

so that

$$
0 = \int_{\tilde{R}} \tilde{\omega}^2\,.
$$

In particular, $\tilde{\omega} = 0$ in $\tilde{R}$.

The existence and uniqueness of problem (3.16) in $\tilde{R}$ can be found in a recent paper by Berestycki, Nirenberg, and Varadhan; see [5]. They show that for given $\tilde{\omega}$ bounded in $\tilde{\tilde{R}}$, there exists a unique solution ψ of the boundary value problem (3.16), where ψ is $W^{2,p}$ near all smooth boundary points and continuous at corner points. Moreover, it is easy to see from their proof that $\nabla\psi \in L^2(\tilde{\tilde{R}})$. In particular, $\psi \in H_0^1(\tilde{R})$ so that (3.17) holds, and consequently, $\tilde{\omega} = 0$.

Appendix

LEMMA (THE CONFORMAL TRANSFORMATION). *There exist two constants C_1 and C_2, and a unique conformal transformation $F(z)\colon \Omega \to R$, with $R = [C_1, C_2] \times [0,1]$, such that $F(\partial_i\Omega) = \partial_i R$ with $\partial\Omega = \bigcup_{i=1}^{4} \partial_i K$, each $\partial_i K$ a side of K; where Ω is the domain characterized in the previous section.*

Proof: It is essential for the regularity of the conformal map that

$$
\partial\Omega = \bigcup_{i=1}^{4} \partial_i\Omega
$$

where each $\partial_i\Omega$ is a (piecewise) $C^{2,\alpha}$-curve that meets the preceding one at the point ω_i making a $\frac{\pi}{2}$ angle; that is, if γ_{i-1} and γ_i are the parametrizations of $\partial_{i-1}\Omega$ and $\partial_i\Omega$, respectively, then $\gamma_{i-1_T}(\omega_i) \perp \gamma_{i_T}(\omega_i)$, where the subindex T denotes tangential derivative.

Under these conditions, the proof is given by the classical result of conformal mapping that corresponds to an incompressible transformation. Take $\ell_2(\vec{x})$ the real harmonic function that solves

$$\begin{cases} \Delta \ell_2 = 0 & \text{in } \Omega \\ \ell_2 \Big|_{\partial_4 \Omega} = 0, \ \ell_2 \Big|_{\partial_2 \Omega} = 1 & \\ \frac{\partial \ell_2}{\partial \nu} \Big|_{\partial_1 \Omega \cup \partial_3 \Omega} = 0 & \end{cases};$$

ℓ_2 is a $C^{2,\alpha}(\bar{\Omega})$ harmonic function as the boundary condition is compatible on the points ω_i, $i = 1, \dots, 4$ because of the assumption of orthogonality on $\partial\Omega$ at ω_i.

Moreover, by the Hopf maximum principle, $0 < \ell_2 < 1$ in Ω and $\ell_{2y} > 0$ on $\partial_4\Omega \cup \partial_2\Omega$.

Now, let ℓ_1 be the conjugate harmonic function obtained by integrating the orthogonal field to the level curves of ℓ_2, that is, ℓ_1

$$\ell_1(x,y) = \int_{(x_0,y_0)}^{(x,y)(t)} (\ell_{2y} - \ell_{2x}) \cdot \gamma'(t) dt + \ell_1(x_0, y_0)$$

where $\gamma(t)$ is a path joining (x_0, y_0) to (x, y) with (x_0, y_0) a fixed point in $\partial\Omega$. Then $\ell_1 \in C^{2,\alpha}(\bar{\Omega})$ satisfies $\frac{\partial \ell_1}{\partial T} = 0$ on $\partial_1\Omega \cup \partial_3\Omega$, then $\ell_1 = C_1 = \ell_1(x_0, y_0)$ if $(x_0, y_0) \in \partial_1 R$ and $\ell_2 = C_2$ where C_2 is fixed by the path integral and by C_1.

Moreover, the function $\ell_{2,y}$ satisfies $\Delta\ell_{2,y} = 0$, $\ell_{2,y} > 0$ on $\partial_4\Omega \cup \partial_2\Omega$ and $\nabla \ell_{2,y} \cdot n = 0$ on $\partial_1\Omega \cup \partial_3\Omega$, then, by Hopf maximum principle, $\inf_\Omega \ell_{2y} \geqq \inf_{\partial\Omega} \ell_{2y} > 0$. Thus, $F(z) = \ell_1 + i\ell_2$ is analytic and $dF/dz = \ell_{1x} + \ell_{2y} + i(\ell_{1y} - \ell_{2x}) = 2(\ell_{2y} - i\ell_{2x})$ has positive real part. Then $F(z)$ is one to one. Hence, $F(z)$ is a conformal transformation that satisfies $F(\bar{\Omega}) = [C_1, C_2] \times [0, 1]$.

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