A discontinuous Galerkin method for the Vlasov-Poisson system

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Abstract

A discontinuous Galerkin method for approximating the Vlasov-Poisson system of equations describing the time evolution of a collisionless plasma is proposed. The method is mass conservative and, in the case that piecewise constant functions are used as a basis, the method preserves the positivity of the electron distribution function. The performance of the method is investigated by computing five example problems. In particular, computed results are benchmarked against established theoretical results for linear advection and the phenomenon of linear Landau damping for both the Maxwell and Lorentz distributions. Moreover, a nonlinear two-stream instability problem is computed to verify that the method conserves mass, momentum, and total energy. The obtained results demonstrate that the discontinuous Galerkin method accurately approximates the Vlasov-Poisson system.

Key words: Discontinuous Galerkin method, Vlasov-Poisson system, Landau damping, Lorentz distribution, two-stream instability

1 Introduction

The single species Vlasov-Poisson system is a kinetic system that can be used to model the time evolution of a collisionless plasma consisting of electrons and
a uniform background of fixed ions under the effects of a self-consistent electrostatic field and a supplied external field. In particular, the Vlasov equation models the transport of the electrons and is nonlinearly coupled to the gradient of the electrostatic potential. The potential satisfies the Poisson equation, which is a continuum description of Coulomb’s law. The unknown electron distribution function is denoted by \( f = f(x, v, t) \), where the independent variables \( x, v, t \) are position, velocity, and time, respectively. For a given \( t > 0 \), the quantity \( f(x, v, t) \, dx \, dv \) is the number of electrons contained in the infinitesimal phasespace volume element \( dx \, dv \) centered about \((x, v)\) at time \( t \). Upon a proper renormalization, \( f \) can be interpreted as a probability distribution function for the electrons over the phasespace.

Due to the nature of the Vlasov-Poisson system, the distribution function can exhibit a variety of dynamic phenomena. One of the most well-known examples of such phenomena is filamentation. Filamentation results when different characteristics of \( f \), where \( f \) is constant along each characteristic, intertwine in the phasespace. This results in stiff gradients, since \( f \) often takes on disparate values along different characteristics. Other interesting phenomena include the interaction between the electric field wave and the electrons, which sometimes results in a net energy transfer from the wave to the electrons leading to an exponential damping of the electric field modes in time.

Many of the numerical techniques for solving the Vlasov-Poisson system can be divided into two groups: (i) those that approximate the system in the phasespace directly and (ii) those that transform the system into a different coordinate space. The numerical approaches that treat the phasespace directly do not, however, usually involve discretizing the Vlasov equation. Rather most of these methods take advantage of the characteristic structure of the Vlasov equation, which implies that the plasma particles evolve along trajectories that satisfy a given set of ordinary differential equations. The most widely used particle scheme is the Particle-in-Cell (PIC) method [4, 12, 17]. The PIC method represents ensembles of micro-particles as a finite number of macro-particles. Each macro-particle is then assumed to evolve along a characteristic trajectory, where the electric field defining the trajectory is computed via any standard scheme. The PIC method seems to give reasonable results in cases where the tail of the distribution is negligible and a large number of particles is not necessary. Otherwise, the method suffers from numerical noise that is proportional to \( 1/\sqrt{n} \), where \( n \) is the number of particles.

Other methods based on the discretization of the phasespace have also been proposed. In [6], an operator splitting method was introduced and shown to be both efficient and accurate for solving a wide range of problems. A continuous finite element method was developed in [35, 36] and was shown to achieve similar results as those obtained in [6]. A positive and mass conservative scheme was employed in [15], both in one- and two-dimensional physical
space, to both linear and nonlinear damping problems. This method is defined at a given time step by first building a piecewise constant approximation over a mesh of the phasespace using the approximation obtained from the previous time step and two correction terms whose values are found by solving two fixed point problems. The piecewise approximation is then used in conjunction with a slope limiter to reconstruct a local polynomial approximation of \( f \) for each cell in the mesh.

In [18, 19], the phasespace was transformed using a Fourier transform and a splitting method was employed to advance the approximation in time. This method also included a filamentation filtering step for the purpose of smoothing the filaments. The numerical results obtained using this method seem reasonable only for problems in which filamentation is not a dominant effect, such as in some two-stream instability problems, where the nonlinearity either slows down or prevents the onset of filamentation. However, this method is inadequate for problems where the physics of interest depend upon the approximation accurately capturing the filamentary nature of the distribution, such as is the case in damping problems.

The objective of this paper is to propose the upwind penalty Galerkin (UPG) method for the approximation of the Vlasov-Poisson system and to evaluate its numerical efficacy. The UPG method is a discontinuous Galerkin (DG) finite element method and was first introduced in [16], where the UPG approximation to the true Vlasov solution was defined to be the unique solution to two variational equations. However, in this work a new definition of the UPG is proposed. In particular, the UPG approximation \( f_h \) to the true electron distribution \( f \) of the Vlasov-Poisson system is defined to be the solution of a first-order, nonlinear, ordinary differential equation (ODE) system. The numerical efficacy of the method is evaluated by performing accuracy, convergence, and conservation tests on computed UPG approximate solutions to a variety of linear and nonlinear perturbation problems. The computed results for these problems are benchmarked against known theoretical results and are compared to results obtained using established methods.

The upwind penalty Galerkin method gives a unified approach for approximating both the hyperbolic and elliptic parts of the Vlasov-Poisson system. Specifically, the Vlasov equation is discretized using the standard upwind Galerkin scheme for conservation laws [8–10] and the Poisson equation is discretized using one of three DG interior penalty schemes [29, 31, 33]. Stability and convergence estimates for the UPG method were presented in [16] for the six-dimensional phasespace. In the same reference, the method was shown to be both locally and globally mass conservative. Moreover, it was proved that the method is positive when piecewise constants are used to approximate the solution to the Vlasov equation.
In the context of computing plasma problems, the UPG method offers substantial gains. In particular, the local nature of the method facilitates adaptive mesh refinements and allows for straightforward parallelization. By taking advantage of these benefits, those regions of the phase space where the electron distribution experiences strong filamentation or boundary layer effects can be resolved by local mesh refinements. The discontinuous nature of the method also helps to resolve the stiff gradients associated with filamentation, since requiring the approximation to be continuous in these cases is too restrictive and would typically lead to excessive numerical diffusion and oscillatory behavior. Due to the fact that the method imposes boundary conditions weakly, a wide variety of boundary conditions can easily be accommodated. Another capability of the method is that it allows for the use of high-order basis functions without numerical stability problems in collisional plasma systems where the electron distribution does not experience rapid variations. This capability was demonstrated in [16], where the UPG method was used to approximate the Vlasov-Poisson-Fokker-Planck system. Thus, the method is well suited to approximate a range of plasmas spanning from the collisionless to the highly collisional regimes.

The initial development of DG methods for hyperbolic and elliptic equations occurred independently, but nearly simultaneously. In 1973, the first DG scheme for linear hyperbolic equations was introduced by Reed and Hill for approximating a neutron transport equation [27]. This work was followed by Lesaint and Raviart in 1974 [21], where a priori error estimates were proved for the DG method applied to two-dimensional, linear hyperbolic problems. The DG schemes for hyperbolic problems were further studied in the series of papers [8–10], which culminated in the introduction of the local discontinuous Galerkin (LDG) method [11]. The generality of the LDG method was further extended to the multidimensional setting under more relaxed assumptions on the data in [7].

One of the first DG schemes for approximating the solutions to second-order elliptic equations was introduced in 1971 by Nitsche [25]. In this work, Dirichlet boundary conditions were enforced weakly rather than strongly through the use of a penalty term. Shortly thereafter, applications of the penalty method to Laplace’s equation were proposed by Babuška et al. in [1–3]. The use of penalty terms across interior faces as a means of enforcing continuity among adjacent elements was introduced in [33] and [34] using a symmetric interior penalty Galerkin (SIPG) finite element method. A nonsymmetric interior penalty method (NIPG) similar to the SIPG method was proposed and analyzed in [29]. The incomplete interior penalty method (IIPG) was introduced in [13, 31, 32] and is very similar to the SIPG and NIPG methods.

The outline of this paper is as follows. In section 2, we describe the Vlasov-Poisson system and give a brief discussion of Landau damping; a proof of
Landau damping for a Lorentz equilibrium is given in appendix A. In section 3, the UPG method for the approximation of the Vlasov-Poisson system is derived. In section 4, numerical results are presented and analyzed. In section 5, conclusions about the efficacy of the UPG method are given.

2 The Vlasov-Poisson system

The Vlasov-Poisson system considered in this work has been scaled by the characteristic time and length scales for which the plasma dynamics of interest are most readily observed, i.e., time is scaled by the inverse plasma frequency \( \omega_p^{-1} \), length by the Debye length \( \lambda_D \), and velocity accordingly by the thermal velocity \( v_{th} = \omega_p \lambda_D \). Using this nondimensionalization, the Vlasov-Poisson system is stated as follows: find the electron distribution function \( f(x,v,t) \), electric field \( E(x,t) \) and potential \( \Phi(x,t) \) such that

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f - E \cdot \nabla_v f &= 0 \quad \Omega \times (0,T], \\
E &= -\nabla_x \Phi \quad \Omega_x \times (0,T], \\
-\Delta_x \Phi &= 1 - \int_{\mathbb{R}^n} f \, dv \quad \Omega_x \times (0,T],
\end{align*}
\]

for fixed \( T > 0 \), subject to an initial condition on \( f \) and boundary conditions on \( f \) and \( \Phi \). The domain \( \Omega \) satisfies \( \Omega = \Omega_x \times \mathbb{R}^n \), where the physical domain \( \Omega_x \subset \mathbb{R}^n \) can either be bounded or all of \( \mathbb{R}^n \). The constant \( n \) is either 1, 2, or 3, depending on the number of spatial dimensions being modeled.

The boundary condition given for \( f \) depends on \( \Omega_x \). If \( \Omega_x = \mathbb{R}^n \), then the condition \( f \to 0 \) must be enforced both as \( |x| \to \infty \) and as \( |v| \to \infty \). If the \( \Omega_x \) is bounded, then a condition must be imposed on \( f \) along the inflow boundary \( \Gamma_I \), where

\[
\Gamma_I = \{ (x,v) \in \partial \Omega_x \times \mathbb{R}^n | v \cdot \nu_x < 0 \},
\]

\( \nu_x \) being the unit outward normal vector to \( \partial \Omega_x \). Often, either \( f = f_I \), for a given \( f_I \), or periodic boundary conditions in \( x \) are imposed along \( \Gamma_I \).

Macroscopic quantities of interest, such as those arising in a continuum description, are easily computed from \( f \). The electron density \( \rho = \rho(x,t) \), current density \( j = j(x,t) \), kinetic energy density \( E_k(x,t) \) and potential energy density
\( \mathcal{E}_e(x, t) \) are defined by

\[
\rho(x, t) = \int_{\mathbb{R}^n} f(x, v, t) \, dv, \tag{3}
\]

\[
j(x, t) = \int_{\mathbb{R}^n} v f(x, v, t) \, dv, \tag{4}
\]

\[
\mathcal{E}_k(x, t) = \frac{1}{2} \int_{\mathbb{R}^n} |v|^2 f(x, v, t) \, dv, \tag{5}
\]

\[
\mathcal{E}_e(x, t) = \frac{1}{2} |E(x, t)|^2. \tag{6}
\]

The above macroscopic quantities satisfy a number of respective conservation laws; see [28].

Interesting properties of the Vlasov-Poisson system result by considering a linear perturbation \( \delta f(x, v, t) \) to an equilibrium distribution \( f_{eq}(v) \) over the 2D-phasespace \([0, L] \times (-\infty, \infty)\), \( L > 0 \). Specifically, suppose that \( f = f_{eq} + \delta f \), where \( f_{eq} \) is a given equilibrium probability distribution, \( \delta f \) and \( \Phi \) are \( L \)-periodic in \( x \), and the initial average value of \( \delta f \) over \( \Omega \) is zero. Equations (1a)-(1c) imply that \( \delta f \) satisfies

\[
\partial_t (\delta f) + v(\delta f)_x - E(\delta f)_v = Ef'_eq \quad [0, L] \times (-\infty, \infty) \times (0, T], \tag{7a}
\]

\[
E = -\Phi_x \quad [0, L] \times (0, T], \tag{7b}
\]

\[
\Phi_{xx} = \int_{\mathbb{R}^n} \delta f \, dv \quad [0, L] \times (0, T]. \tag{7c}
\]

Supposing \( |E(\delta f)_v| \ll 1 \) and dropping this term from (7a) leads to

\[
\partial_t (\delta f) + v(\delta f)_x = Ef'_eq \quad [0, L] \times (-\infty, \infty) \times (0, T], \tag{8a}
\]

\[
E = -\Phi_x \quad [0, L] \times (0, T], \tag{8b}
\]

\[
\Phi_{xx} = \int_{\mathbb{R}^n} \delta f \, dv \quad [0, L] \times (0, T]. \tag{8c}
\]

The linear system (8a)-(8c) was analyzed by Landau [20] for the case when \( f_{eq} = (2\pi)^{-1/2} e^{-v^2/2} \), where it was shown that the electric field mode \( E_k(\omega, t) \) decays exponentially in the time-asymptotic limit; for a proof of Landau damping in this case, see [22]. In particular, if we write the frequency as \( \omega(k) = \omega_R(k) + i\gamma(k) \), where \( \omega_R(k) \) and \( \gamma(k) \) are real-valued, then in the time-asymptotic limit \( E_k(\omega, t) \) decays at a rate \( \gamma(k) \) and oscillates at a frequency \( \omega_R(k) \).

There are equilibria other than the Maxwellian that give rise to exponential damping of the electric field. One such equilibrium is the Lorentz probability distribution, which is defined as
\[ f_{eq}(v) = \frac{v_{\theta}}{\pi} \frac{1}{v^2 + v_{\theta}^2}, \]  

for \( v_{\theta} > 0 \) given. In this case, it can be shown that the decay rate \( \gamma(k) \) of \( n \)-th electric field mode, \( k = 2\pi n/L \), is given by

\[ \gamma(k) = kv_{\theta} \]  

and the corresponding frequency of the electric field oscillations \( \omega(k) \) satisfies

\[ |\omega_R(k)| = 1. \]  

A derivation of (10)-(11) is given in appendix A. It is important to note that formulae (10) and (11) are explicit, which directly results from using the Lorentz equilibrium, whereas the formula for \( \gamma(k) \) and \( \omega_R(k) \) when a Maxwell equilibrium is used are implicitly defined [20].

3 Method Formulation

In this section, we derive the upwind-penalty Galerkin (UPG) method, which is a discontinuous Galerkin method, to approximate the Vlasov-Poisson system. The derivation of the UPG method for the Vlasov-Poisson system proceeds by first discretizing the Vlasov equation using the standard upwind Galerkin (UG) discretization for transport equations [7]. Here it is assumed that the electric field is given and hence the flow field \((v, -E)\) defining the Vlasov equation is known. It is assumed that any mesh for the phasespace, where by mesh we mean a partitioning of the phasespace into convex sets called elements, is the Cartesian cross-product of a mesh for the physical domain and a mesh for the velocity domain. Under this assumption, the physical and velocity domains can be independently refined. Given a mesh for the phasespace, the UG method then defines an approximate solution to the true Vlasov solution in such a way that at any given time the approximate solution restricted to each element of the mesh is a polynomial function. However, the approximate solution is not required to be continuous across the intersections of any two adjacent elements. Therefore, the approximate solution is a piecewise-defined polynomial function with respect to the mesh at any given time.

The next step in the derivation of the UPG method is to discretize the Poisson equation using one of the three interior penalty methods. The three methods are the symmetric interior penalty Galerkin (SIPG) method [33, 34], the nonsymmetric interior penalty Galerkin (NIPG) method [29], and the incomplete interior penalty Galerkin (IIPG) method [31, 32]. A detailed discussion and \textit{a priori} error estimates for each of the methods can be found in the respective
references. The only difference among the three methods is in the value of one specific parameter that arises in the weak formulation that is common to each of the three methods. To discretize the Poisson equation, it will be assumed that the righthandside source function in the equation is given. With this assumption, the penalty Galerkin methods each define a piecewise-polynomial approximation of the true solution to the Poisson equation using a mesh of the spatial domain, where in the context of the Vlasov-Poisson system the true solution to the Poisson equation is the electrostatic potential. The spatial domain mesh used in the discretization of the Poisson equation is required to be the same spatial domain mesh as that used in the UG discretization of the Vlasov equation. This requirement is practical one, as it would be very cumbersome, both in terms of analysis and implementation, to use two distinct meshes for the spatial domain in the discretization the Vlasov-Poisson system. However, the polynomial degree of the potential approximation on a given element of the spatial mesh is not required to equal the degree, with respect to \( x \), of the polynomial approximation of the distribution \( f \).

The UPG method of approximation to the Vlasov-Poisson system is defined by coupling together the UG method of approximation to the Vlasov equation and one of the interior penalty methods of approximation to the Poisson equation. As will be seen, this coupling results in a first-order, nonlinear, ODE system, in which the solution of the system uniquely determines the approximation \( f_h \) of \( f \). This resulting ODE system is readily solved using an explicit time-integrator such as the Runge-Kutta method. Moreover, in the process of computing \( f_h \), both the approximation \( E_h \) of the electric field \( E \) and the approximation \( \Phi_h \) of the potential \( \Phi \) are computed.

3.1 Preliminaries

Let \( \{T_h\}_{h>0} \) be a sequence of successively refined meshes of the bounded domain \( \Omega_x \subset \mathbb{R}^n \) and let \( \{T_h\}_{h>0} \) be a sequence of successively refined meshes of the bounded domain \( \Omega_v \subset \mathbb{R}^n \), where \( n = 1, 2, \) or 3. Implicit in the assumption that \( \Omega_v \) is bounded is that the velocity support of \( f \) is contained in \( \Omega_v \) for all times. Given the meshes \( T_{h_x} = \{K_{j_x}\}_{j_x=1}^{N_{h_x}} \) and \( T_{h_v} = \{K_{j_v}\}_{j_v=1}^{N_{h_v}} \), the elements \( K_{j_x} \) and \( K_{j_v} \) comprising each of the respective meshes are sets of the following types: intervals, if \( n = 1 \); triangles or quadrilaterals, if \( n = 2 \); and tetrahedra, prisms or hexahedra, if \( n = 3 \). Given the meshes \( T_{h_x} \) and \( T_{h_v} \), the spatial refinement level \( h_x \) and the velocity refinement level \( h_v \) are defined by \( h_x = \max_{j_x} \{\text{diam}(K_{j_x})\} \) and \( h_v = \max_{j_v} \{\text{diam}(K_{j_v})\} \).

A sequence of successively refined meshes \( \{T_h\}_{h>0} \) of the domain \( \Omega = \Omega_x \times \Omega_v \) is generated by defining each mesh \( T_h = \{K_{j}\}_{j=1}^{N_h} \) to be \( T_h = T_{h_x} \times T_{h_v} \), where the refinement level \( h \) is given by \( h = (h_x, h_v) \). Thus, for any given element
The derivation of the following UPG method requires the use of the broken Sobolev space $H^s(T_h)$, $s > 1/2$, which is defined as follows:

$$H^s(T_h) = \{ w \in L^2(\Omega) \mid w|_{K_j} \in H^s(K_j), \forall K_j \in T_h \},$$

(12)
i.e., $H^s(T_h)$ is the space of those functions that have elementwise weak derivatives up to, and including, the order $s$. Given nonnegative integers $r_x$ and $r_v$, the discontinuous approximation space $D^{r_x,r_v}(T_h) \subset H^s(T_h)$ considered in this work is

$$D^{r_x,r_v}(T_h) = \{ w \in H^s(T_h) \mid w|_{K_j} \in \mathbb{Q}^{r_x}(K_j) \times \mathbb{Q}^{r_v}(K_j), \forall K_j \in T_h \},$$

(13)
where $\mathbb{Q}^r(K)$ denotes the space of polynomials on a set $K$ that are less than or equal to $r$ in each variable. Thus, $\mathbb{P}^r(K) \subset \mathbb{Q}^r(K)$, where $\mathbb{P}^r(K)$ denotes the space of polynomials satisfying that the sum of the degrees of all the variables is less than or equal to $r$.

The discontinuous nature of the space $H^s(T_h)$ necessitates the introduction of mesh faces. If $K_j$ is a boundary element, then $F_k = \partial K_j \cap \partial \Omega$ is called a boundary mesh face. If $K_1$ and $K_2$ are two intersecting elements whose common intersection lies in the interior of $\Omega$, then $F_k = \partial K_1 \cap \partial K_2$ is said to be an interior mesh face. The set of all mesh faces is denoted by $\mathcal{F}_h = \{F_1, \ldots, F_{P_h}, F_{P_h+1}, \ldots, F_{M_h}\}$, where $F_k$ is an interior face if $1 \leq k \leq P_h$ and a boundary face if $P_h + 1 \leq k \leq M_h$. Each face $F_k \in \mathcal{F}_h$ is associated with a unit normal vector $\nu_k$. For $k > P_h$, $\nu_k$ is chosen to be the outward unit normal to $\partial \Omega$. For $1 \leq k \leq P_h$, we fix $\nu_k$ to be one of the two unit normal vectors to $F_k$. For every interior face $F_k$, the elements $K_1$ and $K_2$ will always be used to denote the two unique elements such that $F_k = K_1 \cap K_2$. Moreover, it is always assumed that $K_1$ satisfies $\nu_{K_1} = \nu_k$ on $F_k$, where $\nu_{K_1}$ denotes the outward unit normal to $\partial K_1$. Then $\nu_{K_2} = -\nu_k$ on $F_k$.

The fact that each mesh $T_h$ is the cross-product of a spatial mesh $T_{h_x}$ and a velocity mesh $T_{h_v}$ gives a specific structure to the boundaries of the elements and to the set of mesh faces $\mathcal{F}_h$. It follows that for $K_j = K_{j_x} \times K_{j_v}$ we have that $\partial K_j = (\partial K_{j_x} \times K_{j_v}) \cup (K_{j_x} \times \partial K_{j_v})$. Let us denote the set of mesh faces for $T_{h_x}$ and $T_{h_v}$ by $\mathcal{F}_{h_x} = \{F^x_1, \ldots, F^x_{P_{h_x}}, F^x_{P_{h_x}+1}, \ldots, F^x_{M_{h_x}}\}$ and $\mathcal{F}_{h_v} = \{F^v_1, \ldots, F^v_{P_{h_v}}, F^v_{P_{h_v}+1}, \ldots, F^v_{M_{h_v}}\}$, respectively, where $F^x_{k_x}$ is an interior face if $1 \leq k_x \leq P_{h_x}$ and a boundary face if $P_{h_x} + 1 \leq k_x \leq M_{h_x}$ and $F^v_{k_v}$ is an interior face if $1 \leq k_v \leq P_{h_v}$ and a boundary face if $P_{h_v} + 1 \leq k_v \leq M_{h_v}$. Then given any arbitrary $F_k \in \mathcal{F}_h$, either there exist a $F^x_{k_x} \in \mathcal{F}_{h_x}$ and a $K_{j_x} \in T_{h_x}$
such that $F_k = F^x_k \cup K_{jx}$ or there exist a $K_{jx} \in \mathcal{T}_{hx}$ and a $F^v_k \in \mathcal{F}_{hv}$ such that $F_k = K_{jx} \cup F^v_k$.

Two useful operators when considering functions in $H^s(\mathcal{T}_h)$ are the average and jump operators, which are respectively defined for $w \in H^s(\mathcal{T}_h)$ along an interior face $F_k$ by

$$
\bar{w} = \frac{1}{2} \left( (w|_{K_1})|_{F_k} + (w|_{K_2})|_{F_k} \right), \quad [w] = (w|_{K_1})|_{F_k} - (w|_{K_2})|_{F_k}.
$$

The above definitions are also valid for vector-valued functions $w \in [H^s(\mathcal{T}_h)]^{2n}$, in which case it follows that $[w] \cdot \nu_k = (w|_{K_1})|_{F_k} \cdot \nu_{K_1} + (w|_{K_2})|_{F_k} \cdot \nu_{K_2}$.

### 3.2 Upwind Galerkin approximation of the Vlasov equation

The derivation the UG method approximation to the Vlasov equation will be done for both for the simple case in which the phasespace is 2D, periodic boundary conditions in $x$ are imposed, and piecewise constants are used to approximate $f$ and for the general case in which the phasespace dimension is $2n$, $n = 1, 2$, or 3, inflow boundary conditions are prescribed, and piecewise polynomials of an arbitrary degree are used to approximate $f$. The derivation of the method for the simple 2D-phasespace case is given since this is an exact description of the method that was used to compute each of the numerical examples in Section 4. We note that in this case, the derivation is given without making use of the set of mesh faces $\mathcal{F}_h$.

For a given $T > 0$ and a data trio $(\alpha, f_0, f_I)$, the Vlasov equation, along with the corresponding initial and boundary conditions, considered in both of the following derivations is

\begin{align*}
\partial_t f + \alpha \cdot \nabla f &= 0 & \Omega \times (0, T], \\
f(t = 0) &= f_0 & \Omega, \\
f &= f_I & \Gamma_I \times (0, T],
\end{align*}

(14a) \quad (14b) \quad (14c)

where $\alpha = (v, -E) \in \mathbb{R}^{2n}$, $E \in \mathbb{R}^n$ is given, $\nabla = (\nabla_x, \nabla_v)$, and $\Omega = \Omega_x \times \Omega_v$. It is assumed that $\Omega_x = \Pi_{i=1}^n [0, X_i]$ and $\Omega_v = [-V_c, V_c]^n$, where $X_1, \ldots, X_n, V_c > 0$ are fixed. The definition of the inflow boundary $\Gamma_I$ is

$$
\Gamma_I = \{ (x, v) \in \partial \Omega \mid \alpha \cdot \nu < 0 \},
$$

(15)

where $\nu$ is the outward unit normal to $\partial \Omega$. Then $\Gamma_o = \partial \Omega / \Gamma_I$. 

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3.2.1 Piecewise constant approximation in 2D-phasespace

Let us assume that the spatial and velocity domains are such that $\Omega_x = [0, L]$ and $\Omega_v = [-V_c, V_c]$. Suppose $0 = x_0 < x_1 < \ldots < x_{N_{hx}} - 1 < x_{N_{hx}} = L$ and $-V_c = v_0 < v_1 < \ldots < v_{N_{hv}} - 1 < v_{N_{hv}} = V_c$, where $N_{hx}, N_{hv} \in \mathbb{N}$. For $j_x = 1, \ldots, N_{hx}$ and $j_v = 1, \ldots, N_{hv}$, define $T_{hx} = \{K_{j_x}\}_{j_x=1}^{N_{hx}}$ and $T_{hv} = \{K_{j_v}\}_{j_v=1}^{N_{hv}}$ by defining each spatial element $K_{j_x}$ to be the closed interval $[x_{j_x-1}, x_{j_x}]$ and each velocity element $K_{j_v}$ to be the closed interval $[v_{j_v-1}, v_{j_v}]$. For each $K_{j_x} \in T_{hx}$ and $K_{j_v} \in T_{hv}$, the quantities $h_{j_x}$ and $h_{j_v}$ are defined as $h_{j_x} = x_{j_x} - x_{j_x-1}$ and $h_{j_v} = v_{j_v} - v_{j_v-1}$. A mesh $T_h = \{K_j\}_{j=1}^{N_h}$ of the phasespace domain $\Omega$ is now generated according to $K_j = K_{j_x} \times K_{j_v}$, where the index $j$ is defined by the element ordering $j = (j_v - 1)N_{hx} + j_x$, for $j_x = 1, \ldots, N_{hx}$ and $j_v = 1, \ldots, N_{hv}$. This implies that $T_h$ contains a total of $N_h = N_{hx}N_{hv}$ elements.

In order to construct basis functions, for $i_x = 1, \ldots, N_{hx}$, let $\theta^{i_x}(x) = 1$, for $x \in K_{i_x}$, and let $\theta^{i_x}(x) = 0$, otherwise, and for $i_v = 1, \ldots, N_{hv}$, let $\chi^{i_v}(v) = 1$, for $v \in K_{i_v}$, and let $\chi^{i_v}(v) = 0$, otherwise. Then, for $i = 1, \ldots, N_h$, define $\psi^i(x, v) = \theta^{i_x}(x)\chi^{i_v}(v)$. It follows that $D^{0,0}(T_h) = \text{span}\{\psi^1, \ldots, \psi^{N_h}\}$.

The derivation of the weak formulation begins by multiplying (14a) by the test function $\psi^i$ and integrating the resulting equation by parts each the element $K_i$ to get

$$
\begin{align*}
\partial_t \left( \int_{K_i} f \, dvdx \right) + \int_{\partial K_i} \mathbf{n} \cdot \nu_{K_i} \, dS + \int_{\partial K_i \cap \Gamma_o} f \mathbf{n} \cdot \nu_{K_i} \, dS &= 0, \\
\int_{\partial K_i \cap \Gamma_t} f_I \mathbf{n} \cdot \nu_{K_i} \, dS &= 0,
\end{align*}
$$

where $f_I$ is defined by

$$
f_I(x, v, t) = \begin{cases} 
  f(L, v, t), & \text{if } x = 0, \\
  f(0, v, t), & \text{if } x = L, \\
  0, & \text{if } |v| = V_c.
\end{cases}
$$

and where $\nu_{K_i}$, the outward unit normal to $K_i$, is given by

$$
\nu_{K_i}(x, v) = \begin{cases} 
  (0, -1), & \text{if } v = v_{i_v-1}, \\
  (1, 0), & \text{if } x = x_{i_x}, \\
  (0, 1), & \text{if } v = v_{i_v}, \\
  (-1, 0), & \text{if } x = x_{i_x-1}.
\end{cases}
$$

Since we want to approximate $f$ by a piecewise constant $f_h$, Eq. (22) needs to be adjusted so that it is defined for functions such as $f_h$ that are discontinuous.
across inter-element boundaries. To this end, the upwind function \( f^u \) on a set \( \partial K_i/\Gamma \) is defined as follows:

\[
f^u(x, v, t; \alpha) = \begin{cases} 
  f^-(x, v, t), & \text{if } \alpha(x, t) \cdot \nu_{K_i} \geq 0, \\
  f^+(x, v, t), & \text{if } \alpha(x, t) \cdot \nu_{K_i} < 0,
\end{cases}
\]  

(19)

where

\[
f^-(x, v, t) = \lim_{s \to 0^-} f((x, v) + s\nu_{K_i}, t)
\]

(20)

and

\[
f^+(x, v, t) = \lim_{s \to 0^+} f((x, v) + s\nu_{K_i}, t).
\]

(21)

Upon replacing \( f \) by \( f^u \) in the integration over the set \( \partial K/\Omega \) in (22), we get that

\[
\partial_t \left( \int_{K_i} f \, dv \, dx \right) + \int_{\partial K_i/\partial \Omega} f^u \alpha \cdot \nu_{K_i} \, dS + \int_{\partial K_i \cap \Gamma_0} f \alpha \cdot \nu_{K_i} \, dS + \int_{\partial K_i \cap \Gamma_I} f I \alpha \cdot \nu_{K_i} \, dS = 0,
\]

(22)

We now proceed to define the piecewise-constant UG approximation \( f_h \) to \( f \). First, \( f_h \) is assumed to have the form

\[
f_h(x, v, t) = \sum_{j=1}^{N_h} \beta^j(t) \psi^j(x, v),
\]

(23)

from which it follows that \( (f_h)_{\mid K_i} = \beta^j(t) \). This implies

\[
\partial_t \left( \int_{K_j} f_h \, dv \, dx \right) = h_{x_j} h_{v_j} \beta^j(t).
\]

(24)

We now use Eqs. (22) and (24) to define the semi-discrete UG approximation \( f_h = \sum_{j=1}^{N_h} \beta^j(t) \psi^j \) to be the unique function in \( C^1([0, T], D^{0,0}(\mathcal{T}_h)) \) satisfying the initial condition \( \int_{K_j} f_h(x, v, 0) = \int_{K_j} f_0, \forall i \in \{1, \ldots, N_h\} \), and, \( \forall t \in (0, T], \)

\[
h_{x_j} h_{v_j} \beta^j(t) + \int_{\partial K_i/\partial \Omega} (f_h)^n(\beta(t)) \alpha \cdot \nu_{K_i} \, dS + \int_{\partial K_i \cap \Gamma_0} f_h(\beta(t)) \alpha \cdot \nu_{K_i} \, dS + \int_{\partial K_i \cap \Gamma_I} (f_h(\beta(t))) I \alpha \cdot \nu_{K_i} \, dS = 0,
\]

for \( i = 1, \ldots, N_h \).

(25)

Eq. (25) is equivalent to a linear, ordinary differential equation system, where the linearity of this system results from the fact that the electric field \( E \) is assumed to be given.

We note that the integration along \( \partial K_i/\Gamma \) in (25) in the case when \( K_i \) is an
interior element, i.e., $\partial K_i \cap \partial \Omega = \emptyset$, becomes

$$
\int_{\partial K_i} f \alpha \cdot \nu_{K_i} \, dS = \int_{x_{i_{v-1}}}^{x_{i_v}} E(x, t) f(x, v_{i_{v-1}}, t) \, dx - \int_{x_{i_{v-1}}}^{x_{i_v}} E(x, t) f(x, v_{i_{v}}, t) \, dx - \int_{v_{i_{v-1}}}^{v_{i_v}} v f(x_{i_{v-1}}, v, t) \, dv + \int_{v_{i_{v-1}}}^{v_{i_v}} v f(x_{i_{v}}, v, t) \, dv
$$

(26)

and the integrations along $\partial K_i \cap \Gamma_O$ and $\partial K_i \cap \Gamma_I$ satisfy

$$
\int_{\partial K_i \cap \Gamma_O} f_h \alpha \cdot \nu_{K_i} \, dS = \int_{\partial K_i \cap \Gamma_I} (f_h)_I \alpha \cdot \nu_{K_i} \, dS = 0.
$$

(27)

### 3.2.2 General formulation

For domains $K \subset \mathbb{R}^{2n}$, let $\langle \cdot, \cdot \rangle_K$ denote the $L^2(K)$-inner product. To distinguish integration over domains $K \subset \mathbb{R}^{2n-1}$, we use the notation $\langle \cdot, \cdot \rangle_K$. To derive a weak formulation for (14a)-(14c), multiply equation (14a) by an arbitrary test function $w \in H^1(T_h)$ and integrate the resulting expression by parts over an arbitrary $K_j \in T_h$. Then, for every $t \in (0, T]$, we obtain

$$
\left( \partial_t f, w \right)_{K_j} - \left( f \alpha, \nabla w \right)_{K_j} + \left( f w^-, \alpha \cdot \nu_{K_j} \right)_{\partial K_j} = 0,
$$

(28)

where $w^-$ denotes the interior trace of $w$ on $K_j$, i.e.

$$
w^-(x, v) = \lim_{s \to 0^-} w \left( (x, v) + s \nu_{K_j} \right),
$$

(29)

with $\nu_{K_j}$ being the outward unit normal to $\partial K_j$. Upon summing equation (28) over all $K_j$ and weakly enforcing the inflow boundary condition, we get

$$
\left( \partial_t f, w \right)_\Omega - \sum_{j=1}^{N_h} \left( f, \alpha \cdot \nabla w \right)_{K_j} + \sum_{k=1}^{P_h} \langle f[w], \alpha \cdot \nu_k \rangle_{F_k} + \sum_{F_k \in \Gamma_O} \langle f w, \alpha \cdot \nu_k \rangle_{F_k}
$$

$$
= - \sum_{F_k \in \Gamma_I} \langle f_I w, \alpha \cdot \nu_k \rangle_{F_k}.
$$

(30)

Since our goal is to approximate $f$ by a function $f_h$ that is discontinuous across the interior faces, equation (30) is now modified so that it is well-defined for such functions. A standard technique is to replace $f$ by its “upwind value” $f^u$ on the interior faces [7], where $f^u$ along each interior face $F_k$ is defined as

$$
f^u(x, v, t; \alpha) = \begin{cases} 
    f_{|K_1}(x, v, t), & \text{if } \alpha(x, t) \cdot \nu_k \geq 0, \\
    f_{|K_2}(x, v, t), & \text{if } \alpha(x, t) \cdot \nu_k < 0.
\end{cases}
$$

(31)
Clearly, (31) is consistent in the sense that since $f$ is continuous across each interior face $F_k$, it follows that $f''(\alpha) = f$ on $F_k$. We also note that $f_u$ depends nonlinearly on $\alpha$. In general, if $\alpha_1$ and $\alpha_2$ are two flow fields having different values on $F_k$ and if $g \in H^s \{ \mathcal{T}_h \}$ is discontinuous across $F_k$, then $g''(\alpha_1 + \alpha_2) \neq g''(\alpha_1) + g''(\alpha_2)$. Replacing $f$ on the interior faces in (30) by its upwind value $f_u$ leads to

$$
(\partial_t f, w)_\Omega - \sum_{j=1}^{N_h} (f, \alpha \cdot \nabla w)_{K_j} + \sum_{k=1}^{P_h} (f''(\alpha)(w), \alpha \cdot \nu_k)_{F_k} + \sum_{F_k \in \Gamma_h} \langle f, w, \alpha \cdot \nu_k \rangle_{F_k} = -\sum_{F_k \in \Gamma_h} \langle f, w, \alpha \cdot \nu_k \rangle_{F_k}.
$$

The upwind Galerkin scheme for the Vlasov equation is now defined by using (32).

Define the bilinear operator $\mathcal{A}$ to be the last three terms in the lefthandside of equation (32), i.e.,

$$
\mathcal{A}(f, w; \alpha) = -\sum_{j=1}^{N_h} (f, \alpha \cdot \nabla w)_{K_j} + \sum_{k=1}^{P_h} (f''(\alpha)(w), \alpha \cdot \nu_k)_{F_k} + \sum_{F_k \in \Gamma_h} \langle f, w, \alpha \cdot \nu_k \rangle_{F_k},
$$

and define the linear operator $\mathcal{L}$ to be the righthandside term in (32), i.e.,

$$
\mathcal{L}(w; \alpha, f) = -\sum_{F_k \in \Gamma_h} \langle f, w, \alpha \cdot \nu_k \rangle_{F_k}.
$$

We now define the following:

**Definition 1** The semi-discrete function $f_h \in C^1([0, T], D^{r_x, r_v}(\mathcal{T}_h))$ is the upwind Galerkin approximation to Vlasov solution $f$ if

\[
\begin{align*}
(i) & \quad \forall \ K_j \in \mathcal{T}_h, \\
& \quad (f_h(x, v, 0), w_h)_{K_j} = (f_a, w_h)_{K_j} \quad \text{and} \quad (35a) \\
(ii) & \quad \forall \ t \in (0, T], \\
& \quad (\partial_t f_h, w_h)_\Omega + \mathcal{A}(f_h, w_h; \alpha) = \mathcal{L}(w_h; \alpha, f), \quad (35b)
\end{align*}
\]

$\forall w \in D^{r_x, r_v}(\mathcal{T}_h)$.

We note that (35b) is equivalent to a first-order ordinary differential equation (ODE) system to be described below.

For each $K_i = K_{ix} \times K_{iv} \in \mathcal{T}_h$, let $\psi_1^i, \ldots, \psi_{n_b}^i$ be a basis for $Q^{r_x}(K_{ix}) \times Q^{r_v}(K_{iv})$, where $n_b = (r_x + 1)^n \times (r_v + 1)^n$. We extend the domain of these
functions to \(\Omega\) by defining each to be identically zero in \(\Omega/K_i\). Let \(\beta(t) = (\beta_1(t), \ldots, \beta_{n_b}(t), \ldots, \beta_{N_h}^N(t), \ldots, \beta_{N_h}^{n_b}(t))\) denote the unique vector such the upwind Galerkin approximation \(f_h\) satisfies

\[
f_h(x, v, t) = \sum_{j=1}^{N_{h}} \sum_{m=1}^{n_b} \beta_{m}^j(t) \psi_{m}^j(x, v). \tag{36}
\]

For \((x, v) \in K_j\), it then follows that \(f|_{K_j} = \sum_{m=1}^{n_b} \beta_{m}^j \psi_{m}^j\). Inserting (36) into (35b) yields that

\[
\sum_{j=1}^{N_{h}} \sum_{m=1}^{n_b} \dot{\beta}_{m}^j(t) (\psi_{m}^j, w_h)_{\Omega} + \sum_{j=1}^{N_{h}} \sum_{m=1}^{n_b} \beta_{m}^j(t) A(\psi_{m}^j, w_h; \alpha) = L(w_h; \alpha, f_I), \quad \forall w_h \in D^{x,v}(T_h). \tag{37}
\]

Since \(\{\psi_{p}^i\}_{p=1, i=1}^{n_b, N_{h}}\) is a basis for \(D^{x,v}(T_h)\), it follows that (37) is equivalent to

\[
\sum_{m=1}^{n_b} \dot{\beta}_{m}^i(t) (\psi_{m}^i, \psi_{p}^i)_{K_i} + \sum_{j \in N(i)} \sum_{m=1}^{n_b} \dot{\beta}_{m}^j(t) A(\psi_{m}^j, \psi_{p}^i; \alpha) = L(\psi_{p}^i; \alpha, f_I), \quad \forall i \in \{1, \ldots, N_h\}, \forall p \in \{1, \ldots, n_b\}, \tag{38}
\]

where \(N(i)\) is the set containing the indices of those neighboring elements of \(K_i\).

Eq. (38) is seen to generate an equivalent matrix system, where \(n_b\) rows of the matrix are generated at a time by sequentially taking \(i\) to equal 1, \ldots, \(N_h\) and for each \(i\) sequentially taking \(p\) to equal 1, \ldots, \(n_b\) in Eq. (38). This procedure results in the matrix ODE system

\[
A_1 \dot{\beta}(t) + A_2(\alpha) \beta(t) = L(\alpha, f_I), \tag{39}
\]

where \(A_1\) is the corresponding constant \(n_bN_h \times n_bN_h\) matrix, \(A_2(\alpha)\) is the corresponding \(n_bN_h \times n_bN_h\) sparse matrix and \(L(\alpha, f_I)\) is a vector of length \(n_bN_h\).

Since the support of the functions \(\psi_{1}^i, \ldots, \psi_{n_b}^i\) is \(K_i\), \(\forall i \in \{1, \ldots, N_h\}\), it follows that \(A_1\) is a block-diagonal matrix, where each block is a \(n_b \times n_b\) matrix. This implies that the inverse of \(A_1\) is easily computed. Thus, the upwind Galerkin approximation \(f_h\) is equivalently defined to be the unique solution to

\[
\dot{\beta}(t) = -A_1^{-1}A_2(\alpha) \beta(t) + A_1^{-1}L(\alpha, f_I), \tag{40}
\]

where the initial condition \(\beta(0)\) is uniquely determined by (35a). To solve this system in time, an explicit time stepping method such as the Runge-Kutta method can be used.
3.3 Interior penalty approximations of the Poisson equation

We now derive the interior penalty formulations for the Poisson equation. As already noted, the mesh $\mathcal{T}_h$ used to discretize the Poisson equation must be the same mesh that was used for the spatial domain to discretize the Vlasov equation, but polynomial degree $r_x$ of the approximation to the Poisson equation is not required to equal the degree in $x$ of $f_h$.

For a given source $G \in L^2(\Omega_x)$ and Dirichlet boundary data $\Phi_D \in L^2(\partial \Omega_x)$, the Poisson equation to be discretized using the symmetric interior penalty (SIPG) method [33, 34], incomplete interior penalty (IIPG) method [13, 31, 32], and nonsymmetric interior penalty (NIPG) [29] method is

$$-\Delta_x \Phi = G \quad \Omega_x, \quad (41a)$$
$$\Phi = \Phi_D \quad \partial \Omega_x. \quad (41b)$$

In this subsection, for $K \subset \mathbb{R}^n$, $n=1,2,3$, we let $(\cdot, \cdot)_K$ denote the $L^2(K)$-inner product and for integrations over sets $K \subset \mathbb{R}^{n-1}$, we use the notation $\langle \cdot, \cdot \rangle_K$. The SIPG, IIPG, and NIPG formulations are all derived by first multiplying (41a) by an arbitrary test function $\theta \in H^{s}(\mathcal{T}_h)$, $s > 1/2$, then integrating by parts on each $K_{j_x} \in \mathcal{T}_h$, and then summing each of the resulting local equations. This leads to

$$\sum_{j_x=1}^{N_{hx}} (\nabla_x \Phi, \nabla_x \theta)_{K_{j_x}} - \sum_{j_x=1}^{N_{hx}} \langle \nabla_x \Phi \cdot \nu_{K_{j_x}}, \theta^- \rangle_{\partial K_{j_x}} = (G, \theta)_{\Omega_x}, \quad (42)$$

where $\nu_{K_{j_x}}$ is the outward unit normal to $\partial K_{j_x}$.

To proceed, we make use of the identity

$$\sum_{j_x=1}^{N_{hx}} (\nabla_x \Phi \cdot \nu_{K_{j_x}}, \theta^-)_{\partial K_{j_x}} = \sum_{k_x=1}^{P_{hx}} \langle \nabla_x \Phi [\theta] + [\nabla_x \Phi \cdot \theta]_{\partial K_{j_x}}, \nu_{k_x} \rangle_{F_{k_x}} + \sum_{F_{k_x} \in \partial \Omega_x} \langle \nabla_x \Phi \cdot \nu_{k_x}, \theta \rangle_{F_{k_x}} \quad (43)$$

in (42) to get that

$$\sum_{j_x=1}^{N_{hx}} (\nabla_x \Phi, \nabla_x \theta)_{K_{j_x}} - \sum_{k_x=1}^{P_{hx}} \langle \nabla_x \Phi [\theta] + [\nabla_x \Phi \cdot \theta]_{\partial K_{j_x}}, \nu_{k_x} \rangle_{F_{k_x}} - \sum_{F_{k_x} \in \partial \Omega_x} \langle \nabla_x \Phi \cdot \nu_{k_x}, \theta \rangle_{F_{k_x}} = (G, \theta)_{\Omega_x}. \quad (44)$$

Since the true solution $\Phi$ has the regularity $\Phi \in H^1(\Omega_x) \cap H^2(\mathcal{T}_h)$, it follows that $[\Phi] \equiv 0$ and $[\nabla_x \Phi] \equiv 0$ along every interior face $F_{k_x}$ [14]. Thus, (44)
reduces to

$$\sum_{j_x=1}^{N_{hx}} (\nabla_x \Phi, \nabla_x \theta)_{K_{j_x}} - \sum_{k_x=1}^{P_{hx}} \langle \nabla_x \Phi [\theta], \nu_{k_x} \rangle_{F_{k_x}} - \sum_{F_{k_x} \in \partial \Omega_x} \langle \nabla_x \Phi, \nu_{k_x} \cdot \theta \rangle_{F_{k_x}}$$

$$= (G, \theta)_{\Omega_x} \quad \quad (45)$$

The differences among the SIPG, IIPG and NIPG methods are now seen through the addition to the lefthandside of (45) of the term

$$c_s \sum_{k_x=1}^{P_{hx}} \langle \nabla_x \theta [\Phi], \nu_{k_x} \rangle_{F_{k_x}}, \quad \quad (46)$$

which equals 0 by the regularity of $\Phi$. The values selected for $c_s$ for the SIPG, IIPG and NIPG methods are -1, 0 and 1, respectively.

The interior penalty formulations are now completed by weakly enforcing both approximate continuity across the interior mesh faces and the Dirichlet boundary condition. Continuity is enforced through the interior penalty term

$$\sum_{k_x=1}^{P_{hx}} \frac{\sigma}{(h_{j_x})^{n/2}} \langle [\Phi], [\theta] \rangle_{F_{k_x}}, \quad \quad (47)$$

and the Dirichlet condition is enforced through the boundary penalty term

$$\sum_{F_{k_x} \in \partial \Omega_x} \frac{\sigma}{(h_{j_x})^{n/2}} \langle \Phi - \Phi_D, \theta \rangle_{F_{k_x}}, \quad \quad (48)$$

where $\sigma > 0$ is an arbitrary penalty parameter that is usually set equal 1. Adding both penalty terms and (46) to the lefthandside of (45) results in

$$\sum_{j_x=1}^{N_{hx}} (\nabla_x \Phi, \nabla_x \theta)_{K_{j_x}} - \sum_{k_x=1}^{P_{hx}} \langle \nabla_x \Phi [\theta] + c_s \nabla_x \theta [\Phi], \nu_{k_x} \rangle_{F_{k_x}}$$

$$+ \sum_{k_x=1}^{P_{hx}} \frac{\sigma}{(h_{j_x})^{n/2}} \langle [\Phi], [\theta] \rangle_{F_{k_x}} - \sum_{F_{k_x} \in \partial \Omega_x} \langle \nabla_x \Phi \cdot \nu_{k_x} \cdot \theta \rangle_{F_{k_x}} + \sum_{F_{k_x} \in \partial \Omega_x} \frac{\sigma}{(h_{j_x})^{n/2}} \langle \Phi, \theta \rangle_{F_{k_x}}$$

$$= (G, \theta)_{\Omega_x} + \sum_{F_{k_x} \in \partial \Omega_x} \frac{\sigma}{(h_{j_x})^{n/2}} \langle \Phi_D, \theta \rangle_{F_{k_x}} \quad \quad (49)$$

Eq. (49) completely defines each of the three interior penalty schemes.

Define the bilinear operator $B(\Phi, \theta)$ to be equal to the five terms in the lefthandside of (49) and define the linear operator $H(\theta; G, \Phi_D)$ to be the two terms in the righthandside of (49). We now give the following definition:
Definition 2 Given $c_s = -1$ (SIPG), 0 (IIPG) or 1 (NIPG), the function $\Phi_h \in D^r_x(T_{h_x})$ is the corresponding interior penalty Galerkin approximation to the Poisson solution $\Phi$ if

$$\mathcal{B}(\Phi_h, \theta_h) = \mathcal{H}(\theta_h; G, \Phi_D),$$

(50)

$\forall \theta_h \in D^r_x\{T_{h_x}\}$.

The linearity of $\mathcal{B}$ implies that (50) is equivalent to a matrix system, which is described below.

To write (50) as a matrix system, let $n_b = (r_x + 1)^n$ and let $\mu = \{\mu_1^1, \ldots, \mu_{n_b}^1, \ldots, \mu_1^{N_{hx}}, \ldots, \mu_{n_b}^{N_{hx}}\}$ be the unique vector such that

$$\Phi_h = \sum_{j_x=1}^{N_{hx}} \sum_{m=1}^{n_b} \mu_m^{j_x} \theta_m^{j_x}(x).$$

(51)

Upon substituting this representation into (50), it can easily be shown that (50) is equivalent to

$$B\mu = H,$$

(52)

where $B$ is an $(n_b + 1) N_{hx} \times (n_b + 1) N_{hx}$ invertible sparse matrix and $H(G, \Phi_D)$ is a vector of length $(n_b + 1) N_{hx}$. However, $B$ is not block diagonal as was the case for $A_1$ in the Vlasov ODE system. Thus, if the spatial domain is two- or three-dimensional then using an iterative solver is, in general, the most efficient means of computing the solution $\mu$ in (52). However, for the one-dimensional spatial domain, $\mu$ can be computed by using an LU-matrix decomposition algorithm to factor $B$. For convenience, we write $\mu = B^{-1}H$, even though in practice $\mu$ might be computed using an iterative method.

3.4 Discontinuous Galerkin approximation of the Vlasov-Poisson system

The upwind penalty Galerkin (UPG) method for the approximation of the Vlasov-Poisson system is now introduced. The method results from combining the upwind Galerkin approximation of the Vlasov equation together with the interior penalty approximation of the Poisson equation.

In order to define the approximation $f_h$ to the solution $f$ of the Vlasov-Poisson system (1a)-(1c), let us suppose for the moment that $f_h$ is given. Then an approximation $\alpha_h = (v, -E_h)$ to $\alpha = (v, -E)$ is determined by first using one of the interior penalty approximation schemes to compute an approximation $\Phi_h$ to the potential $\Phi$ via the formula $\mu = B^{-1}H(G, \Phi_D)$, where $G$ is defined to be $1 - \int f_h dv$. Then, for a given $K_{j_x} \in T_{h_x}$, $(E_h)_{K_{j_x}}$ is computed according to $(E_h)_{K_{j_x}} = -\nabla_x(\Phi_h)_{K_{j_x}}$, which implies that $E_h$ is discontinuous across the
interior faces of $\mathcal{T}_h$. By defining $E_h$ in this way, it follows that $E_h = E_h(f_h)$ and hence $\alpha_h = \alpha_h(f_h)$, or equivalently, $\alpha_h = \alpha_h(\beta(t))$.

Using the above prescription for computing $\alpha_h$, the approximate solution $f_h$ to the Vlasov-Poisson system is computed by solving the ODE (40) with $\alpha$ replaced by $\alpha_h(\beta(t))$. This leads to the following definition:

**Definition 3** Given $c_s = -1, 0, 1$, the semi-discrete function $f_h(\beta(t)) \in C^1([0, T], D^{s.r.v.}(\mathcal{T}_h))$ is the upwind-penalty Galerkin approximation to Vlasov-Poisson solution $f$ if

\begin{align}
(i) \quad & \forall K_j \in \mathcal{T}_h, \forall w \in D^{s.r.v.}(\mathcal{T}_h) \\
& (f_h(\beta(0)), w_h)_{K_j} = (f_0, w_h)_{K_j} \quad \text{and} \\
(ii) \quad & \forall t \in (0, T],
\end{align}

\begin{align}
\dot{\beta}(t) = -A_1^{-1}A_2(\alpha_h(\beta(t)))\beta(t) + A_1^{-1}L(\alpha_h(\beta(t)), f_I).
\end{align}

System (53a)-(53b) can be solved for $\beta(t)$ using any explicit time-integrator.

4 Numerical Results

In this section, numerical results are presented for five example problems to test the accuracy and convergence of the proposed DG method when piecewise constants are used to approximate the distribution $f$ and piecewise quadratic polynomials are used to approximate the potential $\Phi$. In 4.1 and 4.2, we test the ability of the method to solve the Vlasov equation in the case that the electric field is identically zero. The next two examples test the ability of the method to reproduce results consistent with the theoretical results of linear Landau damping. In 4.3, the equilibrium is taken to be a Maxwell distribution. Results are computed using four successively refined uniform meshes in order to demonstrate that the numerical decay rates converge to the theoretical decay rate under mesh refinement. In 4.4, the equilibrium is taken to be a Lorentz distribution. Four different problems are computed in 4.4 in order to show that the accuracy of the DG method is robust across different wavenumbers $k$. Finally, in 4.5 a nonlinear two-stream instability problem is computed which has been studied in [19, 26]. In all of the examples, time is discretized using the fourth-order Runge-Kutta method. In examples two through five, the NIPG method is the penalty method employed to approximate the Poisson system, and the linear system that results from using the NIPG method
is solved using an LU-decomposition algorithm.

Throughout this section, it is assumed that the distribution \( f \) has the form
\[
f(x, v, t) = f_{eq}(v) + \delta f(x, v, t),
\]
(54)

In 4.1 and 4.2, a simple linear advection equation is approximated. In 4.3 and
4.4, the linearized system (8a)-(8c) is approximated. Lastly, in 4.5 the non-
linear system (7a)-(7c) is approximated. The initial and boundary conditions
used in all of the examples are of the form
\[
\delta f(x, v, 0) = A \cos(kx) f_{eq}(v),
\]
(55a)
\[
\delta f(0, v, t) = \delta f(L, v, t),
\]
(55b)
\[
\psi(0, t) = \psi(L, t) = 0,
\]
(55c)
for \((x, v, t) \in (0, L) \times (-V_c, V_c) \times (0, T), \) where \( V_c > 0, L > 0, \) and \( T > 0 \) are
given. The constant \( V_c \) is the cutoff velocity and is chosen large enough so that
the values of \( f \) are negligibly small when \(|v| = V_c\). It follows that each example
problem is completely determined by specifying the governing equations along
with the parameters \( f_{eq}(v), A, k, L, V_c, T. \)

### 4.1 Linear advection in \( x \) with a Maxwell equilibrium

The first example problem approximates the Vlasov equation with the electric
field being identically zero. This test was performed for four standard Vlasov
solvers in [26]. Consider the equation
\[
f_t + vf_x = 0,
\]
(56)
with \( f_{eq} = (2\pi)^{-1/2}\exp(-v^2/2), A = 0.1, k = 0.5, L = 4\pi, V_C = 5 \) and \( T = 40.\)
For any \( t > 0, \) the solution \( f \) is given by \( f(x, v, t) = f_0(x - vt, v). \) In this
particular problem, it can be shown that the net charge density \( \rho_{tot} \) satisfies
\[
\rho_{tot}(x, t) = 1 - \int_{-\infty}^{\infty} f(x, v, t) \, dv = A \cos(kx) e^{-k^2t^2/2}.
\]
(57)
This implies that \( \max_x |\rho_{tot}(x, t)| = 0.1 \, e^{-t^2/8}, \) since \( k = 0.5. \)

To test the accuracy and convergence of the UG method, \( \max_x |\rho_{tot}(x, t)| \) is
computed numerically and the results are plotted in Figure 1. The numerical
results were generated using the five uniform meshes \((N_h, N_v) = (500, 400), \)
\((1000, 800), (2000, 1600), (4000, 400), (8000, 400), \) where \( N_h \) and \( N_v \) denote
the number of partitions of the \( x \)-axis and the \( v \)-axis, respectively.
The first three meshes are such that \( h_x \approx h_v \), whereas the fourth and fifth meshes are such that \( h_x \approx h_v/8 \) and \( h_x \approx h_v/16 \), respectively. The motivation for using the last two meshes comes from the fact that the problem being approximated involves only advection in the \( x \)-direction. Hence, it is reasonable to assume that mesh refinements in \( x \) will improve the numerical accuracy as much as performing refinements in both \( x \) and \( v \), provided the refinement level in \( v \) is sufficiently small so that the error is almost entirely due to the refinement level in \( x \). Figure 1 clearly shows that the UG method is both accurate and numerically convergent under mesh refinements.

4.2 Linear advection in \( x \) with a Lorentz equilibrium

In this example, we again approximate the Vlasov equation with a vanishing electric field. However, a Lorentz equilibrium is used instead of Maxwell equilibrium. Moreover, numerical results are given for four different problems, where each problem corresponds to a different wavenumber. This example corresponds to the linear Landau damping problems approximated in 4.4, with the exception that the electric field is not taken to be zero in 4.4.

Eq. (56) is approximated for \( f_{eq} = \pi^{-1}/(v^2 + 1) \), \( A = 0.01 \), and \( V_c = 30.0 \) for each of the wavenumbers \( k = 1/8, 1/6, 1/4, \) and \( 1/2 \). The corresponding values for \( L \) and \( T \) for \( k = 1/8, 1/6, 1/4, \) and \( 1/2 \) are \( L = 16\pi, 12\pi, 8\pi \) and \( 4\pi \) and \( T = 75, 75, 50, \) and \( 50 \), respectively. The uniform mesh \((N_x,N_v) = (1000,2000)\) was employed in each of the four cases. As in 4.1, for any \( t > 0 \), the solution \( f \) is given by \( f(x,v,t) = \tilde{f}_0(x-vt,v) \). For this case, it can be shown that the net charge density \( \rho_{tot} \) satisfies

\[
\rho_{tot}(x,t) = 1 - \int_{-\infty}^{\infty} f(x,v,t) \, dv = -A \cos(k_x) e^{-kt}. \tag{58}
\]

This implies that \( \max_x |\rho_{tot}(x,t)| = 0.01 \, e^{-kt} \).

The computed results for each of the four wavenumbers \( k \) are shown in Figure 2.

4.3 Linear Landau damping with a Maxwell equilibrium

In this example, we assess the capability of our numerical method to accurately reproduce the damping results predicted by the linear Landau theory for a Maxwell equilibrium. In particular, the numerical convergence of the method is demonstrated by computing the damping rate of the dominant Fourier mode of the electric field using four successively refined meshes. It is shown that the numerical damping rate approaches the theoretical rate upon
mesh refinement. For a comparison to other numerical methods that have been previously employed for this problem, we refer the reader to [6, 15, 36].

The linear system (8a)-(8c) is solved with \( f_{eq} = (2\pi)^{-1/2}e^{-v^2/2} \), \( A = 0.01 \), \( k = 0.5 \), \( L = 4\pi \), \( V_c = 4.5 \) and \( T = 80 \). For this problem, the theoretical decay rate and frequency of oscillations to the third decimal digit are respectively equal to -0.153 and 1.415 [6]. Approximate solutions are computed using the four uniform meshes \( (N_{h_x}, N_{h_y}) = (250, 400), (500, 800), (1000, 1600) \) and \( (2000, 1600) \).

Phase-space contour plots and cross-sectional plots in \( v \) of the approximate solution \( f_h \) for \( (N_{h_x}, N_{h_y}) = (500, 800) \) and \( t = 0, t = 25, t = 50, \) and \( t = 75 \) are displayed in Figure 3. These sequential plots show the increase in filamentation of \( f_h \) as time elapses.

The convergence of numerical decay rates of the dominant electric field Fourier mode is demonstrated in Figure 4. The decay rate resulting for each mesh was computed by calculating the slope of the straight line plotted in the figure. In each case, the line was defined by the point occurring at the peak of the third oscillation and the point occurring at the peak of the ninth oscillation. A time step of \( \Delta t = 0.001 \) was used in order to ensure that the points defining each of the lines were actual computed data points rather than points that were determined using some interpolation of the data. Under mesh refinement, the numerical decay rate is seen to converge, up to three decimal-digit accuracy, to the theoretical decay rate of -0.153. In all four cases, the numerical frequency is observed to correspond to the theoretical frequency up to three decimal-digit accuracy. We also note that, upon refining the mesh, the decay of the dominant mode is sustained for longer times before leveling off. Moreover, in contrast to many of the conventional methods cited above, the DG method is seen not to suffer from numerical, and thus unphysical, recurrence effects.

### 4.4 Linear Landau damping with a Lorentz equilibrium

The ability of the UPG method to achieve accurate damping results across different wavenumbers is now investigated. The Lorentz equilibrium is used in this example since we have explicit formulae for the electric field damping rates and the frequency of the damped oscillations (see Appendix A). Moreover, this example also tests the ability of the method to produce accurate results for an equilibrium that has much heavier tails in \( v \) than does the Maxwellian. The heavy tails of the Lorenztian lead to a faster rate of filamentation since there are more electrons being transported at higher velocities than occurs when using a Maxwellian for the equilibrium.

The linear system (8a)-(8c) is approximated with \( f_{eq} = \pi^{-1}/(v^2+1) \), \( A = 0.01 \),
and \( V_c = 30.0 \) for each of the wavenumbers \( k = 1/8, 1/6, 1/4, \) and \( 1/2 \). The corresponding values for \( L \) and \( T \) for \( k = 1/8, 1/6, 1/4, \) and \( 1/2 \), are \( L = 16\pi, 12\pi, 8\pi \) and \( 4\pi \) and \( T = 75, 75, 50, \) and \( 50 \), respectively. Uniform meshes were employed in all four cases, where in each case \( (N_x, N_v) = (1000, 2000) \).

In Figure 5, plots of the dominant electric field modes for the four cases are given. The observed damping for \( k = 1/2 \) lasts for a shorter time duration than does the damping for the other three wavenumbers, even though the mesh for \( k = 1/2 \) has the finest resolution in \( x \). This result is due to the fact that the filamentation in the velocity direction develops more rapidly than it does in the other three cases. From Appendix A, it follows that for a given wavenumber \( k \) the fundamental mode of the electric field damps exponentially in time at a rate equal to \( \gamma(k) = -kv_\theta \) and the frequency of the damped oscillations satisfies \( |\omega(k)| = 1 \). Given that \( v_\theta = 1 \), the theoretical damping rates corresponding to \( k = 1/8, 1/6, 1/4, \) and \( 1/2 \) are \( \gamma(k) = -1/8, -1/6, -1/4, \) and \( -1/2 \). In all of the four cases shown in Figure 5, the numerical damping rates and frequencies of oscillation are equal to the theoretical values up to the first two decimal-digits.

### 4.5 Nonlinear two-stream instability

In this subsection, we numerically solve the nonlinear two-stream instability problem that has been studied in [19, 26]. In particular, we approximate the nonlinear system (7a)-(7c) for the equilibrium function

\[
 f_{eq}(v) = (2\pi)^{1/2} (2\pi v)^2 e^{-(2\pi v)^2/2},
\]

and for the parameters \( A = 0.05, k = \pi, L = 4\pi, V_c = 5, \) and \( T = 100 \). We use a uniform mesh defined by \( (N_x, N_v) = (1800, 1400) \) and a time step \( \Delta t = 0.00125 \).

Figures 6 and 7 show 3D and 2D contour plots of the solution over phasespace at different instances in time. The initial data consists of two streaks flowing in the opposite direction onto which a small perturbation is super-imposed. As the nonlinear system evolves, a vortex in phasespace forms that traps much of the total electron distribution and exhibits some filamentation within the vortex. Over time, the initially sharp contours of the vortex smooth out due to the numerical diffusion of the UPG method. Figure 8 shows the time evolution of the first four modes. From the figure it is apparent that most of the energy is contained in the first two modes and that, over time, the electric field approaches a stationary value, as is also evidenced by Figure ???. Our results agree favorably with the results reported in the literature.
5 Conclusions

To be finished!
Fig. 1. (Linear advection with Maxwell equilibrium) Plot of $\log_{10}(\max_x |\rho_{\text{tot}}(x,t)|) = -(\log_{10} e) t^2/8 + 1$ vs. $t$: analytic solution (solid), $(N_{h_x}, N_{h_v}) = (500, 400)$ (dot), $(1000, 800)$ (dash-dot), $(2000, 1600)$ (dash-dot-dot), $(4000, 400)$ (short dash), $(8000, 400)$ (long dash).

Fig. 2. (Linear advection with Lorentz equilibrium) Plot of $\log_{10}(\max_x |\rho_{\text{tot}}(x,t)|) = -(\log_{10} e) kt + 2$ vs. $t$: $k=1/8$ (top left), $k=1/6$ (top right), $k=1/4$ (bottom left) and $k=1/2$ (bottom right); analytic solution (dash), numerical solutions (solid).
A Derivation of Landau damping

Following Landau’s original argument [20], assume that $f(x,v,t) = f_{eq}(v) + \delta f(x,v,t)$ is the solution to the linearized system (8a)-(8c) subject to the same conditions given therein. By applying the Fourier transform with respect to $x$ and the Laplace transform with respect to $t$, Landau derived the dispersion relation

$$\epsilon(k,\omega) = 0 , \quad (A.1)$$

where $\epsilon(k,\omega)$, with $k = 2\pi n/L$, is defined by

$$\epsilon(k,\omega) = 1 - k^{-2} \int_{-\infty}^{\infty} \frac{f'_{eq}(v)}{v^2 + \omega^2} \, dv, \quad (\Im(\omega/k) > 0). \quad (A.2)$$

The solution $\omega(k)$ of the dispersion relation (A.1) characterizes the asymptotic-time behavior of the Fourier mode $E_k(\omega, t)$ of the electric field. Specifically, if $\omega = \omega_R + i\gamma$, where $\omega_R$ and $\gamma$ are real-valued, then $\gamma$ is the asymptotic-time rate of decay of the mode $E_k(\omega, t)$ and $\omega_R$ is the frequency of the mode oscillation.

The definition of the Lorentz probability distribution given in (9) implies

$$f'_{eq}(v) = -\frac{2v\theta}{\pi} \frac{v}{(v^2 + \theta^2)^2} . \quad (A.3)$$

Upon defining $u = \omega/k$ and substituting (A.3) into (A.2), we get that

$$\epsilon(k,\omega) = 1 + \frac{2v\theta}{\pi k^2} \int_{\mathbb{R}} \frac{v}{(v^2 + \theta^2)^2(v - u)} \, dv , \quad (A.4)$$

which expresses $\epsilon$ as a function $u$ and $k$.

In order to simplify the above expression for $\epsilon$, decompose the integral term in (A.4) as follows:

$$\int_{\mathbb{R}} \frac{v \, dv}{(v^2 + \theta^2)^2(v - u)} = \int_{\mathbb{R}} \frac{(v - u + u) \, dv}{(v^2 + \theta^2)^2(v - u)}$$

$$= \int_{\mathbb{R}} \frac{dv}{(v - i\theta)(v + i\theta)^2} + u \int_{\mathbb{R}} \frac{dv}{(v - i\theta)(v + i\theta)^2(v - u)}$$

$$= : N + Y(u) . \quad (A.5)$$
Now we recall that Cauchy’s formula for higher-order poles states that if $g$ is an analytic function in a domain $D$ containing a piecewise smooth, simple, closed, directed curve $C$, then for any point $c \in C$ we have that

$$\int_C \frac{g(\beta)}{(\beta - c)^{n+1}} \, d\beta = 2\pi i \frac{d^n}{d\beta^n} g(c), \quad \forall n \in \{0, 1, 2, \ldots\}. \quad (A.6)$$

This formula implies that constant $N$ in (A.5) satisfies

$$N = 2\pi i \frac{d}{dv} \left( \frac{1}{(v + iv\theta)^2} \right)_{v=iv\theta} = \frac{-4\pi i}{(v + iv\theta)^3} \bigg|_{v=iv\theta} = \frac{\pi}{2v^2_\theta} \quad (A.7)$$

and that $Y(u)$ is equal to

$$Y(u) = 2\pi i u \frac{d}{dv} \left( \frac{1}{(v + iv\theta)^2 (v - u)} \right)_{v=iv\theta}$$

$$= 2\pi i u \left( \frac{-2}{(v + iv\theta)^3 (v - u)} - \frac{1}{(v + iv\theta)^2 (v - u)^2} \right)_{v=iv\theta}$$

$$= \frac{\pi i u}{2v^2_\theta (iv\theta - u)} \left( \frac{1}{iv\theta} + \frac{1}{iv\theta - u} \right) = -\pi u \left( \frac{u - 2iv\theta}{2v^2_\theta (u - iv\theta)^2} \right)$$

$$= -N \left( \frac{u^2 - 2iv\theta u}{(u - iv\theta)^2} \right) = -N \left( \frac{u - iv\theta)^2 + v^2_\theta}{(u - iv\theta)^2} \right).$$

We now substitute the righthandside expressions in (A.7) and (A.8) for $N$ and $Y(u)$, respectively, into (A.5). After simplifying the resulting expression, we get the identity

$$\int_{\mathbb{R}} \frac{v \, dv}{(v^2 + v^2_\theta)^2 (v - u)} = \frac{-\pi}{2v_\theta (u - iv_\theta)^2}. \quad (A.9)$$

Substituting equation (A.9) into (A.4) yields the relation

$$\epsilon(k, u) = 1 - \frac{1}{k^2(u - iv_\theta)^2}. \quad (A.10)$$

Eq. (A.10) implies that the dispersion relation (A.1) is equivalent to

$$(u - iv_\theta)^2 = k^{-2} \iff u = iv_\theta \pm k^{-1} \iff \omega = ikv_\theta \pm 1,$$
where the last equality follows since \( u = \omega/k \). Upon decomposing \( \omega \) into its real and imaginary parts \( \omega_R \) and \( \gamma \), it follows that \( \gamma(k) = -kv_\theta \) and \( |\omega_R(k)| = 1 \).

References


Fig. 3. (Linear Landau damping with Maxwell equilibrium) Contour plots (left) and cross-sectional plots (right), $x = 2\pi$, for $\delta f$ at $t = 0, t = 25, t = 50, t = 75$ (descending order).
Fig. 4. (Linear Landau damping with Maxwell equilibrium) Time decay plots of fundamental mode under mesh refinement: \((N_{h_x}, N_{h_y}) = (250, 200) \) (top left), \((500, 400) \) (top right), \((1000, 800) \) (bottom left) and \((2000, 1600) \) (bottom right). The theoretical decay rate is -0.153 up to three decimal-digit accuracy.
Fig. 5. (Linear Landau damping with Lorentz equilibrium) Time decay plots of fundamental modes: $k=1/8$ (top left), $k=1/6$ (top right), $k=1/4$ (bottom left) and $k=1/2$ (bottom right).
Fig. 6. (Nonlinear two-stream instability) 3D plots of the solution $f$ at $t = 0$, $t = 20$, $t = 40$, $t = 60$, $t = 80$ and $t = 100$. 

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Fig. 7. (Nonlinear two-stream instability) Contour plots of the solution $f$ at $t = 0$, $t = 20$, $t = 40$, $t = 60$, $t = 80$ and $t = 100$. 
Fig. 8. (Nonlinear two-stream instability) Time series plots of the first four modes: $k = 1/2$ (top left), $k = 1$ (top right), $k = 3/2$ (bottom left), and $k = 2$ (bottom right).