Existence of Global Weak Solutions to Quasilinear Theory for Electrostatic Plasmas

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Abstract

The quasilinear theory has been widely used to describe the resonant particle-wave interaction in plasmas. The electrostatic case, i.e. the model originating from Vlasov-Poisson system, is the most fundamental and representative one. In this paper, we prove the existence of global weak solutions for electrostatic plasmas in one dimension.

Keywords: quasilinear theory, plasma physics, weak turbulence model, porous medium equation

1 Introduction

The quasilinear theory is a model reduction of the Vlasov-Maxwell(or Vlasov-Poisson) system in weak turbulence regime. Since first proposed by Vedenov et al. [8] and Drummond et al. [3], it has found extensive use in plasma physics to describe the interaction between particles and waves(plasmons). Despite that, not much work from analysis point of view can be found. A recent paper by Bardos and Besse [2] discussed the asymptoticity. The well-posedness of quasilinear systems remains an open problem. This paper is devoted to proving the existence of weak solutions to the one dimensional problem.

Our strategy can be summarized as follows. Firstly, we use the trick in Ivanov et.al.[4] to show that, in one dimensional case, the system is equivalent to a porous medium equation with nonlinear source terms. This allows us to use existing techniques from that field. In particular, the proposed proof is inspired by the book of Vazquez[7]. The basic idea is to analyze a series of approximate problems with parameter n and try to establish regularity estimates uniform in n. Then it is possible to pass the limit to infinity.

In order to pass the limit, the strong convergence of the gradient term is necessary. And that is the most challenging part. Similar problems has been tackled by Abdellaoui, Peral and Walias[1], where the nonlinear source contains a gradient term to some power. A significant difference is that they require positive source, while in our problem, the gradient square term has a minus sign. Nevertheless, their proof for a.e. convergence can still be adopted if we can find an alternative way to prove the prerequisites.

The paper is organized as follows. In Section 2, we derive the equivalent porous medium equation and state the main result. In Section 3, we resort to the book of Ladyzenskaja[5] for the maximum principle and the well-posedness of the strictly coercive approximate problems with parameter n. Section 4 aims at proving the regularity estimates uniform in parameter n, which paves the way to convergence results. In Section 5, the a.e. convergence of gradient term is proved, using the technique from Abdellaoui, Peral and Walias[1]. The proof of main theorem is contained in Section 6. And we will further discuss the nontrivial equilibrium states in Section 7.

2 From Quasilinear System to Porous Medium Equation with Source Term

The goal of this section is to transform the system into a single equation for only one unknown function.

The quasilinear system for particle-wave interaction contains two equations, as there are two unknown functions: the particle probability density function $f(\mathbf{p},t)$ and the wave spectral energy density $W(\mathbf{k},t)$.

The most important feature of the system lies in the fact that particles and waves interact through the absorption/emission kernel, which contains a Dirac delta. In physics this is called "resonance", since particles with certain momentum only interact with waves with some particular wave vectors. The one dimensional plasma is special, because in this case each momentum **p** corresponds to only one wave vector **k**, and vice versa. This is the reason that the following trick is feasible.
In electrostatic case, the equations are as follows,

$$\begin{cases} \partial_t f(\mathbf{p},t) = \nabla_p \cdot \left(\left[\int_{\Omega_k} W(\mathbf{k}) \left(\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \right) \delta(\omega - \nabla_p E(\mathbf{p}) \cdot \mathbf{k}) d\mathbf{k} \right] \cdot \nabla_p f \right), \\ \partial_t W(\mathbf{k},t) = \left[\int_{\Omega_p} (\nabla_p f) \cdot \left(\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \right) \cdot (\nabla_p E(\mathbf{p})) \, \delta(\omega - \nabla_p E(\mathbf{p}) \cdot \mathbf{k}) d\mathbf{p} \right] W, \end{cases}$$

where $E(\mathbf{p}) = \frac{1}{2}\mathbf{p}^2$ is the kinetic energy of a single particle, and the constant ω is the plasma frequency.

In one dimensional plasma, after normalization, the system can be written as

$$\begin{cases} \partial_t f = \partial_p \left(\left[\int_{\Omega_k} W \delta(1 - pk) dk \right] \partial_p f \right), \\ \partial_t W = \left[\int_{\Omega_p} (\partial_p f) \, p \delta(1 - pk) dp \right] W, \end{cases}$$

with initial condition being

$$\begin{cases} f(p,0) = f_0(p), \\ W(k,0) = W_0(k). \end{cases}$$

Here $p = 1/k$ is the so-called resonance condition. Consequently, for $p > 0$ and $k > 0$, after eliminating the Dirac delta by performing integration, the system becomes

$$\begin{cases} \partial_t f(p,t) = \partial_p \left(\left[W(\frac{1}{p}) \frac{1}{p} \right] \partial_p f \right), \\ \partial_t W(k,t) = \left[\partial_p f(\frac{1}{k}) \frac{1}{k^2} \right] W(k,t). \end{cases}$$

Introducing a new function $w(p) = \frac{1}{p^3} W(\frac{1}{p})$ to obtain

$$\begin{cases} \partial_t f = \partial_p \left(p^2 w \partial_p f \right), \\ \partial_t w = p^2 w \partial_p f. \end{cases} \tag{2.1}$$

Compare the right hand side of the above two equations, the second row is identical to the term inside bracket, therefore $\partial_t f = \partial_p (\partial_t w)$, and by fundamental theorem of calculus,

$$f = \partial_p (w - w_0) + f_0.$$

Substitute it into the second row of Equation(2.1), we obtain an equation with only one unknown function $w(p,t)$,

$$\partial_t w = p^2 w \partial_p \left(\partial_p w + f_0 - \partial_p w_0 \right) = p^2 w \partial_p^2 w + g_0 w,$$

where $g_0 = p^2 \partial_p (f_0 - \partial_p w_0)$.
Consider the case where $w_0(p) = 0$ for any $p \in (-\infty, p_a] \cup [p_b, +\infty)$, with $p_a > 0$. This is also the case of interest in physics, for example the "bump on tail" configuration is included in this scenario. The original problem is equivalent to

$$\begin{cases} \partial_t w = p^2 w \partial_p^2 w + g_0 w, & \text{in } \Omega_p = (p_a, p_b) \\ w(p,t) = 0, & \text{on } \partial\Omega_p \\ w(p,0) = w_0(p) \end{cases}$$

In the rest of the article, to keep it consistent with existing literature in mathematics, we will use u, x instead of w, p.
Denote the following problem as problem $(\mathcal{S})$.

$$\begin{cases} \partial_t u = x^2 u \partial_x^2 u + g_0 u = \partial_x \left(x^2 u \partial_x u \right) - x^2 \left(\partial_x u \right)^2 - x \partial_x u^2 + g_0 u, & \text{in } \Omega = (x_a, x_b) \\ u(x) = 0, & \text{on } \partial\Omega \\ u(x,0) = \varphi_0(x) \end{cases} \tag{2.2}$$

This equation belongs to porous medium equations, with nonlinear source terms. We are seeking for a non-negative solution u because otherwise it does not make sense in physics. However the extra term $-x^2 \left(\partial_x u\right)^2 - x\partial_x u^2 + g_0 u$ is not necessarily positive, which means existing techniques for regularity estimates are not applicable anymore. And here lies the main contribution of the proposed work. The details will be presented in Section 4.

Define $Q_T = \Omega \times [0, T]$, the main result of the paper can be stated as follows,

Theorem 1. *If $g_0 \in C^\infty(\overline{\Omega})$, $\varphi_0 \in C^\infty(\overline{\Omega})$, and $\partial_x^2 \varphi_0 = 0$ on $\partial\Omega$, then for problem $(\mathcal{S})$ there exists a weak solution $u \in L^{2l}(0, T; W_0^{1,2l}(\Omega))$ with $l = 1, 2, \cdots$ such that, for any $\eta \in C^\infty(\overline{Q_T})$ that vanishes on $\partial\Omega \times [0,T] \cup \{(x,t) : x \in \Omega, t = T\}$, the following identity holds,*

$$-\left(u, \partial_t \eta\right)_{Q_T} + \left(\frac{x^2}{2} \partial_x u^2, \partial_x \eta \right)_{Q_T} = \left(-x^2 (\partial_x u)^2, \eta \right)_{Q_T} + \left(-x\partial_x u^2, \eta \right)_{Q_T} + \left(g_0 u, \eta \right)_{Q_T} + \left(\varphi_0, \eta(x,0) \right)_\Omega \tag{2.3}$$

3 The Solvability of Approximate Problems

The problem (2.2) is difficult to tackle due to its degeneracy. Therefore we consider a series of approximate problems first. They are arbitrarily close to the original problem, but each one of them is strictly coercive, which ensures the existence of classical solutions. Furthermore, these approximate solutions have enough regularity, allowing us to test with various functions and to obtain bounds that are uniform in the parameter n.
In this section the following approximate problem $(\mathcal{S}_n)$ is considered,

$$\begin{cases} \partial_t u = x^2 \mathcal{P}_n(u) \partial_x^2 u + g_0 u, & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \\ u(x,0) = \varphi_0(x), \end{cases}$$

where $\mathcal{P}_n \in C^\infty(\mathbb{R})$ is a family of functions with the following properties,

(i) $\mathcal{P}_n(y) \geq \frac{1}{2n}, \forall y \in \mathbb{R}$

(ii) $\mathcal{P}_n(y) = y + \frac{1}{n}, \forall y \in \mathbb{R}^+$

(iii) $\mathcal{P}_n'(y) \geq 0$

The following maximum principle can be found in Theorem 2.1, Chapter I of Ladyzenskaja's book[5]. Note that the bounds are independent of the parameter n.

Theorem 2. *(maximum principle) If u_n is a classical solution to the problem $(\mathcal{S}_n)$, then $u_n(x,t)$ satisfies the maximum principle on $\overline{Q_T}$:*

$$\begin{cases} u_n(x,t) \geq 0 \\ u_n(x,t) \leq max(\varphi_0) \exp(\max(|g_0|)t) \end{cases}$$

For the existence of classical solution to the approximate problems $(\mathcal{S}_n)$, we refer the readers to Theorem 6.1, Chapter V of Ladyzenskaja's book[5].

Theorem 3. *(classical solution) For any $T > 0$, the problem $(\mathcal{S}_n)$ admits an unique classical solution $u_n(x,t)$ on $\overline{Q_T}$. Moreover, $u_n(x,t)$ belongs to Hölder space $\mathcal{H}^{2+\beta,(2+\beta)/2}(\overline{Q_T}) \cap \mathcal{H}^{3+\beta,(3+\beta)/2}(Q_T)$ when $g_0, \varphi_0 \in \mathcal{C}^\infty(\overline{\Omega})$.*

Proof. To begin with, write the equation in divergence form.

$$\begin{aligned}\mathcal{L}u \equiv& \partial_t u - x^2\mathcal{P}_n(u)\partial_x^2 u - g_0 u \\ =& \partial_t u - \partial_x\left(x^2\mathcal{P}_n(u)\partial_x u\right) + x^2\mathcal{P}_n'(u)\left(\partial_x u\right)^2 + 2x\mathcal{P}_n(u)\partial_x u - g_0 u\end{aligned}$$

Define

$$\varphi(x,t) \coloneqq \varphi_0(x) + \left[x^2\mathcal{P}_n'(\varphi_0)\partial_x^2\varphi_0 + g_0\varphi_0\right]t$$

Recall the assumption that $\partial_x^2\varphi_0 = 0$ on $\partial\Omega$, the initial-boundary condition can be written as

$$u|_{\Gamma_T} = \varphi|_{\Gamma_T}$$

where $\Gamma_T = \partial\Omega \times [0,T] \cup \{(x,t) : x \in \Omega, t = 0\}$.
In accordance with the notation of Ladyzenskaja, define

$$\begin{aligned}a_1(x,t,u,p) &\coloneqq x^2\mathcal{P}_n(u)p \\ a(x,t,u,p) &\coloneqq x^2\mathcal{P}_n'(u)p^2 + 2x\mathcal{P}_n(u)p - g_0(x)u \\ A(x,t,u,p) &\coloneqq -g_0(x)u\end{aligned}$$

The conditions in Ladyzenskaja's theorem can be easily verified. See Appendix.

□

By Theorem 2, u_n is always non-negative, therefore by the second property of function $\mathcal{P}_n$, the problem $(\mathcal{S}_n)$ in divergence form is as follows,

$$\begin{cases}\partial_t u_n = \partial_x\left(x^2(u_n + \frac{1}{n})\partial_x u_n\right) - x^2\left(\partial_x u_n\right)^2 - x\partial_x(u_n + \frac{1}{n})^2 + g_0 u_n, & \text{in } \Omega \\ u_n(x) = 0, & \text{on } \partial\Omega \\ u_n(x,0) = \varphi_0(x). & \end{cases} \tag{3.1}$$

Define $\tilde{u}_n = u_n + \frac{1}{n}$, then $\tilde{u}_n$ is strictly positive, and it solves the following problem,

$$\begin{cases}\partial_t \tilde{u}_n = \partial_x\left(x^2\tilde{u}_n\partial_x\tilde{u}_n\right) - x^2\left(\partial_x\tilde{u}_n\right)^2 - x\partial_x\tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right), & \text{in } \Omega \\ \tilde{u}_n(x,t) = \frac{1}{n}, & \text{on } \partial\Omega \\ \tilde{u}_n(x,0) = \varphi_0(x) + \frac{1}{n}, & \end{cases} \tag{3.2}$$

4 The Regularity Estimates on Approximate Solutions

As have been mentioned, in order to prove a.e. convergence of the gradient term in the next section, several regularity estimates are of necessity. In the work of Abdellaoui, Peral and Walias[1], the authors used existing results for an elliptic-parabolic problem with measure data. However for the problem we are dealing with, the nonlinear source term is not "measure data", thus it calls for a different approach. And that is the aim of this section.
To begin with, we introduce the following energy estimate.

Proposition 1. *If $\tilde{u}_n$ is a classical solution to problem(3.2) $t \in [0,T]$, then the energy inequality holds,*

$$\|x\partial_x\tilde{u}_n^2\|_{L^2(Q_T)} = \left(\int_0^T\int_\Omega |x\partial_x\tilde{u}_n^2|^2 dxdt\right)^{\frac{1}{2}} \le C(\varphi_0, g_0, T, \Omega),$$

where the constant C is independent of n.

Proof. Test Equation(3.2) with $\tilde{u}_n^2 - \frac{1}{n^2}$, since $\tilde{u}_n^2 - \frac{1}{n^2} = 0$ on $\partial\Omega$, after integration by parts, the equation becomes

$$\begin{aligned}&\left(\partial_t \tilde{u}_n, \tilde{u}_n^2 - \frac{1}{n^2}\right)_\Omega + \left(x^2 \tilde{u}_n \partial_x \tilde{u}_n, \partial_x \left(\tilde{u}_n^2 - \frac{1}{n^2}\right)\right)_\Omega \\ =& \left(-x^2 \left(\partial_x \tilde{u}_n\right)^2, \tilde{u}_n^2 - \frac{1}{n^2}\right)_\Omega + \left(-x\partial_x \tilde{u}_n^2, \tilde{u}_n^2 - \frac{1}{n^2}\right)_\Omega + \left(g_0 \left(\tilde{u}_n - \frac{1}{n}\right), \tilde{u}_n^2 - \frac{1}{n^2}\right)_\Omega\end{aligned}$$

Collecting the terms with $\frac{1}{n}$ to obtain

$$\begin{aligned}&\left(\partial_t \tilde{u}_n, \tilde{u}_n^2\right)_\Omega + \left(\partial_x \tilde{u}_n, x^2 \partial_x \left(\tilde{u}_n^3\right)\right)_\Omega + \left(\partial_x \tilde{u}_n, 2x\tilde{u}_n^3\right)_\Omega \\ =& \left(\partial_t \tilde{u}_n, \frac{1}{n^2}\right)_\Omega + \frac{1}{n^2} \left(\partial_x \tilde{u}_n, x^2 \partial_x \tilde{u}_n\right)_\Omega + \frac{1}{n^2} \left(\partial_x \tilde{u}_n, 2x\tilde{u}_n\right)_\Omega + \left(g_0, (\tilde{u}_n - \frac{1}{n})(\tilde{u}_n^2 - \frac{1}{n^2})\right)_\Omega\end{aligned}$$

By maximum principle for u_n, the shifted solution $\tilde{u}_n(x,t) \geq \frac{1}{n}$, therefore the second term on the right hand side is bounded as follows,

$$\left(\frac{1}{n} x\partial_x \tilde{u}_n, \frac{1}{n} x\partial_x \tilde{u}_n\right)_\Omega \leq \left(\tilde{u}_n x\partial_x \tilde{u}_n, \tilde{u}_n x\partial_x \tilde{u}_n\right)_\Omega = \frac{1}{4} \left(x\partial_x \tilde{u}_n^2, x\partial_x \tilde{u}_n^2\right)_\Omega,$$

Perform integration by parts on all the terms with integrand $x^i \partial_x \tilde{u}_n^j$, the following inequality holds,

$$\frac{1}{2} \left(x\partial_x \tilde{u}_n^2, x\partial_x \tilde{u}_n^2\right)_\Omega \leq - \left(\partial_t \tilde{u}_n, \tilde{u}_n^2\right)_\Omega + \left(\partial_t \tilde{u}_n, \frac{1}{n^2}\right)_\Omega + \frac{1}{2n^4}(x_b - x_a) + \frac{1}{2} \left(\tilde{u}_n^4, 1\right)_\Omega - \frac{1}{n^2} \left(\tilde{u}_n^2, 1\right)_\Omega + \left(g_0, (\tilde{u}_n - \frac{1}{n})(\tilde{u}_n^2 - \frac{1}{n^2})\right)_\Omega.$$

Every integral of $\tilde{u}_n^j$ is bounded as a result of the maximum principle, therefore

$$\int_0^T \int_\Omega |x\partial_x \tilde{u}_n^2|^2 dxdt \leq C(\varphi_0, g_0, T, \Omega).$$

□

According to Ladyzenskaja[5], thanks to the smoothness of data, the solutions are smoother than usual, even the third order derivative is Hölder continuous. And that allows us to study a parabolic equation for the first derivative $\partial_x u_n$, which renders the following regularity estimates.

Proposition 2. *If u_n are classical solutions to the problem(3.1) for $t \in [0,T]$, then their gradient in space are uniformly bounded in $L^{2l}(\Omega)$,*

$$\sup_{t\in[0,T]} \|\partial_x u_n\|_{L^{2l}(\Omega)} \leq C_1(\varphi_0, g_0, T, \Omega, l), \; l = 1, 2, \cdots$$

Consequently,

$$\|\partial_x u_n\|_{L^{2l}(Q_T)} \leq C_2(\varphi_0, g_0, T, \Omega, l), \; l = 1, 2, \cdots$$

and

$$\|u_n\|_{L^{2l}(0,T;W_0^{1,2l}(\Omega))} \leq C_3(\varphi_0, g_0, T, \Omega, l), \; l = 1, 2, \cdots \tag{4.1}$$

Proof. To begin with, rewrite Equation(3.1) in non-divergence form,

$$\partial_t u_n - x^2 \left(u_n + \frac{1}{n}\right) \partial_x^2 u_n - g_0 u_n = 0, \tag{4.2}$$

Let $z_n = \partial_x u_n$, then by Theorem 3, $z_n \in \mathcal{H}^{1+\beta,(1+\beta)/2}(\overline{Q_T}) \cap \mathcal{H}^{2+\beta,(2+\beta)/2}(Q_T)$. Taking first derivative on both sides of Equation(4.2), every term is still continuous. Therefore z_n satisfies the following linear parabolic equation, if we regard u_n as data.

$$\partial_t z_n - \partial_x \left(x^2 \left(u_n + \frac{1}{n} \right) \partial_x z_n \right) - g_0 z_n - u_n \partial_x g_0 = 0$$

In addition, from Equation(4.2) and the boundary condition for u_n, it can be derived that

$$\partial_x z_n = \partial_x^2 u_n = \frac{\partial_t u_n - g_0 u_n}{x^2 \left(u_n + \frac{1}{n} \right)} = 0 \text{ on } \partial\Omega.$$

To summarize, z_n satisfies the following equations,

$$\begin{cases} \partial_t z_n - \partial_x \left(x^2 \left(u_n + \frac{1}{n} \right) \partial_x z_n \right) - g_0 z_n - u_n \partial_x g_0 = 0, & \text{in } \Omega \\ \partial_x z_n = 0, & \text{on } \partial\Omega \\ z_n(x,0) = \partial_x \varphi_0(x) \end{cases}$$

Test the equation with z_n^{2l+1} and perform integration by parts to obtain

$$\left(\partial_t z_n, z_n^{2l+1}\right)_\Omega + \left(x^2 \left(u_n + \frac{1}{n} \right) \partial_x z_n, (2l+1) z_n^{2l} \partial_x z_n \right)_\Omega = \left(g_0 z_n, z_n^{2l+1}\right)_\Omega + \left(u_n \partial_x g_0, z_n^{2l+1}\right)_\Omega .$$

Since the second term on the left hand side is non-negative, the following inequality holds,

$$\frac{1}{2l+2} \frac{d}{dt} \left(\int_\Omega z_n^{2l+2} \right) \le \int_\Omega g_0 z_n^{2l+2} + \int_\Omega u_n \partial_x g_0 z_n^{2l+1}.$$

Again, use the maximum principle for u_n in Theorem 2, and by the assumption on the data g_0,

$$\frac{d}{dt} \left(\int_\Omega z_n^{2l+2} \right) \le C_1(g_0, T, \Omega) \int_\Omega z_n^{2l+2} + C_2(\varphi_0, g_0, T, \Omega) \int_\Omega |z_n|^{2l+1}$$

Let $I = \int_\Omega z_n^{2l+2}$, by Hölder's inequality, the above is equivalent to,

$$\frac{d}{dt} I \le C_1 I + C_3 I^{\frac{2l+1}{2l+2}}$$

Apply Young's inequality on the second term to obtain

$$\frac{d}{dt}(I + C_5) \le C_4 \left(I + C_5\right).$$

Therefore by Grönwall's lemma we have

$$I \le (I_0 + C_5) \exp\left(C_4 t\right) - C_5 \le (I_0 + C_5) \exp\left(C_4 T\right) - C_5.$$

Consequently,

$$\sup_{t \in [0,T]} \|\partial_x u_n\|_{L^{2l}(\Omega)} \le C(\varphi_0, g_0, T, \Omega, l), \ l = 1, 2, \cdots$$

□

Corollary 1. *For any given $l \in \mathbb{N}^+$, up to a subsequence, u_n converge to u weakly in $L^{2l}(0,T; W_0^{1,2l}(\Omega))$.*

Proof. With inequality(4.1), use Banach-Alaoglu Theorem. □

Proposition 3. *The sequence $\tilde{u}_n^2$ is uniformly bounded in $W^{1,2}(Q_T)$.*

Proof. Firstly, test Equation(3.2) with $\partial_t \tilde{u}_n^2$. Integrate by parts in x, and the trace integral vanishes because $\partial_t \tilde{u}_n^2 = 0$ on $\partial\Omega$, hence

$$\left(\partial_t \tilde{u}_n, \partial_t \tilde{u}_n^2\right)_{Q_T} = -\left(\frac{x^2}{2}\partial_x \tilde{u}_n^2, \partial_t \partial_x \tilde{u}_n^2\right)_{Q_T} + \left(-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right), \partial_t \tilde{u}_n^2\right)_{Q_T}.$$

Applying the fundamental theorem of calculus with respect to $t \in (0, T)$ on the first term of the right hand side,

$$\left(\partial_t \tilde{u}_n, \partial_t \tilde{u}_n^2\right)_{Q_T} = -\left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(T)\right)^2\right)_{\Omega} + \left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(0)\right)^2\right)_{\Omega} + \left(-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right), \partial_t \tilde{u}_n^2\right)_{Q_T}$$

The goal is to bound $\left(\partial_t \tilde{u}_n^2, \partial_t \tilde{u}_n^2\right)_{Q_T}$, however the left hand side is $\left(\partial_t \tilde{u}_n, \partial_t \tilde{u}_n^2\right)_{Q_T}$.
Note that by maximum principle, there exists some constant C_1 such that $\tilde{u}_n \in (\frac{1}{2n}, \frac{C_1}{2})$, therefore,

$$\left(\partial_t \tilde{u}_n^2\right)^2 = 4\tilde{u}_n^2 \left(\partial_t \tilde{u}_n\right)^2 \leq C_1 \cdot 2\tilde{u}_n \left(\partial_t \tilde{u}_n\right)^2 = C_1 \left(\partial_t \tilde{u}_n\right)\left(\partial_t \tilde{u}_n^2\right).$$

Integrate both sides on Q_T,

$$\begin{aligned}
\|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)}^2 \leq & C_1 \left(\partial_t \tilde{u}_n, \partial_t \tilde{u}_n^2\right)_{Q_T} \\
= & C_1 \left(-\left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(T)\right)^2\right)_{\Omega} + \left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(0)\right)^2\right)_{\Omega}\right) \\
& + C_1 \left(-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right), \partial_t \tilde{u}_n^2\right)_{Q_T} \\
\leq & C_1 \left(-\left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(T)\right)^2\right)_{\Omega} + \left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(0)\right)^2\right)_{\Omega}\right) \\
& + C_1 \|-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right)\|_{L^2(Q_T)} \cdot \|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)},
\end{aligned}$$

in which we used Hölder's inequality. Then use Young's inequality to bound the last term in the inequality above,

$$\begin{aligned}
& \|-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right)\|_{L^2(Q_T)} \cdot \|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)} \\
\leq & \frac{C_1}{2} \|-x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right)\|_{L^2(Q_T)}^2 + \frac{1}{2C_1}\|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)}^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)}^2 \leq & 2C_1 \left(-\left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(T)\right)^2\right)_{\Omega} + \left(\frac{x^2}{4}, \left(\partial_x \tilde{u}_n^2(0)\right)^2\right)_{\Omega}\right) \\
& + C_1^2 \left\| -x^2 \left(\partial_x \tilde{u}_n\right)^2 - x\partial_x \tilde{u}_n^2 + g_0\left(\tilde{u}_n - \frac{1}{n}\right) \right\|_{L^2(Q_T)}^2
\end{aligned}$$

By Theorem 2, Proposition 1 and Proposition 2, the right hand side is uniformly bounded, hence $\|\partial_t \tilde{u}_n^2\|_{L^2(Q_T)}$ is uniformly bounded. Meanwhile, $\|\partial_x \tilde{u}_n^2\|_{L^2(Q_T)}$ is also uniformly bounded, thus the result follows. □

The following lemma shows that convergence a.e. combined with uniform boundedness implies strong convergence.

Lemma 1. *For a sequence v_n that is uniformly bounded in $L^4(Q_T)$, if v_n converges to $v \in L^4(Q_T)$ almost everywhere in Q_T, then v_n converges to v strongly in $L^2(Q_T)$.*

Proof. By Egorov's theorem, for any $\epsilon > 0$, there exists a measurable set $S_\epsilon \subset Q_T$ such that $|S_\epsilon| \leq \epsilon$ and $v_n \to v$ uniformly in $Q_T \backslash S_\epsilon$. Therefore,

$$\begin{aligned}\int_{Q_T} |v_n - v|^2 &= \int_{S_\epsilon} |v_n - v|^2 + \int_{Q_T\backslash S_\epsilon} |v_n - v|^2 \\ &= \int_{Q_T} |v_n - v|^2 \chi_{S_\epsilon} + \int_{Q_T} |v_n - v|^2 \chi^2_{Q_T\backslash S_\epsilon}\end{aligned}$$

Apply Hölder's inequality on both terms,

$$\begin{aligned}\int_{Q_T} |v_n - v|^2 \leq& \||v_n - v|^2\|_{L^2(Q_T)} \cdot \|\chi_{S_\epsilon}\|_{L^2(Q_T)} + \left(\sup_{(x,t)\in Q_T\backslash S_\epsilon} |v_n - v|^2\right) \cdot \|\chi_{Q_T\backslash S_\epsilon}\|_{L^1(Q_T)} \\ \leq& \|v_n - v\|^2_{L^4(Q_T)} \cdot \epsilon + C_1 \left(\sup_{(x,t)\in Q_T\backslash S_\epsilon} |v_n - v|^2\right)\end{aligned}$$

Since $v_n \to v$ uniformly in $Q_T \backslash S_\epsilon$, taking the limit on both sides,

$$\limsup_{n\to\infty} \int_{Q_T} |v_n - v|^2 \leq \|v_n - v\|^2_{L^4(Q_T)} \cdot \epsilon$$

As ϵ is arbitrary and $\|v_n - v\|_{L^4(Q_T)}$ is uniformly bounded, we conclude that v_n converges to v strongly in $L^2(Q_T)$.

□

Corollary 2. *Up to a subsequence, u_n converge to u strongly in $L^q(Q_T)$, for any $q < \infty$.*

Proof. By compactness, Proposition 3 implies that, for any $q < \infty$, up to a subsequence, $\tilde{u}_n^2$ converge to some function z strongly in $L^q(Q_T)$. Therefore by Lemma 1, $\tilde{u}_n$ converge to $u = \sqrt{z}$ strongly in $L^q(Q_T)$, for any $q < \infty$. □

Combining Corollary 1 and Corollary 2, by taking subsequence of a subsequence, we obtain a limit $u \in L^{2l}(0, T; W_0^{1,2l}(\Omega))$ such that u_n converge to u weakly in $L^{2l}(0, T; W_0^{1,2l}(\Omega))$ and strongly in $L^q(Q_T)$. It remains to show that such a function is indeed a weak solution to the problem. That requires several convergence results, which will be elaborated in the next section.

Before proceeding to the convergence results, we introduce the following estimate, which is a prerequisite for the proof of Theorem 5. The equation of interest calls for an alternative argument to the one proposed by Abdellaoui, Peral and Walias[1].

Proposition 4. *Let $\psi \in C_0^\infty(Q_T)$ be s.t. $\psi \geq 0$ in Q_T, then the sequence $x^2\psi\tilde{u}_n^{-\theta}|\partial_x\tilde{u}_n|$ is uniformly bounded in $L^1(Q_T)$ for any $\theta \in (0, 1/2)$.*

Proof. Let $\psi \in C_0^\infty(Q_T)$ be s.t. $\psi \geq 0$ in Q_T. Test Equation(3.2) with $\psi\tilde{u}_n^{-\delta}$, where $\delta \in (0, 1)$. Since $\psi = 0$ on ∂Q_T, integrating by parts on x, the trace integral vanishes, it follows that,

$$\begin{aligned}&\left(\partial_t\tilde{u}_n, \psi\tilde{u}_n^{-\delta}\right)_{Q_T} + \left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \partial_x\left(\psi\tilde{u}_n^{-\delta}\right)\right)_{Q_T} \\ &\quad= \left(\partial_t\tilde{u}_n, \psi\tilde{u}_n^{-\delta}\right)_{Q_T} + \left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \tilde{u}_n^{-\delta}\left(\partial_x\psi\right)\right)_{Q_T} + \left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \psi\left(\partial_x\tilde{u}_n^{-\delta}\right)\right)_{Q_T} \\ &\quad= \left(-x^2\left(\partial_x\tilde{u}_n\right)^2, \psi\tilde{u}_n^{-\delta}\right)_{Q_T} + \left(-x\partial_x\tilde{u}_n^2, \psi\tilde{u}_n^{-\delta}\right)_{Q_T} + \left(g_0\left(\tilde{u}_n - \frac{1}{n}\right), \psi\tilde{u}_n^{-\delta}\right)_{Q_T}.\end{aligned}$$

Thus, simplifying and rearranging each inner product term in the equation above to obtain

$$\begin{aligned}&\left(\tilde{u}_n^{-\delta}\partial_t\tilde{u}_n, \psi\right)_{Q_T} + \left(\tilde{u}_n^{1-\delta}\partial_x\tilde{u}_n, x^2\left(\partial_x\psi\right)\right)_{Q_T} + (-\delta)\left(\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2, x^2\psi\right)_{Q_T} \\ &\quad= -\left(\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2, x^2\psi\right)_{Q_T} - \left(\tilde{u}_n^{1-\delta}\partial_x\tilde{u}_n, \psi\partial_x(x^2)\right)_{Q_T} + \left(\tilde{u}_n^{-\delta}\left(\tilde{u}_n - \frac{1}{n}\right), \psi g_0\right)_{Q_T}.\end{aligned}$$

Next, collecting the third term on the left hand side and the first term on the right hand side, we have

$$(1-\delta)\left(x^2\psi, \tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)_{Q_T} = -\frac{1}{1-\delta}\left(\partial_t\tilde{u}_n^{1-\delta}, \psi\right)_{Q_T} - \frac{1}{2-\delta}\left(\partial_x\left(x^2\psi\right), \partial_x\tilde{u}_n^{2-\delta}\right)_{Q_T} + \left(\psi g_0, \tilde{u}_n^{1-\delta}\right)_{Q_T} - \left(\psi g_0, \frac{1}{n}\tilde{u}_n^{-\delta}\right)_{Q_T}. \tag{4.3}$$

The first three terms on the right hand side are apparently bounded. Indeed $\tilde{u}_n$ satisfies maximum principle and,

$$\begin{aligned}\left(\partial_t\tilde{u}_n^{1-\delta}, \psi\right)_{Q_T} &= -\left(\tilde{u}_n^{1-\delta}, \partial_t\psi\right)_{Q_T}\\ \left(\partial_x\left(x^2\psi\right), \partial_x\tilde{u}_n^{2-\delta}\right)_{Q_T} &= -\left(\partial_x^2\left(x^2\psi\right), \tilde{u}_n^{2-\delta}\right)_{Q_T}.\end{aligned}$$

In addition, since $\tilde{u}_n = u_n + \frac{1}{n} \geq \frac{1}{n}$, the fourth term (4.3) to be estimated

$$\frac{1}{n}\tilde{u}_n^{-\delta} \leq n^{\delta-1} = \frac{1}{n^{1-\delta}} \leq 1$$

Therefore, the left hand side of (4.3) becomes uniformly bounded in x-space

$$\left(x^2\psi, \tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)_{Q_T} \leq C(n, f_0, g_0) \tag{4.4}$$

Finally, the L^1 norm of the sequence $x^2\psi\tilde{u}_n^{-\theta}|\partial_x\tilde{u}_n|$ implies, by Hölder's inequality, that

$$\begin{aligned}\|x^2\psi\tilde{u}_n^{-\delta/2}|\partial_x\tilde{u}_n|\|_{L^1(Q_T)} &= \left(x^2\psi\tilde{u}_n^{-\delta/2}|\partial_x\tilde{u}_n|, 1\right)_{Q_T}\\ &= \left(\left(x^2\psi\right)^{1/2}, \left(x^2\psi\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)^{1/2}\right)_{Q_T}\\ &\leq \|\left(x^2\psi\right)^{1/2}\|_{L^2(Q_T)}\cdot\|\left(x^2\psi\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)^{1/2}\|_{L^2(Q_T)}\\ &= \|\left(x^2\psi\right)^{1/2}\|_{L^2(Q_T)}\cdot\left(\left(x^2\psi\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)^{1/2}, \left(x^2\psi\tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)^{1/2}\right)_{Q_T}\\ &= \|\left(x^2\psi\right)^{1/2}\|_{L^2(Q_T)}\cdot\left(x^2\psi, \tilde{u}_n^{-\delta}\left(\partial_x\tilde{u}_n\right)^2\right)_{Q_T}.\end{aligned}$$

And so, by inequality (4.4), the sequence $x^2\psi\tilde{u}_n^{-\theta}|\partial_x\tilde{u}_n|$ is uniformly bounded in $L^1(Q_T)$ for any $\theta = \frac{\delta}{2} \in (0, \frac{1}{2})$.

□

5 Convergence Results

The aim of this section is to prove that the sequence $\partial_x u_n$ converge to $\partial_x u$ a.e. in Q_T, where we have adopted the techniques in the work of Abdellaoui, Peral and Walias[1]. The roadmap is as follows:

1. Using Proposition 1, Proposition 2 and Proposition 3 to prove Lemma 2.
2. Theorem 4 is a simple corollary of Lemma 2.
3. Combining Theorem 4 and Proposition 4 to prove Theorem 5.

Lemma 2. *If $\tilde{u}_n$ is the solution to Equation(3.2), then for any $s \in (0,1)$*

$$\lim_{n\to\infty}\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2\right]^s = 0$$

Proof. Recall that $u \in L^2\left(0,T;W_0^{1,2}(\Omega)\right)$, introduce the time–regularization of $u(x,t)$ by Landes et. al.[6],

$$u_\nu(x,t) = \exp(-\nu t)\varphi_0(x) + \nu\int_0^t \exp(-\nu(t-s))u(x,s)ds$$

It is known that

1. $u_\nu(x,t)$ converge to $u(x,t)$ strongly in $L^2\left(0,T;W_0^{1,2}(\Omega)\right)$.
2. u_ν is the solution of the following problem,

$$\begin{cases} \frac{1}{\nu}\partial_t u_\nu + u_\nu = u \\ u_\nu(x,0) = \varphi_0(x) \end{cases} \tag{5.1}$$

Define a cut-off function T_ε as

$$T_\varepsilon(y) = \begin{cases} y, & y \in (-\varepsilon,\varepsilon) \\ \operatorname{sign}(y)\varepsilon, & \text{otherwise} \end{cases} \tag{5.2}$$

And define a non-negative function $J_\varepsilon(y)$, such that $J_\varepsilon'(y) = T_\varepsilon(y)$,

$$J_\varepsilon(y) = \begin{cases} -\varepsilon y - \frac{1}{2}\varepsilon^2, & y \in (-\infty,-\varepsilon) \\ \frac{1}{2}y^2, & y \in (-\varepsilon,\varepsilon) \\ \varepsilon y - \frac{1}{2}\varepsilon^2, & y \in (\varepsilon,\infty) \end{cases} \tag{5.3}$$

It takes two steps to prove that $\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2\right]^s$ converge to zero,

1. prove that $\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2\right]^s \chi\{|u_n - u_\nu| \le \varepsilon\}$ converge to zero
2. prove that $\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2\right]^s \chi\{|u_n - u_\nu| > \varepsilon\}$ converge to zero

For the first step, do the following decomposition

$$\begin{aligned}
&\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2\right]\chi\{|u_n - u_\nu| \le \varepsilon\} \\
=&\int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n - u\right)\right)^2 \\
=&\int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(\tilde{u}_n - u\right) - \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x u\right)\partial_x\left(\tilde{u}_n - u\right) \\
=&\int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(\tilde{u}_n - u\right) \\
&-\int_{Q_T}\left[x^2\left(\tilde{u}_n\chi\{|u_n-u_\nu|\le\varepsilon\} - u\chi\{|u-u_\nu|\le\varepsilon\}\right)\left(\partial_x u\right)\partial_x\left(\tilde{u}_n - u\right)\right] \\
&-\int_{Q_T}\left[x^2 u\chi\{|u-u_\nu|\le\varepsilon\}\left(\partial_x u\right)\partial_x\left(\tilde{u}_n - u\right)\right] \\
=&A_1 + A_2 + A_3
\end{aligned}$$

Start first from A_2 and A_3, as their estimates are relatively simple and straightforward.

Indeed, by Hölder's inequality and Corollary 2, it follows that,

$$\begin{aligned}A_2 =& -\int_{Q_T} \left[x^2 \left(\tilde{u}_n\chi\{|u_n-u_\nu|\le\varepsilon\} - u\chi\{|u-u_\nu|\le\varepsilon\}\right)(\partial_x u)\,\partial_x\left(\tilde{u}_n-u\right)\right]\\ \le& C_1(\Omega,T)\|\tilde{u}_n\chi\{|u_n-u_\nu|\le\varepsilon\} - u\chi\{|u-u_\nu|\le\varepsilon\}\|_{L^2(Q_T)}\|(\partial_x u)\,\partial_x\left(\tilde{u}_n-u\right)\|_{L^2(Q_T)}\\ \le& C_2(\varphi_0,g_0,\Omega,T)\|\tilde{u}_n\chi\{|u_n-u_\nu|\le\varepsilon\} - u\chi\{|u-u_\nu|\le\varepsilon\}\|_{L^2(Q_T)},\end{aligned} \tag{5.4}$$

also, the following term will converge to zero,

$$A_3 = -\int_{Q_T} x^2 u\,(\partial_x u)\left(\partial_x\tilde{u}_n - \partial_x u\right)\chi\{|u-u_\nu|\le\varepsilon\}. \tag{5.5}$$

It remains to bound $A_1 = \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(\tilde{u}_n-u\right)$.
This estimate is performed by first testing Equation(3.2) with $T_\varepsilon(u_n-u_\nu)$, where T_ε is defined in Equation(5.2), to obtain,

$$\begin{aligned}&\left(\partial_t\tilde{u}_n, T_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \partial_x\left(T_\varepsilon(u_n-u_\nu)\right)\right)_{Q_T}\\ =&\left(-x^2\left(\partial_x\tilde{u}_n\right)^2, T_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \left(-x\partial_x\tilde{u}_n^2, T_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \left(g_0\left(\tilde{u}_n-\frac{1}{n}\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T}\end{aligned} \tag{5.6}$$

Since $|T_\varepsilon(u_n-u_\nu)|\le\epsilon$, the right hand side of the above equation can be bounded as follows,

$$\text{RHS} \le \varepsilon\left(\|x^2\left(\partial_x\tilde{u}_n\right)^2\|_{L^1(Q_T)} + \|x\partial_x\tilde{u}_n^2\|_{L^1(Q_T)} + \|g_0\left(\tilde{u}_n-\frac{1}{n}\right)\|_{L^1(Q_T)}\right),$$

with the first term uniformly bounded by Proposition 2, the second one uniformly bounded by the energy inequality in Proposition 1, and the last term by maximum principle. Consequently,

$$\left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \partial_x\left(T_\varepsilon(u_n-u_\nu)\right)\right)_{Q_T} \le C_1(\varphi_0,g_0,\Omega,T)\varepsilon - \left(\partial_t\tilde{u}_n, T_\varepsilon(u_n-u_\nu)\right)_{Q_T}.$$

Since u_ν is a solution of Equation(5.1), $\partial_t u_\nu$ can be replaced with $\nu(u-u_\nu)$,

$$\begin{aligned}\left(\partial_t\tilde{u}_n, T_\varepsilon(u_n-u_\nu)\right)_{Q_T} =& \left(\partial_t\left(u_n-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \left(\partial_t u_\nu, T_\varepsilon(u_n-u_\nu)\right)_{Q_T}\\ =& \left(\partial_t\left(u_n-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T}\\ =& \left(1, \partial_t J_\varepsilon(u_n-u_\nu)\right)_{Q_T} + \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T}\\ =& \left(1, J_\varepsilon(u_n(T)-u_\nu(T))\right)_\Omega - \left(1, J_\varepsilon(u_n(0)-u_\nu(0))\right)_\Omega + \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T},\end{aligned}$$

in which J_ε is defined in Equation(5.3) as the anti-derivative of T_ε.
Each term on the right hand side is bounded from below.
Indeed, by definition of J_ε,

$$\left(1, J_\varepsilon(u_n(T)-u_\nu(T))\right)_\Omega \ge 0. \tag{5.7}$$

Since u_n and u_ν share the same initial condition, the second term is actually zero.

$$\left(1, J_\varepsilon(u_n(0)-u_\nu(0))\right)_\Omega = \left(1, J_\varepsilon(\varphi_0-\varphi_0)\right)_\Omega = 0 \tag{5.8}$$

By the sign-keeping property of T_ε,

$$\begin{aligned}\nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u_\nu)\right)_{Q_T} =& \nu\left(\left(u-u_\nu\right), T_\varepsilon(u-u_\nu-u+u_n)\right)_{Q_T}\\ =& \nu\left(\left(u-u_\nu\right), T_\varepsilon(u-u_\nu)\right)_{Q_T} + \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u)\right)_{Q_T}\\ \ge& \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u)\right)_{Q_T}\end{aligned} \tag{5.9}$$

Therefore, combining inequalities (5.7), (5.8) and (5.9),

$$\begin{aligned}\left(x^2\tilde{u}_n\partial_x\tilde{u}_n, \partial_x\left(T_\varepsilon(u_n-u_\nu)\right)\right)_{Q_T} \le& C_1(\varphi_0,g_0,\Omega,T)\varepsilon - \left(\partial_t\tilde{u}_n, T_\varepsilon(u_n-u_\nu)\right)_{Q_T}\\ \le& C_1(\varphi_0,g_0,\Omega,T)\varepsilon - \nu\left(\left(u-u_\nu\right), T_\varepsilon(u_n-u)\right)_{Q_T}\end{aligned}$$

Therefore,

$$
\begin{aligned}
A_1 &= \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_n-u\right) \\
&= \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_n-u_\nu\right) + \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_\nu-u\right) \\
&= \int_{Q_T} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(T_\varepsilon\left(u_n-u_\nu\right)\right) + \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_\nu-u\right) \\
&\le C_1(\varphi_0,g_0,\Omega,T)\varepsilon - \nu\left(\left(u-u_\nu\right),T_\varepsilon(u_n-u)\right)_{Q_T} + \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_\nu-u\right)
\end{aligned}
\tag{5.10}
$$

Putting together the inequalities (5.10), (5.4) and (5.5),

$$
\begin{aligned}
&\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n-u\right)\right)^2\right]\chi\{|u_n-u_\nu|\le\varepsilon\} \\
=&A_1+A_2+A_3 \\
\le&C_1(\varphi_0,g_0,\Omega,T)\varepsilon - \nu\left(\left(u-u_\nu\right),T_\varepsilon(u_n-u)\right)_{Q_T} + \int_{\{|u_n-u_\nu|\le\varepsilon\}} x^2\tilde{u}_n\left(\partial_x\tilde{u}_n\right)\partial_x\left(u_\nu-u\right) \\
&+C_2(\varphi_0,g_0,\Omega,T)\|\tilde{u}_n\chi\{|u_n-u_\nu|\le\varepsilon\}-u\chi\{|u-u_\nu|\le\varepsilon\}\|_{L^2(Q_T)} \\
&-\int_{Q_T} x^2u\left(\partial_x u\right)\left(\partial_x\tilde{u}_n-\partial_x u\right)\chi\{|u-u_\nu|\le\varepsilon\} \\
=&B_1(n,\nu,\varepsilon)
\end{aligned}
\tag{5.11}
$$

Since $\partial_x\tilde{u}_n$ converge to $\partial_x u$ weakly in $L^2(Q_T)$, the last term of B_1 converges to zero as n goes to infinity, therefore,

$$\lim_{\varepsilon\to0^+}\limsup_{\nu\to\infty}\limsup_{n\to\infty}B_1(n,\nu,\varepsilon)=0$$

For the second step, consider $\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n-u\right)\right)^2\right]^s\chi\{|u_n-u_\nu|>\varepsilon\}$, using Hölder's inequality,

$$
\begin{aligned}
&\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n-u\right)\right)^2\right]^s\chi\{|u_n-u_\nu|>\varepsilon\} \\
\le&C_1(\Omega,T)\|(\partial_x u_n-\partial_x u)^{2s}\|_{L^{\rho'}(Q_T)}\cdot\left(\|\chi\{|u_n-u_\nu|>\varepsilon\}-\chi\{|u-u_\nu|>\varepsilon\}\|_{L^\rho(Q_T)}+\|\chi\{|u-u_\nu|>\varepsilon\}\|_{L^\rho(Q_T)}\right) \\
=&B_2(n,\nu,\varepsilon)
\end{aligned}
\tag{5.12}
$$

Taking the limit,

$$\lim_{\varepsilon\to0^+}\limsup_{\nu\to\infty}\limsup_{n\to\infty}B_2(n,\nu,\varepsilon)=0$$

To summarize,

$$0\le\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n-u\right)\right)^2\right]^s\le B_1(n,\nu,\varepsilon)+B_2(n,\nu,\varepsilon),$$

where B_1 and B_2 are on the right hand side of Equation (5.11) and (5.12). Consequently,

$$\lim_{n\to\infty}\int_{Q_T}\left[x^2\tilde{u}_n\left(\partial_x\left(\tilde{u}_n-u\right)\right)^2\right]^s=0.$$

□

Theorem 4. *The sequence* $\partial_x\tilde{u}_n^2=\partial_x(u_n+\frac{1}{n})^2$ *converge to* $\partial_x u^2$ *strongly in* $L^\sigma(Q_T)$ *for all* $\sigma\in(0,2)$.

Proof. Note that

$$\begin{aligned}
&\int_{Q_T} |\partial_x \tilde{u}_n^2 - \partial_x u^2|^{2s} \\
=&2^{2s} \int_{Q_T} |\tilde{u}_n \partial_x \tilde{u}_n - u\partial_x u|^{2s} \\
=&2^{2s} \int_{Q_T} |\tilde{u}_n \partial_x \tilde{u}_n - \tilde{u}_n \partial_x u + \tilde{u}_n \partial_x u - u\partial_x u|^{2s} \\
=&2^{2s} \int_{Q_T} |(\tilde{u}_n \partial_x \tilde{u}_n - \tilde{u}_n \partial_x u) + \partial_x u\,(\tilde{u}_n - u)|^{2s} \\
\le&C \int_{Q_T} \left(|\tilde{u}_n \partial_x \tilde{u}_n - \tilde{u}_n \partial_x u|^{2s} + |\partial_x u\,(\tilde{u}_n - u)|^{2s}\right)
\end{aligned}$$

By Lemma 2 and Corollary 2, both terms converge to zero if $s \in (0, 1)$. Let $\sigma = 2s$, then $\sigma \in (0, 2)$. ☐

Theorem 5. *The sequence $\partial_x u_n$ converge to $\partial_x u$ a.e. in Q_T*

Proof. Let $\psi \in C_0^\infty(Q_T)$ be s.t. $\psi \ge 0$ in Q_T. To prove convergence a.e., it is sufficient to show that for some $\alpha \in (0, 1)$,

$$\lim_{n\to\infty} \int_{Q_T} |\partial_x u_n - \partial_x u|^\alpha \psi = 0$$

Decompose the domain Q_T,

$$\begin{aligned}
\int_{Q_T} |\partial_x u_n - \partial_x u|^\alpha \psi =& \int_{\{u=0\}} |\partial_x u_n - \partial_x u|^\alpha \psi + \int_{\{u>0\}} |\partial_x u_n - \partial_x u|^\alpha \psi \\
=& \int_{\{u=0\}} |\partial_x u_n|^\alpha \psi + \int_{\{0<u\le \frac{1}{m}\}} |\partial_x u_n - \partial_x u|^s \psi + \int_{\{u>\frac{1}{m}\}} |\partial_x u_n - \partial_x u|^s \psi \\
=&A_1 + A_2 + A_3.
\end{aligned} \tag{5.13}$$

Using Hölder's inequality to get the bound of A_2,

$$\begin{aligned}
A_2 =& \int_{\{0<u\le\frac{1}{m}\}} |\partial_x u_n - \partial_x u|^s \psi \\
\le& \||\partial_x u_n - \partial_x u|^s \psi\|_{L^{2/s}(Q_T)} \|\chi_{\{0<u\le\frac{1}{m}\}}\|_{L^{\frac{2}{2-s}}(Q_T)} \\
\le& C\|\chi_{\{0<u\le\frac{1}{m}\}}\|_{L^{\frac{2}{2-s}}(Q_T)}.
\end{aligned}$$

Note that $\|\chi_{\{0<u\le\frac{1}{m}\}}\|_{L^{\frac{2}{2-s}}(Q_T)}$ can be arbitrarily small.

Next, by Theorem 4, it is known that $\partial_x \tilde{u}_n^2 \to \partial_x u^2$ strongly in $L^\sigma(Q_T)$ for all $\sigma < 2$, therefore A_3 converges to zero, in fact,

$$\begin{aligned}
A_3 =& \int_{\{u>\frac{1}{m}\}} \frac{1}{|u|^s} |u\partial_x u_n - u\partial_x u|^s \psi \\
=& \int_{\{u>\frac{1}{m}\}} \frac{1}{|u|^s} |(u - \tilde{u}_n)\,\partial_x \tilde{u}_n + \frac{1}{2}\left(\partial_x \tilde{u}_n^2 - \partial_x u^2\right)|^s \psi \\
\le& m^s \int_{Q_T} |(u - \tilde{u}_n)\,\partial_x \tilde{u}_n + \frac{1}{2}\left(\partial_x \tilde{u}_n^2 - \partial_x u^2\right)|^s \psi,
\end{aligned}$$

and the limit follows from

$$\limsup_{n\to\infty} A_3(n) \le m^s \limsup_{n\to\infty} \int_{Q_T} |(u - \tilde{u}_n)\,\partial_x \tilde{u}_n + \frac{1}{2}\left(\partial_x \tilde{u}_n^2 - \partial_x u^2\right)|^s \psi = 0$$

Considering A_1 of Equation 5.13, since $u_n \to u$ strongly in $L^{2l}(Q_T)$, by Egorov's Lemma, for every $\epsilon > 0$, there exists a measurable set E_ϵ such that $|E_\epsilon| \leq \epsilon$ and $u_n \to u$ uniformly in $Q_T \backslash E_\epsilon$.

$$\int_{\{u=0\}} |\partial_x u_n|^\alpha \psi = \int_{\{u=0\}\cap E_\epsilon} |\partial_x u_n|^\alpha \psi + \int_{\{u=0\}\cap Q_T\backslash E_\epsilon} |\partial_x u_n|^\alpha \psi$$

The first term is bounded through Hölder's inequality,

$$\int_{\{u=0\}\cap E_\epsilon} |\partial_x u_n|^\alpha \psi = \int_{Q_T} |\partial_x u_n|^\alpha \psi \chi_{\{u=0\}\cap E_\epsilon} \leq \int_{Q_T} |\partial_x u_n|^\alpha \psi \chi_{E_\epsilon} \leq C \| |\partial_x u_n|^\alpha \|_{L^{1/\alpha}(Q_T)} \|\chi_{E_\epsilon}\|_{L^{1/(1-\alpha)}(Q_T)} \leq C\epsilon^{1-\alpha}$$

The second one uses the fact that for any $\mu > 0$, there exists N such that $|u_n - u| = |u_n| < \mu$ for all $n > N$ and for all $x \in \{u = 0\} \cap Q_T \backslash E_\epsilon$. In other words, for $n > N$, $\{u = 0\} \cap Q_T \backslash E_\epsilon$ is a subset of $\{u_n \leq \mu\} \cap Q_T \backslash E_\epsilon$, hence the integral

$$\begin{aligned}
&\int_{\{u=0\}\cap Q_T\backslash E_\epsilon} |\partial_x u_n|^\alpha \psi \\
&\leq \int_{\{u_n\leq \mu\}\cap Q_T\backslash E_\epsilon} |\partial_x u_n|^\alpha \psi \\
&\leq \left(\mu + \frac{1}{n}\right)^{\theta\alpha} \int_{\{u_n\leq \mu\}\cap Q_T\backslash E_\epsilon} \left(\frac{|\partial_x u_n|}{(u_n + \frac{1}{n})^\theta}\right)^\alpha \psi \\
&\leq \left(\mu + \frac{1}{n}\right)^{\theta\alpha} \int_{Q_T} \left(\frac{|\partial_x u_n|}{(u_n + \frac{1}{n})^\theta}\right)^\alpha \psi
\end{aligned}$$

The boundedness of $\int_{Q_T} \left(\frac{|\partial_x u_n|}{(u_n+\frac{1}{n})^\theta}\right)^\alpha \psi$ is secured by Proposition 4. The result follows from taking $\mu \to 0$. $\square$

6 Existence of Global Weak Solution

We are now ready to prove the main result of the paper, i.e. Theorem 1.

Proof. Note that

$$-\left(\tilde{u}_n, \partial_t \eta\right)_{Q_T} + \left(\frac{x^2}{2}\partial_x \tilde{u}_n^2, \partial_x \eta\right)_{Q_T} = \left(-x^2\left(\partial_x \tilde{u}_n\right)^2, \eta\right)_{Q_T} + \left(-x\partial_x \tilde{u}_n^2, \eta\right)_{Q_T} + \left(g_0\left(\tilde{u}_n - \frac{1}{n}\right), \eta\right)_{Q_T} + \left(\varphi_0 + \frac{1}{n}, \eta(x,0)\right)_\Omega$$

It is sufficient to prove that the first term on the right hand side converges to $\left(-x^2(\partial_x u)^2, \eta\right)_{Q_T}$. Take the difference and use Hölder's inequality,

$$\left(x^2\left[\left(\partial_x \tilde{u}_n\right)^2 - \left(\partial_x u\right)^2\right], \eta\right)_{Q_T} \leq C\|\partial_x \tilde{u}_n - \partial_x u\|_{L^2(Q_T)} \cdot \|\partial_x \tilde{u}_n + \partial_x u\|_{L^2(Q_T)}$$

Since $\partial_x \tilde{u}_n$ converge to $\partial_x u$ a.e. in Q_T and $\partial_x \tilde{u}_n$ is uniformly bounded in $L^4(Q_T)$, by Lemma 1,

$$\lim_{n\to\infty} \|\partial_x \tilde{u}_n - \partial_x u\|_{L^2(Q_T)} = 0$$

$\square$

7 Nontrivial Equilibrium State

If $u_\infty(x)$ is an equilibrium state for Problem $(\mathcal{S})$, then we have

$$\begin{cases} 0 = x^2 u_\infty \partial_x^2 u_\infty + g_0 u_\infty, & \forall x \in \Omega = [x_a, x_b] \\ u_\infty(x) = 0, & \forall x \in \partial\Omega \end{cases}$$

Solve the equation

$$x^2\partial_x^2\mathcal{M} + g_0 = x^2\partial_x^2\mathcal{M} + x^2\partial_x\left(f_0 - \partial_x u_0\right) = 0$$

We obtain that

$$\mathcal{M}(x) = u_0(x) + \frac{1}{\int_{x_a}^{x_b} ds}\left(\left(\int_{x_a}^{x_b} f_0(s)ds\right)\cdot\left(\int_{x_a}^{x} ds\right) - \left(\int_{x_a}^{x} f_0(s)ds\right)\cdot\left(\int_{x_a}^{x_b} ds\right)\right)$$

If $\mathcal{M}(x) > 0$ for any $x \in \Omega$, then it is the unique nontrivial equilibrium state. Otherwise $u_\infty(x) = \mathcal{M}^+(x)$ is one of the possible nontrivial equilibrium states in weak sense.

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Appendix

According to Theorem 6.1 in Chapter V of Ladyzenskaja et. al.'s book[5], the following conditions (a) to (f) are sufficient for Theorem 3.
Recall that

$$\begin{aligned} a_1(x,t,u,p) &\coloneqq x^2\mathcal{P}_n(u)p \\ a(x,t,u,p) &\coloneqq x^2\mathcal{P}_n'(u)p^2 + 2x\mathcal{P}_n(u)p - g_0(x)u \\ A(x,t,u,p) &\coloneqq -g_0(x)u \end{aligned}$$

and

$$\varphi(x,t) \coloneqq \varphi_0(x) + \left[x^2\mathcal{P}_n'(\varphi_0)\partial_x^2\varphi_0 + g_0\varphi_0\right] t \tag{7.1}$$

We will verify the conditions one by one.

(a) For $(x,t) \in \overline{Q_T}$ and arbitrary u, the diffusion term is strictly coercive,

$$\frac{\partial a_1}{\partial p}(x,t,u,p) = x^2\mathcal{P}_n(u) \geq \frac{x_a^2}{2n} > 0,$$

and the reaction term has the following lower bound,

$$A(x,t,u,0)u = -g_0(x)u^2 \geq -\max(|g_0|)u^2.$$

(b) For $(x,t) \in \overline{Q_T}$, when $|u| \leq M$, for arbitrary p, the operators are bounded in the following sense.

$$\frac{\partial a_1}{\partial p}(x,t,u,p) = x^2\mathcal{P}_n(u) \leq x_b^2\,(M+1)\,,$$

and

$$
\begin{aligned}
&\left(|a_1| + |\frac{\partial a_1}{\partial u}|\right)(1+|p|) + |\frac{\partial a_1}{\partial x}| + |a| \\
=&\left(x^2\mathcal{P}_n(u)|p| + x^2\mathcal{P}_n'(u)|p|\right)(1+|p|) + 2x\mathcal{P}_n(u)|p| + |x^2\mathcal{P}_n'(u)p^2 + 2x\mathcal{P}_n(u)p - g_0(x)u| \\
\le&\left(x^2\mathcal{P}_n(u)|p| + x^2\mathcal{P}_n'(u)|p|\right)(1+|p|) + 2x\mathcal{P}_n(u)|p| + x^2\mathcal{P}_n'(u)p^2 + 2x\mathcal{P}_n(u)|p| + |g_0(x)u| \\
\le&\mu(M, x_b, \max(|g_0|))\,(1+|p|)^2\,.
\end{aligned}
$$

(c) For $(x,t) \in \overline{Q_T}$, $|u| \le M$ and $|p| \le M_1$, the functions a_1, a, $\frac{\partial a_1}{\partial p}$, $\frac{\partial a_1}{\partial u}$, and $\frac{\partial a_1}{\partial x}$ are arbitrarily smooth in x, t, u and p, therefore they satisfy any Hölder continuity condition.

(d) Note that

$$
\begin{aligned}
\frac{\partial a_1}{\partial u} &= x^2\mathcal{P}_n'(u)p, \\
\frac{\partial a}{\partial p} &= 2x^2\mathcal{P}_n'(u)p + 2x\mathcal{P}_n(u), \\
\frac{\partial a}{\partial u} &= x^2\mathcal{P}_n''(u)p^2 + 2x\mathcal{P}_n'(u)p - g_0(x).
\end{aligned}
$$

For $(x,t) \in \overline{Q_T}$, $|u| \le M$ and $|p| \le M_1$, all the above terms are bounded by a constant $C(M, M_1, \mathcal{P}_n, g_0, \Omega)$. In addition, neither a nor a_1 depend on t, therefore condition (d) is satisfied.

(e) By definition of ψ in Equation(7.1), ψ is arbitrarily smooth in $\overline{Q_T}$. In addition, for $x \in \partial\Omega$ and $t = 0$, the following identity holds,

$$
\partial_t\varphi(x,t) = x^2\mathcal{P}_n'(\varphi_0)\partial_x^2\varphi_0 + g_0\varphi_0 = x^2\mathcal{P}_n'(\varphi)\partial_x^2\varphi + g_0\varphi
$$

(f) It is trivial that the boundary $\partial\Omega$ satisfies any Hölder continuity condition.