ON THE CAUCHY PROBLEM FOR BOLTZMANN EQUATION MODELING A POLYATOMIC GAS

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Abstract. In the present manuscript we consider the Boltzmann equation that models a polyatomic gas by introducing one additional continuous variable, referred to as microscopic internal energy. We establish existence and uniqueness theory in the space homogeneous setting for the full non-linear case, under an extended Grad assumption on transition probability rate, that comprises hard potentials for both the relative speed and internal energy with the rate in the interval $(0, 2]$, which is multiplied by an integrable angular part and integrable partition functions. The Cauchy problem is resolved by means of an abstract ODE theory in Banach spaces, for an initial data with finite and strictly positive gas mass and energy, finite momentum, and additionally finite k_* polynomial moment, with k_* depending on the rate of the transition probability and the structure of a polyatomic molecule or its internal degrees of freedom. Moreover, we prove that polynomially and exponentially weighted Banach space norms associated to the solution are both generated and propagated uniformly in time.

Contents

1. Introduction

In this manuscript we consider a single polyatomic gas. The more complex structure of a molecule that may have more than one atom causes new phenomena at the level of molecular collisions. In particular, besides the translational motion in the physical space as in the classical case of monatomic elastic collisions, there appear possibilities of molecular rotations and/or vibrations, referred to as internal degrees of freedom. Collisional kinetic theory captures this feature by introducing the so-called microscopic internal energy of a molecule. Then, an elastic collision of polyatomic gases means conservation of the total – kinetic plus internal – energy of the two colliding molecules if binary interactions are taken into account.

Within the kinetic theory there is no unique way how to approach the microscopic internal energy, which can be understood as a measure of deviation from the classical case of a single monatomic gas. For example, in the semi-classical approach [20, 29, 43, 21], internal energy is assumed to take discrete values. The idea is to prescribe one distribution function to each energy level, resulting in a system of equations describing a gas. This model uses experimental data and is adequate for computational tasks. On the other hand, there exist continuous kinetic models that take the another path – the idea is to introduce one additional variable, a continuous microscopic internal energy, and to parametrize both molecular velocity and internal energy, using the so-called Borgnakke-Larsen procedure, which leads to one single Boltzmann equation [19, 24, 23]. Among continuous models, there are subtle differences causing by the choice of a functional space as an environment where physical intuition is provided, which are reflected on the structure of the cross-section, as pointed out in [25].

Both semi-classical and continuous models abound with many formal results. For instance, the Champan-Enskog method was developed in [1] for the semi-classical models and recently in [11] for the continuous model from [24]. Many macroscopic models of extended thermodynamics are derived starting from the continuous kinetic model [24]. In fact, the additional variable of internal energy fitted naturally into the two hierarchies of moment equations for a polyatomic gas, as first observed in [36, 37]. The maximum entropy principle was the main tool to close system of equations corresponding to six and fourteen moments [39, 15, 35, 41], and to numerically test those models [30]. An interested reader can consult [40] and references therein. In addition, the formal results spread to the mixture of polyatomic gases [34, 12, 13], that may be reactive as well [14, 9].

Despite the great interest of research community, so far there are few rigorous mathematical results addressing analytical properties of polyatomic gas kinetic model. A basic question is what are the differential cross section and transition probability rates for which a solution is admissible in a suitable functional space depending on the nature of the polyatomic Boltzmann flow operator defined by their particle molecular velocity and exchange of internal energy laws, or whether the initial value problem in such spaces is solvable, if solutions are well defined globally in time, and what is their high energy tail behavior.

In this manuscript we first establish the existence and uniqueness theory for the solution of the space homogeneous Boltzmann equation for a polyatomic gas continuous kinetic model introduced in [19]. The underlying assumption on the transition probability rate is of extended Grad type, meaning that besides the positive power of relative speed, we need the very same contribution of the internal energy. Moreover, the relative speed and internal energy are combined additively, and not multiplicatively as it was used in literature so far. Surprisingly enough, such a model of transition function perfectly embeds into physical interpretation, providing the total agreement with the models of extended thermodynamics and experimental data, as shown in [25].

The existence and uniqueness result is obtained by an application of the theory of ODEs in Banach spaces [32], that has been successfully used in many frameworks, such as mixture setting [26], polymers kinetic problems [3], quantum Boltzmann equation for bosons in very low temperature [8] and the weak wave turbulence models for stratified flows [28].

In the present manuscript we propose to study the scalar Boltzmann flow for interacting polyatomic gases interchanging pairs of pre and post molecular velocities $v \in \mathbb{R}^3$ and internal energies $I \in [0, \infty)$ as one additional continuous variable. This collisional model describes the statistical time evolution of probability distribution density $f(t, v, I)$ in the Banach space $L^1_k(\mathbb{R}^{3^+})$, of integrable functions in the upper half space $\mathbb{R}^{3+} := \mathbb{R}^3 \times [0, \infty)$ with the Lebesgue weight type function $\langle v, I\rangle^k := (1 + \frac{|v|^2}{2} + \frac{I}{m})^{k/2}$, for molecular velocities $v \in \mathbb{R}^3$ and internal energies

$I \in [0, \infty)$. Consequently, $\|f\|_{L^1_k(\mathbb{R}^{3+})}(t)$ are also referred as the $k-$Lebesgue moments associated to the solution of the Boltzmann flow.

A keypoint in this analysis consists in showing that that the dissipation built in the collision operator for polyatomic gasses is manifested by the decay of k-polynomial moment of the collisional form for a sufficiently large k, depending on data with finite initial mass, energy and a moment of order bigger or equal that k_*, with $k_* > 1$, since, in our notation $k = 1$ is macroscopic mass plus the total (kinetic + internal) energy. For this collisional gas model, such property is warranted by the control, from above and below, of transition rates associated to the velocity and internal energy interactions laws. This proposed approach requires detailed averaging over parameters in a compact manifold that distributes scattering mechanisms as functions of the scattering angle over the sphere of influence of the interaction, as much as the parameters, or partition functions, that distribute total energy to molecular variables.

In the classical case of a single monatomic gas, this result is known as sharp Povzner Lemma by angular averaging over the sphere, introduced for the first time in [16] for hard spheres in three dimensions and extended in [27] to variable hard potentials in velocity dimension bigger or equal than three, and used in many results for kinetic collisional binary transport in inelastic interactions theories, such as granular flow [18], and gas mixtures [26]. For this classical single monatomic gas describing binary elastic interactions of molecular velocities $v \in \mathbb{R}^d$, dissipation has an immediate effect: the evolution of the positive contribution of the Boltzmann operator k-Lebesgue moments, or equivalently classical k-moments, occurs for any $k > 1$, where the macroscopic gas mass plus energy corresponds to $k = 1$ that is a conserved quantity, since the local mass and energies associated the gain and loss operator balance to zero. Another example of this behavior was recently shown by the authors in the gas mixture setting [26] corresponding to a system of Boltzmann equations with disparate masses, with their corresponding energy identity showing that the positive contribution of k-moments coming from each pairs of gain operators associated to the system yields an analog dissipative effect, albeit for $k > k_*$, with k_* depending on the mass species ratios when taken by pairwise interactions, and shown k_* to grow with the disparateness of molecular masses.

This averaging on compact manifolds ambient spaces, where transition rate functions are defined, only involve moments of the positive contribution to the Boltzmann flow dynamics, that translates into a property for the moments associated to the gain operator. Namely, such compact manifold averaging property produces a dissipative mechanism for a large enough moment k, depending on the potential rate of transition probabilities and on molecules internal modes related to the complexity of molecular structure.

Once the dissipation of the collision operator gain part is shown, it remains to treat its loss term. In fact, it is a crucial aspect of our analysis to find a bound from below for the negative contribution of the polyatomic Boltzmann flow, which supports the coerciveness estimate in the natural Banach space $L^1_k(\mathbb{R}^{3^+})$ associated to polyatomic gas model. This result may be understood as the analogue to coerciveness property in the classical theory for diffusion type equations in continuum mechanics where the control of the energies or suitable Banach norms is done in Sobolev spaces, while in the framework of statistical mechanics the suitable norms are non-reflexive Banach spaces, as in this case $L^1_k(\mathbb{R}^{3^+})$. The *coercive factor*

A_{k_*} which is the strictly positive constant associated to be scalar Boltzmann flow of the order k_* at which the positive contribution of Lebesgue moments becomes submissive is identified by the smallest positive constant A_{k_*} to be characterized in Section 6, which depends on the averaging manifold Lemma associated to the potential rate of the transition probabilities, on internal modes of molecules and on the initial data. One can view A_{k_*} as the analog to the role of coercive form associated to elliptic and parabolic flows in continuum mechanics modelling where coerciveness is crucial for the existence and uniqueness theories in Sobolev spaces. The coerciveness estimate is based on a functional inequality that controls from below the convolution of any function $f \in L^1_{1^+}$ with a potential function of rate γ by the corresponding Lebesgue bracket of order γ, for any $\gamma \in [0, 2]$. The proof is inspired by the work [5] for the classical Boltzmann equation. In addition, the coercive estimates we present here are fundamental to obtain in $W^{m,2}$ Sobolev (Hilbert) spaces as shown in [6].

The decay of the $k-$Lebesgue moment of the Boltzmann operator positive contribution with respect to $k > k_*$ and the coerciveness estimate for the negative contribution allows for a priori k^{th}-moment estimates that are sufficient to generate infinitely many Ordinary Differential Inequalities (ODIs) with a negative superlinear term proportional to A_{k_*}. That is a sufficient condition for the Boltzmann flow under consideration in suitable invariant region Ω of Banach space $L^1_k(\mathbb{R}^{3^+})$ to be solvable globally in time.

We emphasize that conditions on initial data excludes singular measures and the need of entropy bounds. Yet the resulting construction of a unique solution in the space on polynomial moments yields entropy boundedness at any time, if initially so.

Our analytical approach focus on studying norms in the Banach space $L^1(\mathbb{R}^{3+})$ with $\mathbb{R}^{3+} := \mathbb{R}^3 \times \mathbb{R}_+$ associated to the solution of the time evolution problem for Boltzmann equation, in the space probability density functions defined over the classical metric space $\mathbb{R}^{3+}$, with both polynomial and exponential weights, by following the usual analytical path established for the classical Boltzmann equation for the single elastic monatomic gas model. This research started with [22, 44] in the case of polynomial moments and with [16] where the concept of exponential moments is presented, as much as in the techniques for moments summability, leading to the understanding high energy tail behavior of with inverse Gaussian in velocity space, associated to the solution of the Boltzmann equation for hard spheres (i.e. power exponent $\gamma = 1$) and constant angular part. These results were developed in [18] for inelastic interactions and non-Gaussian weight moments, and later, in [27] to collision kernels for hard potentials (i.e. $\gamma \in (0, 1]$) for any angular section with L^{1+}-integrability. Further, generation of exponential moments of order $\gamma/2$ with bounded angular section were shown in [33]. New approach was taken in [4] based on partial sums summability techniques, that extended the results to collision kernels for hard potentials with $\gamma \in (0, 2]$, for any angular section, with just L^1-integrability. In particular, exponential moments of order γ are shown to generate in a finite time, while Gaussian moments propagate if initially are finite, which holds independently of γ. Moreover, all these result were generalized when the angular part is not integrable (in the angular non-cutoff regime) [42, 31, 17, 38, 2].

The manuscript is organized as follows. First we introduce a kinetic model describing a polyatomic gas in Section 2, together with the notation and main definitions. Then in Section 3 we make precise sufficient properties for establishing existence and uniqueness theory, which comprises assumption on the form of transition function as the additive form of relative speed and microscopic internal energy with a potential $\gamma \in (0, 2]$ multiplied by some factors, together with its upper and lower bounds. Then in Section 4 we prove the Coerciveness Estimate for the loss operator, and in Section 5 we state and prove the two fundamental lemmas, namely the Energy Identity Decomposition and the Polyatomic Compact Manifold Averaging Lemma. These estimates identify the k_*-moment that will yield the coercive constant A_{k_*}. Section 6 deals with the statements and proofs to a priori estimates for k^{th}-moments of any order $k \geq k_*$ and defines the explicit form of A_{k_*}. These results enable us to identify an invariant region in which the collision operator will satisfy all the properties needed for existence and uniqueness result proved theory in Section 7 by means of solving an time evolution ODE in a suitable invariant region Ω in the Banach space $L^1(\mathbb{R}^{3+})$. Then, in Section 8 we show the solution of the Boltzmann equation for polyatomic gases has a property of summability of moments that is expressed, both, in the generation and propagation of exponential moments. In the propagation case, the unique solution of the Boltzmann flow for polyatomic gases initial data having an exponential moment of prescribed order $2s$, $s \in (0, 1]$ and rate β_0, there exists a smaller rate β such that the exponential moment of order β and same rate s is bounded uniformly in time (i.e. propagation holds), with such bound being three times the initial exponential moment. The rate β satisfies a minimum of three conditions, one of them degenerates as the coercive constant A_{k_*} and with a the maximum of an estimate of a few finite number of $k-$Lebesgue bounds for the moments propagation estimates. The case of generation of exponential, or summability of, moments holds for initial data with just k_*-Lebesgue moments. In this case, the order is $2s$, with $s \in (0, \gamma/2]$ and the rate $\beta = \beta(t)$, for positive time, also depends on the coercive factor A_{k_*} and with a the maximum of an estimate of a few finite number of $k-$Lebesgue bounds for the moments generation estimates as much as on the k_*-Lebesgue moments of the initial data. The generation of the exponential moment bound is achieved in short time, and remains uniformly bounded in time by the initial's data k_*-Lebesgue moment, exhibiting an invariant region estimate. Finally, the Appendix contains some technical results needed for the theory.

2. Kinetic model for a polyatomic gas

In this Section we will describe the Boltzmann equation for a polyatomic gas. We adapt the *continuous* approach, which introduces a *single continuous* variable I that we call *microscopic internal energy*, supposed to capture all the phenomena related to a more complex structure of a polyatomic molecule. The main feature is the presence of internal degrees of freedom that a molecule undertakes on an interaction, or collision. Besides the usual translational motion, a polyatomic molecule may experience rotations and/or vibrations, referred to as internal modes.

At the microscopic level of collisions, such motions cause appearance of the microscopic internal energy, apart from the usual kinetic energy in the conservation of energy law, under the assumption of elastic collisions. On the other side, at the macroscopic level, internal modes reflect on the energy law as well. As in this manuscript, we restrict to polytropic gases (meaning that macroscopic internal energy is linear with respect to the temperature), the caloric equation of state reads

$$e = D\,\frac{k\,T}{2\,m}, \tag{2.1}$$

where e is the internal energy, k the Boltzmann constant, m mass and T temperature of the gas. The constant D is related to the degrees of freedom. In the classical case for elastic interactions, only translation is taken into account, corresponding to D taking the value of the space dimension and the kinetic collisional model of the classical Boltzmann equation. In general D is determined by the sum of the total degrees of freedom, translational as much as rotational and vibrational motion associated to the collision. That means, in space dimension three, this constant takes at least the value $D = 3$, which is the classical case for monatomic gases modeled by the scalar Boltzmann equation, but for the polyatomic model the constant D must be larger than the dimension of the space of motion, $D > 3$.

2.1. **Collision modelling.** The starting point is to model a collision process between two interacting molecules. We suppose that a colliding pair of molecules have velocities and microscopic internal energies (v', I'), $(v'_*, I'_*) \in \mathbb{R}^{3+} := \mathbb{R}^3 \times [0, \infty)$ before the interaction, that became (v, I) and (v_*, I_*), respectively, after such interaction. Under the assumption of elastic interactions, these quantities are linked through the conservation laws of local momentum and total (kinetic + microscopic internal) molecular energy, namely,

$$\begin{aligned} v + v_* &= v' + v'_*, \\ \frac{m}{2}\left|v\right|^2 + I + \frac{m}{2}\left|v_*\right|^2 + I_* &= \frac{m}{2}\left|v'\right|^2 + I' + \frac{m}{2}\left|v'_*\right|^2 + I'_*. \end{aligned} \tag{2.2}$$

It is often more convenient to work in the center of mass reference frame by the introduction of center of mass V and relative velocity u as follows

$$V := \frac{v + v_*}{2}, \quad u := v - v_*. \tag{2.3}$$

Then, the total molecular energy law from (2.2) can be simply written by

$$\frac{m}{4}\left|u\right|^2 + I + I_* = \frac{m}{4}\left|u'\right|^2 + I' + I'_* =: E. \tag{2.4}$$

since clearly conservation of local momentum implies conservation of center of mass velocity

$$V = V'. \tag{2.5}$$

Collisional laws express pre-collisional quantities v', I', v'_*, I'_* in terms of the post-collisional ones. This is achieved via a parametrization of local conservation equations (2.2), according to the Borgnakke-Larsen procedure. To this end, we focus on energy (2.4) and first introduce a parameter $R \in [0, 1]$ that distributes local energy proportion of the total energy E into a pure kinetic part RE and a pure internal part of energy, proportional to $(1 - R)E$, according to

$$\frac{m}{4}\left|u'\right|^2 = RE, \quad I' + I'_* = (1 - R)E.$$

In addition, we set a parameter $r \in [0, 1]$ to distribute the proportion of total internal energy $(1 - R)E$ to each interacting states corresponding to the incoming molecular internal energy pair I', I'_* as follows

$$I' = r(1 - R)E, \qquad I'_* = (1 - r)(1 - R)E. \tag{2.6}$$

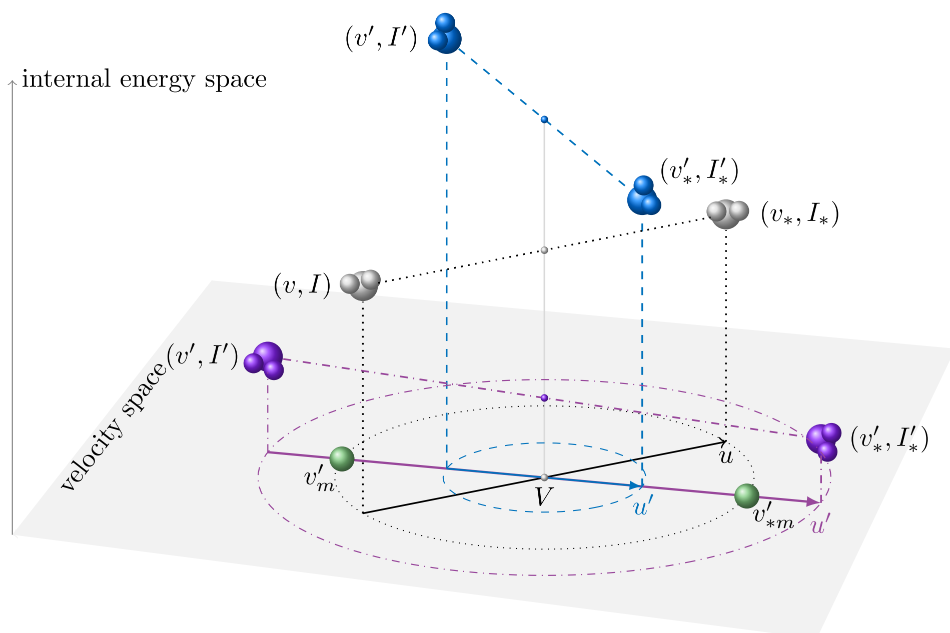

FIGURE 1. Given the two coliding polyatomic molecules with velocities and internal energies (v, I) and (v_*, I_*) rendered in gray tone, we prescribe a scattering direction σ and partition functions r and R. Then states (v', I') and (v'_*, I'_*) can be calculated using the collision transformation (2.6) and (2.8). The violet dash-dotted state (- - - -) corresponds to the choice $R = 0.8$ and the blue dashed one (- - -) to $R = 0.1$. The total microscopic energy E weights a bigger proportion to the internal energy for smaller values of $R \in [0,1]$, while larger values of $R \in [0,1]$ would give a bigger proportion to the kinetic energy of molecules. One can view that the limit $R = 1$ shrinks the total energy E to be just the kinetic energy, while the limit $R = 0$ renders the total energy as the pure internal energy. The green state corresponds to the collisions without internal energy, or classical monatomic elastic collision.

Finally, we introduce the classical scattering direction associated to classical collisional elastic theory, $\sigma \in S^2$, in order to parametrize pre-collisional relative molecular velocity u',

$$u' = |u'|\,\sigma = 2\sqrt{\frac{RE}{m}}\sigma. \tag{2.7}$$

We note that this relation holds for a classical monatomic single species model in the absence of internal energy modes for which $|u'| = |u|$.

This representation introduces the fundamental set of coordinates in center of mass and the *pure* kinetic energy. The last equation together with moment conservation law from (2.2) yields expressions for velocities,

$$v' = V + \sqrt{\frac{RE}{m}}\sigma, \quad v'_* = V - \sqrt{\frac{RE}{m}}\sigma. \tag{2.8}$$

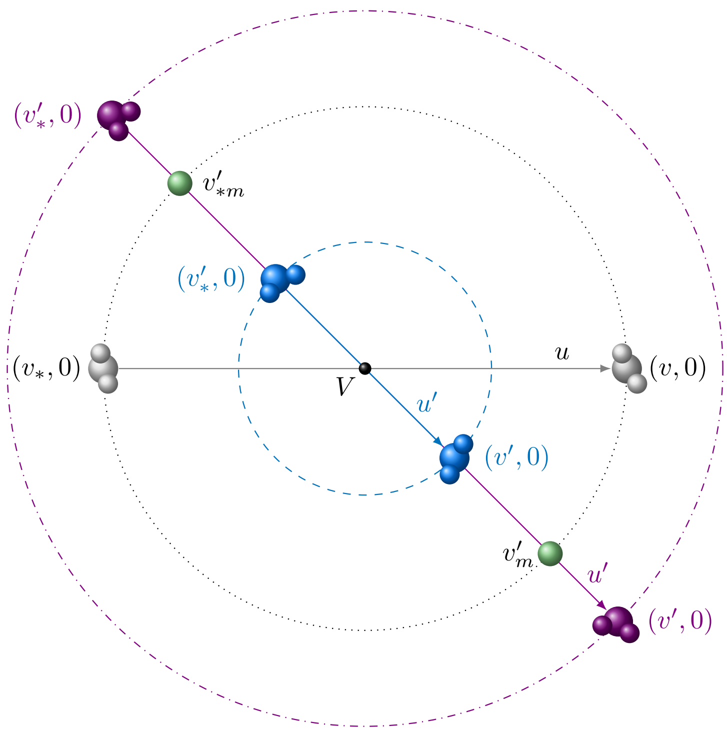

FIGURE 2. A projection of the polyatomic graph of Figure 1 into the plane corresponding to the coplanar direction of $\mathbb{R}^3$ containting the vector u and $\sigma = u'/|u'|$. Note that the scattering direction is an invariant of the polyatomic molecule collision law regardless of partition functions r, R, with $|u'|$ is proportional to R.

2.2. **The collision transformation.** The first step in modelling the collision operator is to study transformation from post- to pre-collisional quantities. In particular, we need to compute Jacobian of this transformation, in order to ensure invariance of the measure appearing in the weak form of collision operator.

Lemma 2.1. *The Jacobian of transformation*

$$T : (v, v_*, I, I_*, r, R, \sigma) \mapsto (v', v'_*, I', I'_*, r', R', \sigma'), \tag{2.9}$$

where velocities v' and v'_* are defined in (2.8), energies I' and I'_* in (2.6), and

$$r' = \frac{I}{I + I_*} = \frac{I}{E - \frac{m}{4}|u|^2}, \quad R' = \frac{m|u|^2}{4E}, \quad \sigma' = \frac{u}{|u|}, \tag{2.10}$$

is given by

$$J_T = \frac{(1-R)R^{1/2}}{(1-R')R'^{1/2}} = \frac{(1-R)|u'|}{(1-R')|u|}. \tag{2.11}$$

The proof of this Lemma can be found in Appendix A.

For later purposes we also prove the following Lemma, which finds a function invariant with respect to the collision process that contains the factor $I^\alpha I_*^\alpha$, crucial for polyatomic modelling. As we will see, α will be related to the degrees of freedom D from macroscopic caloric equation of state (2.1).

We first introduce the following functions, referred by *partition functions* for the kinetic-internal energy split, and internal molecular energy split, respectively, given by

$$\varphi_\alpha(r) := (r(1-r))^\alpha, \qquad \psi_\alpha(R) := (1-R)^{2\alpha}, \tag{2.12}$$

which ensure the expected invariance property for the conservative polyatomic gas model.

Lemma 2.2. *Let functions $\varphi_\alpha(r)$ and $\psi_\alpha(R)$ be from (2.12). The following invariance holds*

$$I^\alpha I_*^\alpha \varphi_\alpha(r)\psi_\alpha(R) = I'^\alpha I_*'^\alpha \varphi_\alpha(r'), \psi_\alpha(R'),$$

for any power $\alpha \in \mathbb{R}$, where the involved quantities are linked via the mapping (2.9).

Proof. We first write

$$r(1-R) = \frac{I'}{E},\ I = r'(1-R')E,\ (1-r)(1-R) = \frac{I'_*}{E},\ I_* = (1-r')(1-R')E,$$

so that

$$r(1-R)\, I\, (1-r)(1-R)\, I_* = I'\, r'(1-R')\, I'_*\, (1-r')(1-R').$$

To conclude the proof, it remains to raise this equation to the power α. □

2.3. **The Boltzmann collision operator for binary polyatomic gases.** In this manuscript, we follow the definition of collision operator from [19]. Then the natural working functional framework for the evolutions of probability densities is the Banach space $L^1(\mathbb{R}^{3+})$ in the variables v and I.

This Boltzmann type collision operator, written in strong bilinear form, is modeled by the non-local operator acting on a pair of probability density measures $(f,g)(v,I)$ defined as follows

$$Q(f,g)(v,I) = \int_{\mathbb{R}^{3+}\times K} \left(f'\, g'_* \left(\frac{I\, I_*}{I'\, I'_*}\right)^\alpha - f g_* \right) \times \mathcal{B}\, \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R)\, R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma \mathrm{d}I_*\, \mathrm{d}v_*, \tag{2.13}$$

$\alpha > -1$, with functions $\varphi_\alpha(r)$, $\psi_\alpha(R)$ from (2.12). The region of integration is $\mathbb{R}^{3+} \times K$, where $\mathbb{R}^{3+}$ denotes the upper half 3-dimensional space of unbounded regions of definition of molecular velocity v and internal energy I, and K a compact manifold embedded in the four dimensional space, that is,

$$\mathbb{R}^{3+} := \mathbb{R}^3 \times [0,\infty), \qquad \text{and} \qquad K := [0,1]^2 \times S^2\,. \tag{2.14}$$

We have used standard abbreviations $f' := f(v', I')$, $g'_* := g(v'_*, I'_*)$, $f := f(v,I)$, $g_* := g(v_*, I_*)$.

The transition probability rates are, in part, quantified by probability measures denoted by

$$\mathcal{B} := \mathcal{B}(v, v_*, I, I_*, R, r, \sigma) \geq 0, \tag{2.15}$$

that are assumed to be invariant with respect to the following two changes of variables

$$(v, v_*, I, I_*, R, r, \sigma) \leftrightarrow (v', v'_*, I', I'_*, R', r', \sigma'), \tag{2.16}$$

$$(v, v_*, I, I_*, R, r, \sigma) \leftrightarrow (v_*, v, I_*, I, R, 1-r, -\sigma), \tag{2.17}$$

which secures microreversibility. That means the transition function $\mathcal{B}$ is invariant for such exchange of state satisfying

$$\mathcal{B}(v, v_*, I, I_*, R, r, \sigma) = \mathcal{B}(v', v'_*, I', I'_*, R', r', \sigma') = \mathcal{B}(v_*, v, I_*, I, R, 1-r, -\sigma). \tag{2.18}$$

Besides these usual assumptions on the transition function $\mathcal{B}$, we will have some additional ones, as stated in the Section 3.1 below.

In addition, it is worthwhile to rewrite strong form (2.13) in the symmetric form,

$$Q(f,g)(v,I) = \int_{\mathbb{R}^{3+}\times K} \left(\frac{f'\, g'_*}{(I'\, I'_*)^{\alpha}} - \frac{f g_*}{(I\, I_*)^{\alpha}} \right) \times \mathcal{B}\, \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R)\, R^{1/2}\, I^\alpha I_*^\alpha \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma \mathrm{d}I_*\, \mathrm{d}v_*, \tag{2.19}$$

obtained by just pulling out the factor $(I\, I_*)^\alpha$ from the gain term.

We explain a role of each term involved in the definition of collision operator (2.13) or equivalently (2.19). First, renormalization of a distribution function f by the factor I^α will allow to obtain a correct macroscopic energy law from (2.1), and we shall see below that there is a link between α and degrees of freedom D introduced in (2.1). Because of the additional factor $(I\, I_*)^\alpha$ we need to incorporate functions $\varphi_\alpha(r)$ and $\psi_\alpha(R)$ in order to have invariance property by virtue of the Lemma 2.2. Finally, term $(1-R)\, R^{1/2}$ is coming from the Jacobian of collision transformation computed in the Lemma 2.1, which ensures good definition of the weak form.

In particular, when the transition probability measure

$$\mathcal{B}(v, v_*, I, I_*, R, r, \sigma)\, \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R)\, R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma$$

is a finite function of the pre and post collisional states (v, v_*, I, I_*), the weak form (2.13) or, equivalently, its symmetrized form (2.19) can be split into the difference of two positive parts, referred as to the gain Q^+ and loss Q^- collisional forms, namely,

$$Q(f,g)(v,I) = Q^+(f,g)(v,I) \; - \; Q^-(f,g)(v,I),$$

with

$$\begin{aligned} Q^+(f,g)(v,I) &= \int_{\mathbb{R}^{3+}\times K} \frac{f'\, g'_*}{(I'\, I'_*)^{\alpha}} \mathcal{B}\, \varphi_\alpha(r) \psi_\alpha(R)\, (1-R) R^{1/2}\, I^\alpha I_*^\alpha \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma\, \mathrm{d}I_*\, \mathrm{d}v_*, \\ Q^-(f,g)(v,I) &= f(v,I)\, \nu[g](v,I), \end{aligned} \tag{2.20}$$

where the loss term is local in $f(v,I)$, proportional to the *collision frequency*, $\nu[g](v,I)$, defined by

$$\nu[g](v,I) := \int_{\mathbb{R}^{3+}\times K} g_*\, \mathcal{B}\, \varphi_\alpha(r) \psi_\alpha(R)\, (1-R) R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma\, \mathrm{d}I_*\, \mathrm{d}v_*. \tag{2.21}$$

Remark 1. *It should be noted that the transition probability rate form* (2.15) *is a more general form than a* differential cross section*, which is the usual expression for classical elastic collisional theory given by just* $|u|\, b(\hat{u}\cdot\sigma)$ *in three dimensions. In this work, the form of* $\mathcal{B}$ *may not only include such* differential cross section *factor, but also needs to include other factors in order to obtain an invariant measure that describes the transition states* (2.16) *and* (2.17) *involving internal energies that* *characterized the modelling of polyatomic gases. Because of this fact, we refer to* $\mathcal{B}\varphi_\alpha(r)\psi_\alpha(R)(1-R)R^{1/2}I^\alpha I_*^\alpha$ *as the transition probability rate form.*

In addition, the roll of this factor in the Boltzmann collisional theory is crucial for the theory of existence and uniqueness, as much as decay rates to equilibrium.

2.4. **Weak form of collision operator.** We first describe an invariant measure which ensures well defined weak form of the collision operator, by means of the following Lemma.

Lemma 2.3. *For any* $\alpha > -1$*, the following measure is invariant with respect to the changes* (2.16)-(2.17)

$$\mathrm{d}A = \mathcal{B}(v, v_*, I, I_*, R, r, \sigma)\, \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R)R^{1/2} I^\alpha I_*^\alpha \,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}I_*\,\mathrm{d}v_*\,\mathrm{d}I\,\mathrm{d}v. \tag{2.22}$$

Proof. The proof for (2.16) easily follows using invariant properties of the transition probability rate associated to $\mathcal{B}$, Lemma 2.2 and Jacobian of the collision transformation from Lemma 2.1. On the other hand, invariance (2.17) clearly holds. □

Lemma 2.4. *For any test function* $\chi(v, I)$ *that makes the following left hand side finite, the collision operator* (2.13) *takes the following weak form*

$$\begin{aligned}
&\int_{\mathbb{R}^{3+}} Q(f,g)(v,I)\,\chi(v,I)\,\mathrm{d}I\,\mathrm{d}v \\
&\quad= \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} f g_* \left(\chi(v',I') + \chi(v'_*,I'_*) - \chi(v,I) - \chi(v_*,I_*)\right) \\
&\qquad\qquad \times \mathcal{B}\,\varphi_\alpha(r)\,\psi_\alpha(R)\,(1-R)R^{1/2}\,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}I_*\,\mathrm{d}v_*\,\mathrm{d}I\,\mathrm{d}v \\
&\quad= \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} \frac{f g_*}{(I I_*)^\alpha} \left(\chi(v',I') + \chi(v'_*,I'_*) - \chi(v,I) - \chi(v_*,I_*)\right)\mathrm{d}A,
\end{aligned} \tag{2.23}$$

with the measure $\mathrm{d}A$ *from* (2.22).

Proof. We integrate the collision operator (2.13) against a suitable test function $\chi(v, I)$ with respect to v and I variables and then perform changes of variables (2.16) and (2.17). Using invariance properties of the measure $\mathrm{d}A$ (2.22) stated in Lemma 2.3, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{3+}} Q(f,g)(v,I)\chi(v,I)\mathrm{d}I\,\mathrm{d}v \\
&\qquad= \int_{(\mathbb{R}^{3+})^2\times K} \frac{f g_*}{I^\alpha I_*^\alpha}\left(\chi(v',I') - \chi(v,I)\right)\mathrm{d}A \\
&\qquad\qquad= \int_{(\mathbb{R}^{3+})^2\times K} \frac{f g_*}{I^\alpha I_*^\alpha}\left(\chi(v'_*,I'_*) - \chi(v_*,I_*)\right)\mathrm{d}A,
\end{aligned}$$

which yields desired estimate (2.23). □

2.5. **The Boltzmann equation.** In order to describe a polyatomic gas, list of arguments of a distribution function is extended by *microscopic internal energy* I, i.e. we take

$$f := f(t, v, I).$$

The evolution of f is governed by the Boltzmann equation

$$\partial_t f = Q(f,f)(v,I), \tag{2.24}$$

where the collision operator is written in (2.13) or equivalently (2.19) and (2.21), namely,

$$Q(f,f)(v,I), = Q^{+}(f,f)(v,I) - Q^{-}(f,f)(v,I) = Q^{+}(f,f)(v,I) - f\,\nu[f](v,I).$$

2.6. **$\mathcal{H}$-theorem.** A natural dissipative quantity that is minimized at the statistical equilibrium is usually given by the concept of entropy in the associated evolution of the probability density function, solutions to the associated Boltzmann equation for binary polyatomic gases. In this case, as in the classical case of elastic monatomic gases, such quantity is is given by the entropy functional, written in the space space homogeneous setting,

$$\mathcal{H}(f)(t) := \int_{\mathbb{R}^{3+}} f(t,v,I)\log(f(t,v,I)I^{-\alpha})\,\mathrm{d}I\mathrm{d}v\,. \tag{2.25}$$

Then, by means of the weak formulation associated to the equation (2.24) defined above, the evolution of the entropy (2.25) is obtained when multiplying both sides of equation (2.24) by $\log(f(t,v,I)I^{-\alpha})$ and integrating with respect to the pair v and I. This results in the entropy production functional $\mathcal{D}(f)(t)$ associated to the Boltzmann collision operator for binary polyatomic gases, that is

$$\mathcal{D}(f)(t) := \int_{\mathbb{R}^{3+}} Q(f,f)(t,v,I)\log(f(t,v,I,)I^{-\alpha})\,\mathrm{d}I\mathrm{d}v\,. \tag{2.26}$$

The following theorem focuses on the properties of this entropy dissipation functional.

Theorem 2.5 (The $\mathcal{H}$-theorem)**.** *Let the transition function $\mathcal{B}$ be positive function almost everywhere, and let $f \geq 0$ such that the collision operator $Q(f,f)$ and entropy production $\mathcal{D}(f)$ are well defined. Then the following properties hold*

i. Entropy production is non-positive, that is

$$\mathcal{D}(f) \leq 0.$$

ii. The three following properties are equivalent

(1) $\mathcal{D}(f) = 0$,

(2) $Q(f,f) = 0$ for all $(v,I) \in \mathbb{R}^{3+}$,

(3) There exists $n \geq 0, U \in \mathbb{R}^3$, and $T > 0$, such that the unit mass renormalized Maxwellian equilibrium for polyatomic gases is

$$M_{eq}(v,I) = \frac{n}{Z(T)}\left(\frac{m}{2\pi k_B T}\right)^{3/2} I^{\alpha}\ e^{-\frac{1}{kT}\left(\frac{m}{2}|v-U|^2+I\right)}, \tag{2.27}$$

where $Z(T)$ is a partition (normalization) function

$$Z(T) = \int_{[0,\infty)} I^{\alpha} e^{-\frac{I}{kT}}\,\mathrm{d}I = (kT)^{\alpha+1}\Gamma(\alpha+1),$$

with Γ as gamma function.

The proof can be found in [19].

2.7. **Functional space.** The choice of a functional space is crucial for the framework to construct solutions with a good physical and mathematical meaning. Given the description of the dynamics of the polyatomic gas Boltzmann model for the evolution of a non-negative probability density measure describing the binary interaction of particles exchanging the states of their molecular velocities and internal energies, a natural functional normed Banach space is the one of integrable functions with polynomial weights, whose norms describe the observables or moments associated to such measures.

The natural polynomial weight to generate a weighted Banach space containing the same spaces with a lower polynomial degree weight, is given by

$$\langle v, I \rangle = \sqrt{1 + \frac{1}{2} |v|^2 + \frac{I}{m}}, \tag{2.28}$$

all associated to the velocity $v \in \mathbb{R}^3$ and microscopic internal energy $I \in [0, \infty)$, which is independent of mass units. We refer to this weight function as the Lebesgue bracket and will look for a unique solution to the Cauchy problem associated to the Boltzmann equation (2.24) in the Banach space weighted by powers of this Lebesgue weight form. More precisely, we define

$$L_k^1 = \left\{ f \text{ measurable} : \int_{\mathbb{R}^{3+}} |f(v, I)| \langle v, I \rangle^{2k} \, \mathrm{d}I \mathrm{d}v < \infty, \ k \geq 0 \right\}, \tag{2.29}$$

with the range of integration $\mathbb{R}^{3+} = \mathbb{R}^3 \times [0, \infty)$ from (2.14). Its associated norm is

$$\|f\|_{L_k^1} = \int_{\mathbb{R}^{3+}} |f(v, I)| \langle v, I \rangle^{2k} \, \mathrm{d}I \mathrm{d}v. \tag{2.30}$$

We recall the monotonicity property for norms weighted with (2.28),

$$\|f\|_{L_{k_1}^1} \leq \|f\|_{L_{k_2}^1} \text{ whenever } 0 \leq k_1 \leq k_2. \tag{2.31}$$

We also define the associated semi-norm, with the classical notation,

$$\dot{L}_k^1 = \left\{ f \text{ measurable} : \int_{\mathbb{R}^{3+}} |f(v, I)| \left(\frac{1}{2} |v|^2 + \frac{I}{m} \right)^k \mathrm{d}I \mathrm{d}v < \infty, \ k \geq 0 \right\}. \tag{2.32}$$

Yet, when we refer to the distribution function, without loss of generalization, the norm (2.30) is called the polynomial moment, as the following Definition 2.6 precises.

Definition 2.6. *The Lebesgue polynomial moment of order $k \geq 0$ associated to any integrable function $g(t, v, I)$ is defined with*

$$\mathfrak{m}_k[g](t) = \int_{\mathbb{R}^{3+}} g(t, v, I) \langle v, I \rangle^{2k} \, \mathrm{d}I \mathrm{d}v$$

with the Lebesgue bracket weight from (2.28).

Note that the Lebesgue polynomial moment of any order of any non-negative probability density function $f(v, I)$ coincides with $\|f\|_{L_k^1}$, and when $k = 0$ is the macroscopic number density associated to the probability density $f(t, v, I)$, hence $\mathfrak{m}_0[f](t) = \|f\|_{L_0^1}(t) = \|f\|_{\dot{L}_0^1}(t)$. However, the semi-norm $\|f\|_{\dot{L}_k^1}(t) < \mathfrak{m}_k[f](t)$, for any $k > 0$.

The particular case $k = 1$ is associated to the classical total kinetic and internal energy of a non-negative $f(t, v, I)$ is smaller that the one generated by the Lebesgue polynomial moment, that is $\|f\|_{\dot{L}_1^1}(t) < \mathfrak{m}_1[f](t)$. It is important to notice that if the classical energy of a non-negative probability density measure is positive, then such $f(t,v,I)$ can not be a an absolutely singular measure.

In this manuscript, we also study exponentially weighted L^1-norms, by means of exponential moments defined as follows.

Definition 2.7. *Exponential moment for an integrable function g, of rate $\beta > 0$ and order $2s$, $0 < s \le 1$, is defined by*

$$\mathcal{E}_s[g](\beta,t) := \int_{\mathbb{R}^{3+}} g(t,v,I)\, e^{\beta \langle v,I\rangle^{2s}}\, \mathrm{d}I\, \mathrm{d}v. \tag{2.33}$$

Note that, for any non-negative probability density function $f(v,I)$, their associated exponential moments coincide with their exponential weighted L^1-norm.

In the upcoming Section, we provide a physical interpretation to some polynomial moments.

2.8. **Macroscopic observables.** We first note that for certain test functions weak form (2.23) annihilates. This is encoded in the collision conservation laws (2.2). Namely, the following Lemma holds.

Lemma 2.8. *The collision invariants for the collision operator* (2.13)*, i.e. functions $\chi(v,I)$ for which the weak form* (2.23) *annihilates*

$$\int_{\mathbb{R}^{3+}} Q(f,g)(v,I)\, \chi(v,I)\, \mathrm{d}I\, \mathrm{d}v = 0,$$

are linear combination of the following functions

$$\chi_1(v,I) = m, \qquad \chi_k(v,I) = m\, v_k,\ k = 1,2,3, \qquad \chi_5(v,I) = \frac{m}{2}\,|v|^2 + I. \tag{2.34}$$

Macroscopic observables are defined as moments of the distribution function f against functions of molecular variables, velocity $v \in \mathbb{R}^3$ and internal energy $I \in [0,\infty)$. For example, when test functions are the collision invariants (2.34) then we define mass density ρ, momentum density ρU and total energy density $\frac{\rho}{2}\,|U|^2 + \rho\, e$ of a polyatomic gas as the following moments

$$\rho = \int_{\mathbb{R}^{3+}} m f\, \mathrm{d}I\, \mathrm{d}v, \quad \rho\, U = \int_{\mathbb{R}^{3+}} m v f\, \mathrm{d}I\, \mathrm{d}v, \quad \frac{\rho}{2}\,|U|^2 + \rho\, e = \int_{\mathbb{R}^{3+}} \left(\frac{m}{2}\,|v|^2 + I\right) f\, \mathrm{d}I\, \mathrm{d}v.$$

We highlight the relation between collision invariants and the Lebesgue weight (2.28),

$$\langle v, I\rangle^2 = \frac{1}{m}\left(\chi_1 + \chi_5\right).$$

Therefore, polynomial moment of the order $k = 0$ multiplied by mass, $m\, \mathfrak{m}_0(t)$ is interpreted as gas mass density, while for $k = 1$, the moment $m\, \mathfrak{m}_1(t)$ is the sum of mass density plus total energy density of the gas,

$$m\, \mathfrak{m}_0 = \rho, \qquad m\, \mathfrak{m}_2 = \rho + \frac{\rho}{2}\,|U|^2 + \rho\, e.$$

We can finally find connection with the caloric equation of state for a polytropic gas (2.1). By introducing the peculiar velocity $c = v - U$, we can define the internal energy density,

$$\rho\, e = \int_{\mathbb{R}^{3+}} \left(\frac{m}{2}\,|c|^2 + I\right) f\, \mathrm{d}I\, \mathrm{d}v.$$

In the local equilibrium state, when distribution function has a shape of local Maxwellian (2.27), internal energy takes the form

$$\rho\, e = \left(\alpha + \frac{5}{2}\right) n\, k\, T. \tag{2.35}$$

Now relation to caloric equation of state for a polytropic gas (2.1) becomes evident. Then we can connect D from (2.1) and α from (2.35), that also appears in the definition of collision operator (2.13),

$$\alpha = \frac{D-5}{2}.$$

Since for polyatomic gases $D > 3$, we obtain the overall condition $\alpha > -1$. We recall that $\alpha = -1$ corresponds to a monatomic gas ($D = 3$), when we obtain the classical relation $\rho\, e = (3/2) nkT$.

The following Table 1 shows a relation between different modes of a polyatomic gas molecule with its number of degrees of freedom D as much as the corresponding value of the parameter α.

Table 1. Degrees of freedom D and the value of α that correspond to different modes (combinations of translation/rotation/vibration), where $N \geq 2$ stands for the number of atoms in a polyatomic gas molecule.

	Translation and rotation		Translation, rotation and vibration
	Linear molecule	Non-linear molecule	
D	5	6	$3N$
α	0	$\frac{1}{2}$	$\frac{1}{2}(3N-5)$

3. Sufficient properties for existence and uniqueness theory

In this Section we describe sufficient tools needed to build the Existence and Uniqueness theory to be presented in Section 7. Namely, at the first place we choose an appropriate transition function $\mathcal{B}$ that corresponds to an extended Grad assumption. Namely, we assume the form of hard potentials with a rate $\gamma \in (0, 2]$ for both relative speed and molecular internal energies combined additively,

$$|v - v_*|^\gamma + \left(\frac{I + I_*}{m}\right)^{\gamma/2}, \qquad \gamma \in (0, 2],$$

with a control, from above and below, with respect to the parameters R, r, σ belonging to the compact manifold K. Moreover, we find relevant physical examples, namely the three models for $\mathcal{B}$, that satisfy the imposed assumptions on $\mathcal{B}$. With this form of the transition function, we prove essential estimates, its upper and lower bounds in molecular velocities and internal energies, that allow to conclude the coerciveness estimate for the loss part of the collision operator in Section 4 and the decay of $k-$Lebesgue moment of its gain part with respect to k in Section 5. Then in Section 6 we are going to have a priori estimates for any solution of the

Boltzmann equation for polyatomic gases in L^1_k for $k > k_*$ with k_* determined by the bounds for transition function $\mathcal{B}$ and the conditions of fundamental Lemmas of Sections 4 and 5. Finally, we define an *invariant region* $\Omega \subset L^1_1$ for the Boltzmann equation, in which the collision operator $Q : \Omega \to L^1_1$ satisfies (i) Hölder continuity, (ii) Sub-tangent and (iii) one-sided Lipschitz conditions. These are sufficient conditions to obtain existence and uniqueness of a global in time solution with a regularity to be described in Section 7.

3.1. **Transition function $\mathcal{B}$.** One of the essential ingredients for building existence and uniqueness theory is an assumption on transition function $\mathcal{B}$, that quantifies the collision frequency through scattering mechanisms and partition functions as a function of the total molecular energy (2.4). In this manuscript, we want to keep the transition function $\mathcal{B}$ as general as possible in order to allow our kinetic model to cover a broad class of physical interpretations. To that end, apart from its positivity and micro-reversibility requirements stated in (2.18), we assume the following minimal mathematical requirements to ensure existence and uniqueness properties associated to the initial value problem for the Boltzmann equation for binary interaction of polyatomic gases as defined in (2.24).

Assumption 3.1 (The form of the transition function $\mathcal{B}$)**.** *Let $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}(|u|, I, I_*)$ be defined as*

$$\tilde{\mathcal{B}}(|u|, I, I_*) := |u|^\gamma + \left(\frac{I + I_*}{m}\right)^{\gamma/2}, \quad u := v - v_*, \quad \gamma \in (0, 2]. \tag{3.1}$$

We assume that the transition function $\mathcal{B} := \mathcal{B}(v, v_*, I, I_*, r, R, \sigma)$ satisfies the following extended Grad assumption for collision kernels,

$$d^{lb}_\gamma(r)\, e^{lb}_\gamma(R)\, b(\hat{u}\cdot\sigma)\, \tilde{\mathcal{B}}(|u|, I, I_*) \le \mathcal{B} \le d^{ub}_\gamma(r)\, e^{ub}_\gamma(R)\, b(\hat{u}\cdot\sigma)\, \tilde{\mathcal{B}}(|u|, I, I_*), \tag{3.2}$$

for every $v, v_* \in \mathbb{R}^3$, $I, I_* \in [0,\infty)$, $r, R \in [0,1]$, $\sigma \in S^2$, with $\hat{u} = u/|u|$, where functions $b(\hat{u}\cdot\sigma)$, $d^{ub}_\gamma(r)$, $d^{lb}_\gamma(r)$, $e^{ub}_\gamma(R)$, and $e^{lb}_\gamma(R)$ satisfy the following integrability conditions,

(1) the angular function $b(\hat{u}\cdot\sigma)$ is integrable with respect to the measure $\mathrm{d}\sigma$,

$$b(\hat{u}\cdot\sigma) \in L^1(S^2; \mathrm{d}\sigma), \tag{3.3}$$

(2) functions $d^{ub}_\gamma(r)$ and $d^{lb}_\gamma(r)$ are integrable with respect to the measure $\varphi_\alpha(r)\mathrm{d}r$, with $\varphi_\alpha(r)$ from (2.12), more precisely

$$d^{ub}_\gamma(r)\varphi_\alpha(r), \quad d^{lb}_\gamma(r)\varphi_\alpha(r) \in L^1([0,1]; \mathrm{d}r), \tag{3.4}$$

and additionally

$$d^{ub}_\gamma(r) = d^{ub}_\gamma(1-r), \quad d^{lb}_\gamma(r) = d^{lb}_\gamma(1-r),$$

which ensures the second microreversibility property (2.18),

(3) functions $e^{ub}_\gamma(R)$ and $e^{lb}_\gamma(R)$ are integrable with respect to the measure $\psi_\alpha(R)(1-R)R^{1/2}\mathrm{d}R$, where ψ_α is introduced in (2.12), namely

$$e^{ub}_\gamma(R)\psi_\alpha(R)(1-R)R^{1/2}, \quad e^{lb}_\gamma(R)\psi_\alpha(R)(1-R)R^{1/2} \in L^1([0,1]; \mathrm{d}R). \tag{3.5}$$

Remark 2. *We observe that conditions (3.4) and (3.5) involve the weighted averages of the factors $d^{lb}_\gamma(r)$ and $d^{ub}_\gamma(r)$ product to the partition function for the* *molecular energy $\varphi_\alpha(r)$; as well as $e^{lb}_\gamma(R)$ and $e^{ub}_\gamma(R)$ product to the partition function for the split of kinetic and internal energy $\psi_\alpha(R)$; respectively. We introduce the short hand notation to these averages by defining the following constants,*

$$\begin{aligned} c^{lb}_{\gamma,\alpha} &:= \int_0^1 d^{lb}_\gamma(r)\varphi_\alpha(r)\mathrm{d}r, \quad C^{lb}_{\gamma,\alpha} := \int_0^1 e^{lb}_\gamma(R)\psi_\alpha(R)(1-R)R^{1/2}\mathrm{d}R, \\ c^{ub}_{\gamma,\alpha} &:= \int_0^1 d^{ub}_\gamma(r)\varphi_\alpha(r)\mathrm{d}r, \quad C^{ub}_{\gamma,\alpha} := \int_0^1 e^{ub}_\gamma(R)\psi_\alpha(R)(1-R)R^{1/2}\mathrm{d}R. \end{aligned} \tag{3.6}$$

Moreover,

$$\kappa^{lb} = \|b\|_{L^1(\mathrm{d}\sigma)}\, c^{lb}_{\gamma,\alpha} C^{lb}_{\gamma,\alpha}, \qquad \kappa^{ub} = \|b\|_{L^1(\mathrm{d}\sigma)}\, c^{ub}_{\gamma,\alpha} C^{ub}_{\gamma,\alpha}. \tag{3.7}$$

Note that assumption 3.1 stresses the transition probability associated to the differential cross section is an extended Grad assumption for hard potentials with a dependance to the internal energy exchange which characterizes polyatomic collisional gas models. As such, the assumption of transition function $\mathcal{B}$ will be shown to be achievable for at least three choices of models of $\mathcal{B}(v, v_*, I, I_*, r, R, \sigma)$ satisfying conditions (3.1-3.5), that are sufficient to rigorously prove the existence of finite upper and strictly positive lower bounds sufficient for solving the Cauchy problem associated to a natural Banach space defined by (2.29) with a natural norm characterized by the Lebesgue weight function (2.28) and norm by (2.30). Such upper and lower estimates must allow to control infinity system of ordinary Differential Inequalities (ODI) associated to solutions to the initial value problem (2.24) uniformly in time. This strategy is rather elaborated and shall be presented in several steps.

3.2. **Models for transition function $\mathcal{B}$.** Next we propose three different choices of the transition function $\mathcal{B}$ and prove that each choice satisfies all conditions on Assumption 3.1. We also focus our attention to the multiplicative factors depending on r and R, to the term $b(\hat{u}\cdot\sigma)\,\tilde{\mathcal{B}}$, in the upper and lower bounds of $\mathcal{B}$ from (3.2).

For each choice of the extended Grad decomposition that follows we specify functions $d^{ub}_\gamma(r)$, $d^{lb}_\gamma(r)$, $e^{ub}_\gamma(R)$ and $e^{lb}_\gamma(R)$ that, not only fulfill the integrability conditions, but also provide explicit expressions for controlling constants, from above and below (3.6). These integrals are used in the Section 4. Here we calculate the constants (3.6) only in the first example.

3.2.1. *Model 1 (The total energy).* We first consider the total energy in the relative velocity-center of mass velocity framework, that is

$$\mathcal{B}(v, v_*, I, I_*, r, R, \sigma) = b(\hat{u}\cdot\sigma)\left(\frac{m}{4}\left|v - v_*\right|^2 + I + I_*\right)^{\gamma/2}, \qquad \gamma \in (0, 2]. \tag{3.8}$$

Then $\mathcal{B}$ is of the form (3.2) with

$$d^{lb}_\gamma(r) = d^{ub}_\gamma(r) = 1, \quad e^{lb}_\gamma(R) = m^{\gamma/2} 2^{-(\gamma/2+1)}, \quad e^{ub}_\gamma(R) = m^{\gamma/2},$$

as proven is in the Appendix B.1.1, B.2.1. Therefore, using (2.12), performing the integration to calculate the constants (3.6) by means of the Gamma function, for this Model 1 it follows

$$c^{lb}_{\gamma,\alpha} = c^{ub}_{\gamma,\alpha} = \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)}, \quad C^{lb}_{\gamma,\alpha} = m^{\gamma/2}\, 2^{-(\gamma/2+1)} \frac{\sqrt{\pi}\Gamma(2\alpha+2)}{2\Gamma\left(2\alpha+\frac{7}{2}\right)},$$

$$C^{ub}_{\gamma,\alpha} = m^{\gamma/2} \frac{\sqrt{\pi}\Gamma(2\alpha+2)}{2\Gamma\left(2\alpha+\frac{7}{2}\right)}, \tag{3.9}$$

for $\alpha > -1$, where Γ represents the Gamma function.

3.2.2. *Model 2 (kinetic and microscopic internal energy detached).* In this model, we split kinetic and microscopic internal energy for the colliding pair of particles, by using parameter $R \in [0, 1]$,

$$\mathcal{B}(v, v_*, I, I_*, r, R, \sigma) = b(\hat{u}\cdot\sigma) \left(R^{\gamma/2} |v - v_*|^\gamma + (1 - R)^{\gamma/2} \left(\frac{I + I_*}{m} \right)^{\gamma/2} \right), \tag{3.10}$$

for $\gamma \in (0, 2]$. As proven in the appendix sections B.1.2 and B.2.2, this model satisfies the form (3.2), with

$$d_\gamma^{lb}(r) = d_\gamma^{ub}(r) = 1, \quad e_\gamma^{lb}(R) = \min\{R, (1-R)\}^{\gamma/2}, \quad e_\gamma^{ub}(R) = \max\{R, (1-R)\}^{\gamma/2}. \tag{3.11}$$

Another possible choice is

$$d_\gamma^{lb}(r) = d_\gamma^{ub}(r) = 1, \quad e_\gamma^{lb}(R) = R^{\gamma/2}(1 - R)^{\gamma/2}, \quad e_\gamma^{ub}(R) = 1.$$

3.2.3. *Model 3 (kinetic and particle's microscopic internal energies detached).* In this model we separate kinetic and microscopic internal energy with the parameter $R \in [0, 1]$. Furthermore, internal energy is divided among colliding particles with the help of parameter $r \in [0, 1]$. More precisely, we consider,

$$\begin{aligned} \mathcal{B}(v, v_*, I, I_*, r, R) = b(\hat{u} \cdot \sigma) \Big(& R^{\gamma/2} |v - v_*|^\gamma \\ & + \left(r(1 - R)\frac{I}{m} \right)^{\gamma/2} + \left((1 - r)(1 - R)\frac{I_*}{m} \right)^{\gamma/2} \Big), \end{aligned} \tag{3.12}$$

for $\gamma \in (0, 2]$. Then the form (3.2) is satisfied with

$$\begin{aligned} & d_\gamma^{lb}(r) = \min\{r, (1 - r)\}^{\gamma/2}, \quad d_\gamma^{ub}(r) = 1, \\ & \quad e_\gamma^{ub}(R) = 2^{1-\gamma/2} \max\{R, (1 - R)\}^{\gamma/2}, \quad e_\gamma^{lb}(R) = \min\{R, (1 - R)\}^{\gamma/2}, \end{aligned} \tag{3.13}$$

or with

$$d_\gamma^{lb}(r) = r^{\gamma/2}(1 - r)^{\gamma/2}, \quad d_\gamma^{ub}(r) = 1, \quad e_\gamma^{ub}(R) = 2^{1-\gamma/2}, \quad e_\gamma^{lb}(R) = R^{\gamma/2}(1 - R)^\gamma,$$

as shown in B.1.3 and B.2.3.

This Model 3 for the transition function $\mathcal{B}$ is of particular importance for establishing macroscopic moments models starting from the Boltzmann equation. Namely, in [25] it is shown that the six moments model with this transition function provides a source term which satisfies the macroscopic residual inequality on the whole range of six moments model validity, that provides the total agreement with the extended thermodynamics theory of six moments, as one of the rare systems admitting a non-linear or *far from equilibrium* closure of the governing equations using the entropy principle. Moreover, for this model 3 of the transition function $\mathcal{B}$, the macroscopic fourteen moments model achieves the matching with experimental data as well. More precisely, [25] shows that the model 3 yields transport coefficients (shear and bulk viscosities and heat conductivity) for which both experimentally measured dependence of the shear viscosity upon temperature is recovered and the Prantdtl number coincides with its theoretical value at a satisfactory level. This remarkable success relies on the new *additive* form of the transition function that we propose in (3.1), instead of the multiplicative one used so far in the literature.

4. Coerciveness estimates

In this and next Sections we prove fundamental lemmas, that should be used sequentially, as they are presented.

All of them are motivated by the search of a proof showing that k-th polynomial moment of the solution will satisfy an Ordinary Differential Inequality (ODI) in the Banach Space $L^1_k(\mathbb{R}^{3+})$ with a negative super-linear term, that is for any $\gamma \in (0, 2]$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{L^1_k}(t) &= \int_{\mathbb{R}^{3+}} Q(f,f)(v,I) \langle v, I\rangle^{2k} \mathrm{d}I \mathrm{d}v \\ &= \mathfrak{m}_k \left[Q(f,f)\right] \leq -A_{k_*} \|f\|_{L^1_k}^{1+\frac{\gamma/2}{k}}(t) + B_k \|f\|_{L^1_k}(t), \end{aligned} \tag{4.1}$$

where the sequence of strictly positive constants $\{A_k\}_k$ is strictly decreasing and is such that $0 < A_k < A_{k_*}$ for all $k > k_*$, for large enough k_*, to be determined later. This A_{k_*} depends on the transition probability and partition functions, on γ and the initial mass and is independent of $k > k_*$. Our aim is to gather sufficient a priori estimates that will show the existence of global in time solutions to an initial value problem associated to to (4.1).

This Section 4 focuses on the fundamental question of obtaining coerciveness in the natural Banach space norm for the collisional form associated to polyatomic gas models. This result can be viewed as the analogue to a coercive constant in the classical theory for diffusion type equations in continuum mechanics where the control of the energies or suitable Banach norms is done in Sobolev spaces, while in the framework of statistical mechanics the suitable norms are non-reflexive Banach spaces, as in this case L^1_k.

Coercive estimate is fundamental for the existence of global solutions for the Boltzmann flow. This property provides the following functional inequality that controls the convolution of any function $f \in L^1_{1+}$ with a potential function of the form (3.1) from below by the corresponding Lebesgue bracket (2.28). We are inspired by the work [5] for the classical Boltzmann equation, as such inequality is sufficient to control from below the collision frequency associated (2.21), and the loss collision operator associated to the polyatomic gas model, under assumptions on the transition probability states as described in Section 3. In addition, the coercive estimates we present here are fundamental to obtain in $W^{m,2}$ Sobolev (Hilbert) spaces as shown in [6].

Lemma 4.1 (Lower bound)**.** *Let $\gamma \in [0, 2]$. For some function $0 \leq \{f(t)\}_{t\geq 0} \subset L^1_1$ we assume that it satisfies*

$$\mathcal{M}_l \leq \int_{\mathbb{R}^{3+}} f(t,v,I) \mathrm{d}I \mathrm{d}v \leq \mathcal{M}_u, \quad \mathcal{E}_l \leq \int_{\mathbb{R}^{3+}} f(t,v,I) \left(\frac{1}{2}|v|^2 + \frac{I}{m}\right) \mathrm{d}I \mathrm{d}v \leq \mathcal{E}_u, \tag{4.2}$$

for some positive constants $\mathcal{M}_l$, $\mathcal{M}_u$, $\mathcal{E}_l$ and $\mathcal{E}_u$, and

$$\int_{\mathbb{R}^{3+}} f(t,v,I)\, v \,\mathrm{d}I \,\mathrm{d}v = 0. \tag{4.3}$$

Assume also boundedness of the moment

$$\int_{\mathbb{R}^{3+}} f(t,v,I)\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\frac{2+\delta}{2}} \mathrm{d}I\,\mathrm{d}v \le \Delta, \quad \delta>0. \tag{4.4}$$

Then there exists a constant $c_{lb}>0$ which depends on constants $\mathcal{M}_l$, $\mathcal{M}_u$, $\mathcal{E}_l$, $\mathcal{E}_u$ Δ, δ and γ from the assumptions (4.2)-(4.4) above such that the convolutional form of any $f \in L^1_1$ satisfying conditions (4.2) and (4.4) with a power law function of order γ from (3.1), is controlled from below by the local Lebesgue weight of order γ, namely

$$f*\left(|v|^\gamma+\left(\frac{I}{m}\right)^{\frac{\gamma}{2}}\right) := \int_{\mathbb{R}^{3+}} f(t,w,J)\left(|v-w|^\gamma+\left(\frac{I+J}{m}\right)^{\frac{\gamma}{2}}\right)\mathrm{d}J\mathrm{d}w \ge c_{lb}\,\langle v,I\rangle^\gamma, \tag{4.5}$$

It takes the form

$$c_{lb}=\frac{\min\{\mathcal{M}_l,\mathcal{E}_l\}}{8}\left(2^{4+\delta}\left(\frac{\max\{\mathcal{M}_u,\Delta\}}{\mathcal{E}_l}\right)\left(1+\left(\frac{2(\mathcal{M}_u+\mathcal{E}_u)}{\mathcal{M}_l\left(2^{\gamma-2}-\frac{1}{8}\right)}\right)^{\frac{2}{\gamma}}\right)^{\frac{2+\delta}{2}}\right)^{\frac{-2+\gamma}{\delta}} \times\left(1+\left(\frac{2(\mathcal{M}_u+\mathcal{E}_u)}{\mathcal{M}_l\left(2^{\gamma-2}-\frac{1}{8}\right)}\right)^{\frac{2}{\gamma}}\right)^{-\gamma/2}, \tag{4.6}$$

for $\gamma \in (0,2]$, and if $\gamma=0$ then $c_{lb}=2$.

Remark 3. *It is also important to notice that both constants $\mathcal{M}_l \le \|f\|_{L^1_0}$ and $\mathcal{E}_l \le \|f\|_{\dot{L}^1_1}$ coincide with the semi-norms defined in (2.32), for a non-negative distribution density $f(t,v,I)$, from (4.2) and (4.4) can not be both zero, otherwise the lower bound c_{lb} from (4.6) would vanish and the coercive estimate (4.32) would fail, and consequently the existence theory of global in time solutions, we are about to develop, would also fail. It is also clear that singular measures can not be solutions in this Banach topology, as both $\mathcal{M}_l$ and $\mathcal{E}_l$ cannot be zero.*

Proof of Lemma 4.1. The point of this proof is to bound from below a convolution of a $f \in L^1_1$, satisfying conditions (4.2) and (4.4) integrated with the weight defined in (3.1), by a constant depending on parameters from those conditions and on γ multiplied by the Lebesgue weight

$$\langle v,I\rangle^\gamma=\left(1+\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}, \tag{4.7}$$

where the integral's weight function is given by

$$\tilde{\mathcal{B}}(|v-w|,I,J)=|v-w|^\gamma+\left(\frac{I+J}{m}\right)^{\frac{\gamma}{2}} \in \mathbb{R}. \tag{4.8}$$

This task entices to study the control from below and above of (4.8) which will generate, after performing the integration with $f(w,J)$, the desired lower bound involving (4.7). Such calculation requires a major extension of the proof proposed in [5] for the classical Boltzmann in $\mathbb{R}^d$, as for the current model in $\mathbb{R}^{3+}$ needs to be adjusted to the topological distances in half spaces.

We work out the details of this proof in two parts after we introduce the notion of upper semi-spheres suitable for the estimates.

We start by setting the following set in $\mathbb{R}^{3+} = \mathbb{R}^3 \times [0,\infty)$. More precisely, we introduce the semi-sphere in the upper half space $\mathbb{R}^{3+} := \mathbb{R}^3 \times [0,\infty)$ by

$$B_\rho(v,I) := \left\{ (w,J) \in \mathbb{R}^{3+} : \sqrt{\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}} \leq \rho \right\}, \tag{4.9}$$

for any $(v,I) \in \mathbb{R}^{3+}$.

Moreover, the following useful pointwise estimates are sufficient to connect the weight convolution factor in (4.8) to the Lebesgue weights (4.7). On one hand, by Minkowski-type inequalities and the concavity of $\gamma/2$-power functions, $0 < \gamma \leq 2$, we obtain the following crucial estimate from below for the integrand (4.8),

$$\begin{aligned}
\tilde{\mathcal{B}}(|v-w|, I, J) &\geq \min\left\{1, 2^{1-\gamma}\right\} |v|^\gamma - |w|^\gamma + 2^{\frac{\gamma}{2}-1}\left(\left(\frac{I}{m}\right)^{\gamma/2} + \left(\frac{J}{m}\right)^{\gamma/2} \right) \\
&\geq 2^{\gamma/2-1}\left(\left(\frac{1}{2}|v|^2\right)^{\gamma/2} + \left(\frac{I}{m}\right)^{\gamma/2} \right) - 2^{\gamma/2}\left(\left(\frac{1}{2}|w|^2\right)^{\gamma/2} + \left(\frac{J}{m}\right)^{\gamma/2} \right) \\
&\geq 2^{\gamma-2}\left(\frac{1}{2}|v|^2 + \frac{I}{m}\right)^{\gamma/2} - 2\left(\frac{1}{2}|w|^2 + \frac{J}{m}\right)^{\gamma/2}.
\end{aligned} \tag{4.10}$$

In addition, clearly, the concavity of the $\gamma/2$ root function, $0 < \gamma \leq 2$, implies for the integrand factor (4.8),

$$\tilde{\mathcal{B}}(|v-w|, I, J) \geq \left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^{\gamma/2}. \tag{4.11}$$

On the other hand, the base of the power expression from the right hand side in (4.11) can be estimated from above by invoking the local Jensen's inequality for convex k-power functions, for any real valued number $k \geq 1$,

$$\begin{aligned}
\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^k &\leq 2^k\left(\frac{1}{2}\left(|v|^2 + |w|^2\right) + \frac{(I+J)}{m}\right)^k \\
&\leq 2^{2k-1}\left(\left(\frac{1}{2}|v|^2 + \frac{I}{m}\right)^k + \left(\frac{1}{2}|w|^2 + \frac{J}{m}\right)^k \right).
\end{aligned} \tag{4.12}$$

These constructions enable us to first obtain preliminary estimates over the semi-sphere $B_S(v,I)$ for a parameter S that depends on the data parameters $\mathcal{E}_l$, $\mathcal{M}_u$, Δ, δ and the radius ρ_* of a semi-sphere $B_{\rho_*}(0,0)$. Such estimates will allow us to control the integral over $\mathbb{R}^{3+}$, by splitting the integration domain into the semi-sphere $B_{\rho_*}(0,0)$ and its complement for a good choice of radius ρ_* depending on the data, namely $\rho_* = \rho_*(\mathcal{M}_l, \mathcal{M}_u, \mathcal{E}_u, \gamma)$. Estimates for the integration over the complement $B^c_{\rho_*}(0,0)$ will fix the value of ρ_* that ensures the lower bound (4.5) with an explicitly calculated constant c_{lb} as a function of the data.

Since the case $\gamma = 0$ is trivial, we just consider $\gamma \in (0,2]$. Setting the shorter notation for our convolution notation in (4.5) by $\Lambda_\gamma(v,I)$, and using the inequality

(4.11) for the integrands, we obtain

$$\begin{aligned}\Lambda_\gamma(v,I) &:= \int_{\mathbb{R}^{3+}} f(t,w,J)\left(|v-w|^\gamma + \left(\frac{I+J}{m}\right)^{\gamma/2}\right) \mathrm{d}J\,\mathrm{d}w \\ &> \int_{\mathbb{R}^{3+}} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^{\gamma/2} \mathrm{d}J\,\mathrm{d}w =: \bar{\Lambda}_\gamma(v,I).\end{aligned} \tag{4.13}$$

Then we make of use the semi-sphere defined in (4.9) with an arbitrary radius $S>0$ to connect the integrals $\bar{\Lambda}_\gamma(v,I)$ and $\bar{\Lambda}_2(v,I)$ which is obtained by setting $\gamma=2$. Namely, by splitting the integration domain $\mathbb{R}^{3+} = B_S(v,I)\cup B_S^c(v,I)$ and since the integral $\bar{\Lambda}_\gamma(v,I)$ with respect to the whole space $\mathbb{R}^{3+}$ is bigger than the same integral with respect to its one part $B_S(v,I)$, and noting that

$$\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^{\gamma/2-1} \geq S^{\gamma-2} \quad \text{on the semi-sphere } B_S(v,I),$$

we obtain

$$\begin{aligned}\bar{\Lambda}_\gamma(v,I) &\geq S^{\gamma-2}\int_{B_S(v,I)} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)\mathrm{d}J\,\mathrm{d}w, \\ &\geq S^{\gamma-2}\left(\bar{\Lambda}_2(v,I) - \int_{B_S^c(v,I)} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)\mathrm{d}J\,\mathrm{d}w\right).\end{aligned} \tag{4.14}$$

For the first integral $\bar{\Lambda}_2(v,I)$ we develop the square $|v-w|^2$ and use the assumption (4.3), as much as the integrability properties on f from conditions (4.2), to bound it from below by the Lebesgue brackets,

$$\begin{aligned}\bar{\Lambda}_2(v,I) &= \int_{\mathbb{R}^{3+}} f(t,w,J)\left(\frac{1}{2}\left(|v|^2+|w|^2\right) + \frac{(I+J)}{m}\right)\mathrm{d}J\,\mathrm{d}w \\ &\geq \mathcal{M}_l\left(\frac{1}{2}|v|^2 + \frac{I}{m}\right) + \mathcal{E}_l \geq \mathcal{E}_l.\end{aligned} \tag{4.15}$$

On the other hand, for the second integral we invoke integrability conditions (4.2) and (4.4), and the pointwise estimate from (4.12), applied to $k=\frac{2+\delta}{2}>1$,

$$\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^{\frac{2+\delta}{2}} \leq 2^{1+\delta}\left(\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\frac{2+\delta}{2}} + \left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\frac{2+\delta}{2}}\right) \tag{4.16}$$

to obtain the control from above of the integral on the complement of the semi-sphere $B_S^c(v,I)$ by the Lebesgue weight $\langle v,I\rangle^{2+\delta}$, since

$$\begin{aligned}&\int_{B_S^c(v,I)} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)\mathrm{d}J\,\mathrm{d}w \\ &\quad\leq \frac{1}{S^\delta}\int_{B_S^c(v,I)} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)^{\frac{2+\delta}{2}}\mathrm{d}J\mathrm{d}w \\ &\quad\leq \frac{1}{S^\delta}2^{1+\delta}\left(\mathcal{M}_u\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\frac{2+\delta}{2}} + \Delta\right) \\ &\quad\leq \frac{1}{S^\delta}2^{1+\delta}\max\{\mathcal{M}_u,\Delta\}\langle v,I\rangle^{2+\delta}.\end{aligned} \tag{4.17}$$

Thus, combining (4.15) and (4.17) for the term in (4.14) we obtain

$$\bar{\Lambda}_2(v,I) - \int_{B^c_S(v,I)} f(t,w,J)\left(\frac{1}{2}|v-w|^2 + \frac{(I+J)}{m}\right)\mathrm{d}J\,\mathrm{d}w \geq \mathcal{E}_l - \frac{2^{1+\delta}}{S^\delta}\max\{\mathcal{M}_u,\Delta\}\langle v,I\rangle^{2+\delta}. \tag{4.18}$$

Therefore, in order to end the argument, we need to choose S and a radius ρ_* for the semi-spheres $B_S(v,I)$ and $B_{\rho_*}(0,0)$, such that the lower bound for estimate (4.18) is strictly positive. That means , it is enough to choose S such that

$$\frac{2^{1+\delta}}{S^\delta}\max\{\mathcal{M}_u,\Delta\}\langle v,I\rangle^{2+\delta} \leq \frac{2^{1+\delta}}{S^\delta}\max\{\mathcal{M}_u,\Delta\}(1+\rho_*^2)^{\frac{2+\delta}{2}} < \frac{\mathcal{E}_l}{8},$$

for any $(v,I)\in B_{\rho_*}(0,0)$, which amounts to choose S

$$S(\rho_*) > \left(2^{4+\delta}\frac{\max\{\mathcal{M}_u,\Delta\}}{\mathcal{E}_l}(1+\rho_*^2)^{\frac{2+\delta}{2}}\right)^{\frac{1}{\delta}} > 1\,. \tag{4.19}$$

Hence, for $\bar{\Lambda}_\gamma(v,I)$ from (4.14) we obtain

$$\Lambda_\gamma(v,I) > \bar{\Lambda}_\gamma(v,I) > \frac{\mathcal{E}_l}{8}\,S(\rho_*)^{\gamma-2}, \quad \text{for } (v,I)\in B_{\rho_*}(0,0). \tag{4.20}$$

Now, we are in good conditions to estimate the convolutional form $\Lambda_\gamma(v,I)$ as defined in (4.13), or equivalently (4.5), form below in terms of for any $(v,I)\in B_{\rho_*}(0,0)$.

In particular, involving the estimate from below (4.10) of the integrand $\tilde{\mathcal{B}}(|v-w|,I,J)$ from (4.8) of $\Lambda_\gamma(v,I)$, we obtain that the integral $\Lambda_\gamma(v,I)$ itself is bounded below, after using the assumption (4.2), by

$$\begin{aligned}\Lambda_\gamma(v,I) &= \int_{\mathbb{R}^{3+}} f(t,w,J)\left(|v-w|^\gamma + \left(\frac{I+J}{m}\right)^{\gamma/2}\right)\mathrm{d}J\,\mathrm{d}w \\ &\geq \mathcal{M}_l\left(2^{\gamma-2}\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}\right) \\ &\quad - 2\int_{\mathbb{R}^{3+}} f(t,w,J)\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2}\mathrm{d}J\,\mathrm{d}w.\end{aligned} \tag{4.21}$$

Then, the integral by the negative sign is easily estimated from above by splitting the integration domain $\mathbb{R}^{3+} = B_1(0,0)\cup B_1^c(0,0)$ by

$$\begin{aligned}\int_{\mathbb{R}^{3+}} f(t,w,J)\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2}\mathrm{d}J\,\mathrm{d}w &\leq \int_{B_1(0,0)} f(t,w,J)\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2}\mathrm{d}J\,\mathrm{d}w \\ &\quad + \int_{B_1^c(0,0)} f(t,w,J)\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2}\mathrm{d}J\,\mathrm{d}w.\end{aligned} \tag{4.22}$$

The first integral is on $B_1(0,0)$, which implies $\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)\leq 1$, hence it is controlled by $\mathcal{M}_u$ using the first inequality from (4.2). For the second one on $B_1^c(0,0)$, since $\gamma\in(0,2]$ then $\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2} \leq \left(\frac{1}{2}|w|^2+\frac{J}{m}\right)$, and the second estimate in

(4.2) shows that is bounded by $\mathcal{E}_u$. Thus, the integral from (4.21) is controlled from above by

$$\int_{\mathbb{R}^{3+}} f(t,w,J)\left(\frac{1}{2}|w|^2+\frac{J}{m}\right)^{\gamma/2} \mathrm{d}J\,\mathrm{d}w \le \mathcal{M}_u+\mathcal{E}_u.$$

It follows then, that the integral $\Lambda_\gamma(v,I)$, is controlled from below after using the previous considerations, as follows

$$\Lambda_\gamma(v,I) \ge 2^{\gamma-2}\mathcal{M}_l\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2} - 2\left(\mathcal{M}_u+\mathcal{E}_u\right). \tag{4.23}$$

So now we need to choose ρ_* to be large enough such that whenever $(v,I)\in B^c_{\rho_*}(0,0)$, the following condition holds

$$\Lambda_\gamma(v,I) \ge \frac{\mathcal{M}_l}{8}\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}, \quad \text{for } (v,I)\in B^c_{\rho_*}(0,0). \tag{4.24}$$

This amounts to take

$$\rho_* = \left(\frac{2\left(\mathcal{M}_u+\mathcal{E}_u\right)}{\mathcal{M}_l\left(2^{\gamma-2}-\frac{1}{8}\right)}\right)^{1/\gamma} \ge 1. \tag{4.25}$$

We conclude the proof by gathering estimates (4.24) and (4.20),

$$\begin{aligned}\Lambda_\gamma(v,I) &\ge \left(\frac{\mathcal{E}_l}{8}\,S^{\gamma-2}\mathbb{1}_{B_{\rho_*}(0,0)}(v,I)+\frac{\mathcal{M}_l}{8}\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}\mathbb{1}_{B^c_{\rho_*}(0,0)}(v,I)\right)\\ &\ge \frac{\min\{\mathcal{M}_l,\mathcal{E}_l\}}{8}\,S^{\gamma-2}\left(\mathbb{1}_{B_{\rho_*}(0,0)}(v,I)+\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}\mathbb{1}_{B^c_{\rho_*}(0,0)}(v,I)\right),\end{aligned} \tag{4.26}$$

the last inequality is because $S=S(\rho_*)\ge 1$ from (4.19). Therefore, it is possible to find an explicit constant $c_{lb}>0$ such that the lower bound (4.5) holds.

In the sequel we construct the constant c_{lb} that can be view as an analog to Poincaré constant in the classical Sobolev embedding theorem in connection to the diffusion problems. For the first term in (4.26) we obtain

$$\langle v,I\rangle^2 = 1+\frac{1}{2}|v|^2+\frac{I}{m}\le 1+\rho_*^2, \quad \text{for } (v,I)\in B_{\rho_*}(0,0). \tag{4.27}$$

The second one corresponding to $(v,I)\in B^c_{\rho_*}(0,0)$ with $\rho_*\ge 1$ by (4.25), it follows

$$\frac{1}{2}|v|^2+\frac{I}{m}\ge \rho_*^2 \ge \frac{1}{\rho_*^2},$$

or

$$\rho_*^2\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)\ge 1. \tag{4.28}$$

Therefore, we get

$$\langle v,I\rangle^\gamma \le \left(1+\rho_*^2\right)^{\gamma/2}\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}, \quad \text{for } (v,I)\in B^c_{\rho_*}(0,0). \tag{4.29}$$

Gathering (4.27) and (4.29), we obtain the following estimate,

$$\langle v, I\rangle^{\gamma} = \langle v, I\rangle^{\gamma}\left(\mathbb{1}_{B_{\rho_*}(0,0)}(v,I) + \mathbb{1}_{B^c_{\rho_*}(0,0)}(v,I)\right)$$
$$\leq \left(1+\rho_*^2\right)^{\gamma/2}\left(\mathbb{1}_{B_{\rho_*}(0,0)}(v,I) + \left(\frac{1}{2}|v|^2 + \frac{I}{m}\right)^{\gamma/2}\mathbb{1}_{B^c_{\rho_*}(0,0)}(v,I)\right). \quad (4.30)$$

Therefore, going back to the final estimate (4.26), we obtain,

$$\Lambda \geq \frac{\min\{\mathcal{M}_l, \mathcal{E}_l\}}{8}\,\frac{S(\rho_*)^{\gamma-2}}{(1+\rho_*^2)^{\gamma/2}}\langle v, I\rangle^{\gamma} =: c_{lb}\langle v, I\rangle^{\gamma}\,. \quad (4.31)$$

Using inequalities (4.19) and (4.25) we obtain the final expression for the constant c_{lb} as announced in (4.6), which completes the proof of Lemma 4.1. □

The following corollary holds immediately by the definition of the Banach space $L^1_k(\mathbb{R}^{3+})$ of integrable functions with respect to the Lebesgue weight, as defined in (2.28), (2.29) and (2.30).

Corollary 1 (Coercive Estimate)**.** *The loss collision operator from* (2.24)*, acting on any function* $f \in L^1_k$*, for* $k \geq 1+\gamma/2$*,* $\gamma \in [0,2]$*, satisfies*

$$\int_{\mathbb{R}^{3+}} Q^-(f,f)\,\langle v, I\rangle^{2k}\,\mathrm{d}I\mathrm{d}v \geq \kappa^{lb} c_{lb}\,\|f\|_{L^1_{k+\frac{\gamma}{2}}}\,, \quad (4.32)$$

with c_{lb} *independent of moment order* k *stated in* (4.6) *and* κ^{lb} *is from* (3.7)*.*

Proof. For the transition function $\mathcal{B}$ satisfying the assumption 3.2, the following lower bound for the collision frequency defined in (2.21) holds

$$\nu[f](v,I) \geq \kappa^{lb}\int_{\mathbb{R}^{3+}} f_*\tilde{B}\mathrm{d}I_*\mathrm{d}v_* \geq \kappa^{lb}c_{lb}\langle v, I\rangle^{\gamma},$$

with the constant κ^{lb} coming out from the integration over the compact set K is introduced in (3.7) and the last inequality is from the lower bound (4.5). Now the loss collision operator (2.20) can be bounded from below as follows

$$\int_{\mathbb{R}^{3+}} Q^-(f,f)\,\langle v, I\rangle^{2k}\,\mathrm{d}I\mathrm{d}v = \int_{(\mathbb{R}^{3+})^2\times K} f(t,v,I)\,\nu[f](v,I)\,\langle v, I\rangle^{2k}\,\mathrm{d}I\mathrm{d}v$$
$$\geq \kappa^{lb}\,c_{lb}\int_{\mathbb{R}^{3+}} f\,\langle v, I\rangle^{2k+\gamma}\mathrm{d}I\mathrm{d}v = \kappa^{lb}c_{lb}\,\|f\|_{L^1_{k+\frac{\gamma}{2}}}\,,$$

which concludes the proof. □

5. Fundamental lemmas for the gain operator

This section is devoted to obtain the control of the positive contributions of the collision operator moments for the polyatomic case. The following results are obtained by gathering identities and estimates that exalt the nature of binary collisional operator in weak form as a dissipative single species particle mixing operator whose collisional transition probability component varying on a compact manifold is controlled by an averaging operator on such manifold whose volume remains invariant by the particle.

The most fundamental result, The Polyatomic Compact Manifold Averaging Lemma, is presented in Lemma 5.3. Such calculation is performed in the weak formulation of the polyatomic collisional form after presented the conservation of energy decomposition identity that enables this proof of Lemma 5.3. This Lemma can be viewed as a sharper form of the Povzner Lemma developed by [16] for the classical Boltzmann equation for hard spheres in three dimensions, by [18] for inelastic hard spheres in three dimensions [27] for hard potentials in any dimension above or equal to three, and recently revisited in [5]. The presentation that follows in unedited for the Boltzmann equation model of polyatomic gases.

5.1. **The energy identity.** We first define the total energy of the two colliding molecules using the Lebesgue weight (2.28).

Definition 5.1 (The total energy in the Lebesgue weight form)**.** *Let v', v'_*, I' and I'_* be functions of v, v_*, I, I_*, r, R and σ as given in* (2.8) *and* (2.6)*. Then we define the total energy in the Lebesgue weight* (2.28) *form as follows*

$$E^{\langle\rangle} := \langle v, I\rangle^2 + \langle v_*, I_*\rangle^2 = \langle v', I'\rangle^2 + \langle v'_*, I'_*\rangle^2 = 2 + |V|^2 + \frac{E}{m}, \tag{5.1}$$

with E from (2.4)*.*

In order to encode behavior of a polyatomic gas, we first need to understand energy recombination during a collision process, using transformations (2.8) and (2.6). This knowledge is crucial for expressing pre-collisional quantities $\langle v', I'\rangle^2$ and $\langle v'_*, I'_*\rangle^2$ in terms of particular partitions of the total energy, as shows the following Lemma.

Lemma 5.2 (Energy Identity Decomposition)**.** *Let v', v'_*, I' and I'_* be defined in collision transformations* (2.8) *and* (2.6)*. There exists convex conjugate factors $p = p(v, v_*, I, I_*, R)$ and $q = q(v, v_*, I, I_*, R)$, i.e. $p + q = 1$, and a function $\lambda = \lambda(v, v_*, I, I_*, R)$ such that the following representation holds*

$$\langle v', I'\rangle^2 = E^{\langle\rangle}\left(p + \lambda \hat{V}\cdot\sigma\right), \qquad \langle v'_*, I'_*\rangle^2 = E^{\langle\rangle}\left(q - \lambda \hat{V}\cdot\sigma\right).$$

Moreover, this representation preserves the total molecular energy,

$$\langle v', I'\rangle^2 + \langle v'_*, I'_*\rangle^2 = E^{\langle\rangle}\left((p + \lambda \hat{V}\cdot\sigma) + (q - \lambda \hat{V}\cdot\sigma)\right) \equiv E^{\langle\rangle}. \tag{5.2}$$

Proof. We consider partitions of the energy $E^{\langle\rangle}$ obtained by introducing convex combinations associated to functions Θ and Σ that may depend on v, v_*, I, I_* and R, as follows

(i) for $\Theta \in [0, 1]$ we have

$$\Theta E^{\langle\rangle} = 1 + |V|^2 \quad \Rightarrow \quad (1 - \Theta)\, E^{\langle\rangle} = 1 + \frac{E}{m}, \tag{5.3}$$

(ii) for $\Sigma \in [0, 1]$ we get

$$\Sigma\,(1 - \Theta)\, E^{\langle\rangle} = 1 + R\frac{E}{m} \quad \Rightarrow \quad (1 - \Sigma)\,(1 - \Theta)\, E^{\langle\rangle} = (1 - R)\frac{E}{m}. \tag{5.4}$$

Now, using collisional rules (2.8) and (2.6) yield the associated Lebesgue weights for the calculation of total molecular energy of the postcollisional (primed) states

$$\langle v', I'\rangle^2 = 1 + \frac{1}{2}|V|^2 + \frac{1}{2}R\frac{E}{m} + \sqrt{\frac{RE}{m}}\,|V|\,\hat{V}\cdot\sigma + r(1 - R)\frac{E}{m},$$
$$\langle v'_*, I'_*\rangle^2 = 1 + \frac{1}{2}|V|^2 + \frac{1}{2}R\frac{E}{m} - \sqrt{\frac{RE}{m}}\,|V|\,\hat{V}\cdot\sigma + (1 - r)(1 - R)\frac{E}{m},$$

which can be rewritten in terms of functions Θ and Σ as in (5.3)-(5.4), the parameter $r \in [0,1]$ from (2.6) and the dot product $\hat{V} \cdot \sigma$ as follows

$$\begin{aligned} \langle v', I' \rangle^2 &= E^{\langle\rangle} \left(\frac{1}{2}\Theta + \frac{1}{2}\Sigma(1-\Theta) + r(1-\Sigma)(1-\Theta) \right) \\ &\qquad + \sqrt{(\Theta E^{\langle\rangle} - 1)(\Sigma(1-\Theta)E^{\langle\rangle} - 1)}\, \hat{V} \cdot \sigma, \\ \langle v'_*, I'_* \rangle^2 &= E^{\langle\rangle} \left(\frac{1}{2}\Theta + \frac{1}{2}\Sigma(1-\Theta) + (1-r)(1-\Sigma)(1-\Theta) \right) \\ &\qquad - \sqrt{(\Theta E^{\langle\rangle} - 1)(\Sigma(1-\Theta)E^{\langle\rangle} - 1)}\, \hat{V} \cdot \sigma \,. \end{aligned} \tag{5.5}$$

Now set the convex factors from (5.5), to be

$$\begin{aligned} p &:= \frac{1}{2}\Theta + \frac{1}{2}\Sigma(1-\Theta) + r(1-\Sigma)(1-\Theta) = \frac{s}{2} + r(1-s), \\ q &:= \frac{1}{2}\Theta + \frac{1}{2}\Sigma(1-\Theta) + (1-r)(1-\Sigma)(1-\Theta) = \frac{s}{2} + (1-r)(1-s), \end{aligned}$$

where the dependence of p and q upon velocities v, v_*, internal energies I, I_* and variable R is through the function $s := s(v, v_*, I, I_*, R)$ defined with

$$s = \Theta + \Sigma(1-\Theta) \quad \Rightarrow \quad (1-s) = (1-\Sigma)(1-\Theta), \tag{5.6}$$

with Θ, Σ from (5.3)-(5.4). Since $\Theta, \Sigma \in [0,1]$ it also follows

$$s \in [0,1], \quad \text{for any} \quad v, v_* \in \mathbb{R}^3,\ I, I_* \in [0,\infty),\ \text{and } R \in [0,1]. \tag{5.7}$$

Clearly p and q add up to unity. In addition set

$$\lambda := \sqrt{(\Theta E^{\langle\rangle} - 1)(\Sigma(1-\Theta)E^{\langle\rangle} - 1)}. \tag{5.8}$$

Hence, adding the two left hand sides of identities from (5.5), the conservation of the total, i.e. kinetic plus internal molecular energy is given by

$$\begin{aligned} \langle v', I' \rangle^2 + \langle v'_*, I'_* \rangle^2 &= E^{\langle\rangle}(p + \lambda \hat{V} \cdot \sigma + q - \lambda \hat{V} \cdot \sigma) = E^{\langle\rangle}(p+q) \\ &= E^{\langle\rangle} = \langle v, I \rangle^2 + \langle v_*, I_* \rangle^2, \end{aligned}$$

if we recall that the total molecular energy for a polyatomic state interacting (or colliding) pairs (v, I) and (v_*, I_*) is given by $E^{\langle\rangle} := \langle v, I \rangle^2 + \langle v_*, I_* \rangle^2$. Thus, the energy identity (5.2) holds. □

5.2. **The Polyatomic Compact Manifold Averaging Lemma.** The energy identity (5.2) allows to find a dissipation effect of the collision operator. Namely, we will prove that k-th moment of the gain term decreases with respect to k, allowing the moment of the same order k of the loss term to prevail in the dynamics, when sufficiently large order of moments k is taken into account. The decay of the gain term is attained by averaging k^{th}-power of the postcollisional total molecular energies, that is $\langle v', I' \rangle^{2k} + \langle v'_*, I'_* \rangle^{2k}$. Due to an additional variable I in the polyatomic gas model, the averaging needs to be performed with respect to the compact manifold that contains a domain of the two parameters: *(i)* one angular parameter (scattering direction) σ that splits the kinetic energy on molecular velocities, *(ii)* one additional parameter r that distributes the total internal energy among colliding molecules, and parameter R that splits the total molecular energy into the kinetic and internal part, whose averaging does not contribute to the decay of the gain part and integration gives the constant. This result can be viewed as an extension of the angular averaging Povzner lemma used for classical elastic and inelastic collisional theory for single of multiple mixture of monatomic gases.

Lemma 5.3 (The Polyatomic Compact Manifold Averaging Lemma). *Let v', v'_*, I' and I'_* be given as in (2.8) and (2.6). Suppose that functions $b(\sigma\cdot\hat{u})$, $d^{ub}_\gamma(r)$ and $e^{ub}_\gamma(R)$ satisfy the integrability conditions (3.3), (3.4) and (3.5), respectively. Then the following estimate holds*

$$\int_K \left(\langle v', I'\rangle^{2k} + \langle v'_*, I'_*\rangle^{2k}\right) b(\hat{u}\cdot\sigma)\, d^{ub}_\gamma(r)\, \varphi_\alpha(r)\, e^{ub}_\gamma(R)\, \psi_\alpha(R)\, (1-R)R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma \\ \leq \mathcal{C}_k \left(\langle v, I\rangle^2 + \langle v_*, I_*\rangle^2\right)^k, \quad (5.9)$$

with the contracting constant $\mathcal{C}_k$, that is $\mathcal{C}_k \searrow 0$, as $k\to\infty$. In addition, there exists a $\bar{k}_* > 1$ such that

$$\mathcal{C}_k < \kappa^{lb}, \quad \text{for all } k > \bar{k}_*, \quad (5.10)$$

where κ^{lb} is given in (3.7).

Moreover, when $b(\sigma\cdot\hat{u}) \in L^p(S^2;\mathrm{d}\sigma)$ and $d^{ub}_\gamma(r)\varphi_\alpha(r) \in L^p([0,1];\mathrm{d}r)$, $p\in(1,\infty]$, the contracting constant $\mathcal{C}_k$ can be explicitly computed with the known decay rate,

$$\mathcal{C}_k \leq 2\, C^{ub}_{\gamma,\alpha} \|b\|_{L^p(\mathrm{d}\sigma)} \|d^{ub}_\gamma \varphi_\alpha\|_{L^p(\mathrm{d}r)} (4\pi)^{1/p'} \\ \times \left(\frac{1}{kp'+1} + \frac{2kp'}{(kp'+1)(kp'+2)}\left(1 - \left(\frac{1}{2}\right)^{kp'+2}\right)\right)^{1/p'}. \quad (5.11)$$

with p and p' being the classical conjugates $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. In order to prove this Lemma, we first use energy identity and representation (5.5), that generated the λ factor in (5.8),

$$\lambda := \sqrt{(\Theta E^{\langle\rangle} - 1)(\Sigma(1-\Theta)E^{\langle\rangle} - 1)}.$$

Using the Young inequality we get an estimate

$$\lambda \;\leq\; \frac{1}{2}\left(\Theta E^{\langle\rangle} + \Sigma(1-\Theta)E^{\langle\rangle} - 2\right) \leq \frac{(\Theta + \Sigma(1-\Theta))}{2} E^{\langle\rangle} = \frac{s}{2}\, E^{\langle\rangle},$$

with the function $s\in[0,1]$ defined in (5.6)-(5.7). Thus,

$$\pm\,\lambda\,\hat{V}\cdot\sigma \;\leq\; \frac{(\Theta+\Sigma(1-\Theta))}{2} |\hat{V}\cdot\sigma|\, E^{\langle\rangle} = \frac{s}{2}\, |\hat{V}\cdot\sigma|\, E^{\langle\rangle}.$$

Therefore, the convex form (5.5) can be estimated pointwise

$$\begin{aligned}
\langle v', I'\rangle^2 &\leq E^{\langle\rangle}\left((\Theta+\Sigma(1-\Theta))\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right) + (1-\Sigma)(1-\Theta)r\right) \\
&= E^{\langle\rangle}\left(s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right) + (1-s)r\right), \\
\langle v'_*, I'_*\rangle^2 &\leq E^{\langle\rangle}\left((\Theta+\Sigma(1-\Theta))\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right) + (1-\Sigma)(1-\Theta)(1-r)\right) \\
&= E^{\langle\rangle}\left(s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right) + (1-s)(1-r)\right).
\end{aligned} \quad (5.12)$$

Moreover, since functions

$$s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right)+(1-s)r, \quad \text{and} \quad s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right)+(1-s)(1-r) \tag{5.13}$$

are linear with respect to s and the range of s is $[0,1]$ by (5.7), it follows that the maximum of both functions (5.13) is attained at the boundary i.e. for either $s=0$ or $s=1$. Therefore, we can write

$$s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right)+(1-s)r \le \max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, r\right\},$$

and

$$s\left(\frac{1+|\hat{V}\cdot\sigma|}{2}\right)+(1-s)(1-r) \le \max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, 1-r\right\}.$$

This allows to estimate (5.12) as follows

$$\langle v', I'\rangle^2 \le E^{\langle\rangle} \max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, r\right\}, \qquad \langle v'_*, I'_*\rangle^2 \le E^{\langle\rangle} \max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, 1-r\right\}.$$

With these estimates and using the symmetry properties associated to the partition function with respect to r by the fact $\varphi_\alpha(r)=\varphi_\alpha(1-r)$ and $d_\gamma^{ub}(r)=d_\gamma^{ub}(1-r)$, the left-hand side of (5.9) becomes

$$\begin{aligned}\int_K \left(\langle v', I'\rangle^{2k}+\langle v'_*, I'_*\rangle^{2k}\right) b(\hat{u}\cdot\sigma)\, d_\gamma^{ub}(r)\, \varphi_\alpha(r)\, e_\gamma^{ub}(R)\, \psi_\alpha(R)\, (1-R)R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma \\ \le 2\left(E^{\langle\rangle}\right)^k \int_K \left(\max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, r\right\}\right)^k b(\hat{u}\cdot\sigma)\, d_\gamma^{ub}(r)\, \varphi_\alpha(r) \\ \times e_\gamma^{ub}(R)\, \psi_\alpha(R)\, (1-R)R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma =: \mathcal{K}.\end{aligned} \tag{5.14}$$

It is interesting to note that decay of $\mathcal{K}$ from (5.14) in k is warranted by the averaging over σ and r, and not necessarily on R, which implies that integration with respect to R comes out as a constant. Thus, using the notation (3.6), (5.14) becomes

$$\begin{aligned}\mathcal{K} \le 2\, C_{\gamma,\alpha}^{ub}\left(E^{\langle\rangle}\right)^k \int_{S^2}\int_0^1 \left(\max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, r\right\}\right)^k b(\hat{u}\cdot\sigma) d_\gamma^{ub}(r)\, \varphi_\alpha(r) \mathrm{d}r\, \mathrm{d}\sigma, \\ \le \mathcal{C}_k \left(E^{\langle\rangle}\right)^k,\end{aligned} \tag{5.15}$$

where we have denoted

$$\mathcal{C}_k = 2\, C_{\gamma,\alpha}^{ub} \sup_{\{\hat{V}\in S^2, \hat{u}\in S^2\}} \int_{S^2}\int_0^1 \left(\max\left\{\frac{1+|\hat{V}\cdot\sigma|}{2}, r\right\}\right)^k b(\hat{u}\cdot\sigma)\, d_\gamma^{ub}(r)\, \varphi_\alpha(r) \mathrm{d}r\, \mathrm{d}\sigma. \tag{5.16}$$

The following estimates of the constant $\mathcal{C}_k$ are inspired and follow the analog ones described in [16], [18], [27], [7], [4], and more recently revisited in [5] for the classical space homogeneous Boltzmann binary elastic interacting particle model for monatomic gases in the case of the angular transition $b(\hat{u}\cdot\sigma)$ just an integrable function on the sphere S^2.

Indeed, if the angular transition $b(\hat{u}\cdot\sigma) \in L^1(S^2; \mathrm{d}\sigma)$ and the partition function $d^{ub}_\gamma(r)\varphi_\alpha(r) \in L^1([0,1]; \mathrm{d}r)$, then the integral $\mathcal{C}_k$ is monotonically decreasing in k, without necessarily a specific decay rate in the parameter k.

This statement follows from expressing $\hat{V}$ in polar coordinates with zenith $\hat{u}$. That makes the integral on the sphere that characterizes $\mathcal{C}_k$ to be a continuous function on in the vectors $\hat{V}$ and $\hat{u}$, whose integrand, with respect to the measure $b(\hat{u}\cdot\sigma)\mathrm{d}\sigma d^{ub}_\gamma(r)\varphi_\alpha(r)\mathrm{d}r$ is strictly decreasing in $k > 1$ up to a set of measure zero (namely at $\sigma = \{\pm\hat{V}\}$ or at $r = 1$). Then $\mathcal{C}_{k_1} > \mathcal{C}_{k_2}$ for any $k_1 < k_2$ follows by taking the supremum from the continuity in $\hat{V}$ and $\hat{u}$ property. In particular, we conclude by monotone convergence Theorem that the constant $\mathcal{C}_k$ is contracting

$$\mathcal{C}_k \searrow 0, \quad \text{as } k \to \infty. \tag{5.17}$$

In particular, with κ^{ub} from (3.7), it follows that

$$\mathcal{C}_k < 2\kappa^{ub}, \quad \text{for any } k > 1.$$

In addition, since $\kappa^{lb} \leq \kappa^{ub}$ with the constant κ^{lb} also defined in (3.7), we conclude by (5.17) there exists $\bar{k}_*$ such that

$$\mathcal{C}_k < \kappa^{lb} \quad \text{for any } k > \bar{k}_*, \tag{5.18}$$

where $\bar{k}_*$ is the smallest k such that (5.18) holds.

On the other side, for the classical Boltzmann model for binary interactions it was also shown in [16], [18], [27] and in [7], applied to both elastic or inelastic collisions, that in the case when $b \in L^\infty(S^2; \mathrm{d}\sigma)$ or $b \in L^p(S^2; \mathrm{d}\sigma)$, for $p > 1$, it is possible to calculate the decay rate of $\mathcal{C}_k$ as a function of k.

For a polyatomic gas, we describe in detail these two cases for the angular transition $b(\hat{u}\cdot\sigma)$ and the upper bound of partition function $d^{ub}_\gamma(r)\varphi_\alpha(r)$ for which the constant $\mathcal{C}_k$ has an explicit decay with respect to k and the integrability rate p, as follows.

(i) If the angular transition rate $b(\hat{u}\cdot\sigma) \in L^p(S^2; \mathrm{d}\sigma)$, and the partition function $d^{ub}_\gamma(r)\varphi_\alpha(r) \in L^\infty([0,1]; \mathrm{d}r)$ with $p > 1$, then a straight forward application of the Hölder inequality to the integral apart of (5.16) yields the estimate

$$\mathcal{C}_k \leq 2\, C^{ub}_{\gamma,\alpha} \|b\|_{L^p(\mathrm{d}\sigma)} \|d^{ub}_\gamma \varphi_\alpha\|_{L^p(\mathrm{d}r)} \times \left(\int_{S^2} \int_0^1 \left(\max\left\{ \frac{1 + |\hat{V}\cdot\sigma|}{2}, r \right\} \right)^{kp'} \mathrm{d}r\, \mathrm{d}\sigma \right)^{1/p'}, \tag{5.19}$$

with p and p' the pairs satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. In the last integral we change variables $\sigma \mapsto \mu$, $\mu = \hat{V}\cdot\sigma$ and obtain

$$\int_{S^2} \int_0^1 \left(\max\left\{ \frac{1 + |\hat{V}\cdot\sigma|}{2}, r \right\} \right)^{kp'} \mathrm{d}r\, \mathrm{d}\sigma = 4\pi \int_0^1 \int_0^1 \left(\max\left\{ \frac{1+\mu}{2}, r \right\} \right)^{kp'} \mathrm{d}r\, \mathrm{d}\mu.$$

Therefore, from (5.19), for $\mathcal{C}_k$ we get the following estimate

$$\mathcal{C}_k \leq 2\, C^{ub}_{\gamma,\alpha} \|b\|_{L^p(\mathrm{d}\sigma)} \|d^{ub}_\gamma \varphi_\alpha\|_{L^p(\mathrm{d}r)} (4\pi)^{1/p'} \mathcal{C}^p_k,$$

with

$$\mathcal{C}_k^p = \left(\int_0^1 \int_0^1 \left(\max\left\{ \frac{1+\mu}{2}, r \right\} \right)^{kp'} \mathrm{d}r \, \mathrm{d}\mu \right)^{1/p'} .$$

Then we can explicitly compute this constant $\mathcal{C}_k^p$, as shown in Appendix Section C, by setting $n = kp'$ in the expression (C.1), namely,

$$\mathcal{C}_k^p = \left(\frac{1}{kp'+1} + \frac{2kp'}{(kp'+1)(kp'+2)} \left(1 - \left(\frac{1}{2}\right)^{kp'+2} \right) \right)^{1/p'} . \tag{5.20}$$

(ii) However, if the angular transition rate $b(\hat{u}\cdot\sigma) \in L^\infty(S^2; \mathrm{d}\sigma)$, that is $b(\hat{u}\cdot\sigma)$ is bounded on the sphere S^2 and $d_\gamma^{ub}(r)\varphi_\alpha(r) \in L^\infty([0,1]; \mathrm{d}r)$, the decay rate is faster for $k > 1$. Indeed, (5.16) can be upper bounded by

$$\mathcal{C}_k \leq 2\, C_{\gamma,\alpha}^{ub} \, \|b\|_{L^\infty(\mathrm{d}\sigma)} \left\| d_\gamma^{ub}\, \varphi_\alpha \right\|_{L^\infty(\mathrm{d}r)} \times \sup_{\{\hat{V}\in S^2, \hat{u}\in S^2\}} \int_{S^2} \int_0^1 \left(\max\left\{ \frac{1+|\hat{V}\cdot\sigma|}{2}, r \right\} \right)^k \mathrm{d}r \, \mathrm{d}\sigma$$
$$= 8\pi\, C_{\gamma,\alpha}^{ub} \, \|b\|_{L^\infty(\mathrm{d}\sigma)} \left\| d_\gamma^{ub}\, \varphi_\alpha \right\|_{L^\infty(\mathrm{d}r)} \mathcal{C}_k^\infty , \tag{5.21}$$

where, after the change of variables $\sigma \mapsto \mu$, $\mu = \hat{V}\cdot\sigma$ in the last expression (5.21), $\mathcal{C}_k^\infty$ is given with

$$\mathcal{C}_k^\infty := \int_0^1 \int_0^1 \left(\max\left\{ \frac{1+\mu}{2}, r \right\} \right)^k \mathrm{d}r \, \mathrm{d}\mu. \tag{5.22}$$

This double integral (5.22) is computed in the Appendix Section C, setting $n = k$ in (C.1), and the final expression is

$$\mathcal{C}_k^\infty = \frac{1}{k+1} + \frac{2k}{(k+1)(k+2)} \left(1 - \left(\frac{1}{2}\right)^{k+2} \right), \quad k > 1. \tag{5.23}$$

□

Therefore, the total energy identity (5.2) enables to obtain a partial crucial result that controls the averaging on the compact manifold K of the k^{th}-power of the postcollisional total molecular energies, that is $\langle v', I' \rangle^{2k} + \langle v'_*, I'_* \rangle^{2k}$ by the k^{th}-power of the molecular energy, i.e. $E^{\langle\rangle k}$ time a factor $\mathcal{C}_k$ is '*contracting*", that means it decays as k grows to infinity.

This result is an imperative for proving decay of the k-th moment of collision operator gain term when averaged over the compact manifold K. This fact allows for the corresponding k-th moment of the loss term to prevail in the dynamics, when sufficiently large order of moment $k > \bar{k}_*$ is taken into account in order to ensure (5.10). It is worthwhile to mention that the Averaging Lemma ensures the existence of such $\bar{k}_*$, since only the contracting constant $\mathcal{C}_k$ depends on k. This order of moment $\bar{k}_*$ needed to guarantee this property is studied in the upcoming Remark 4.

Remark 4 (Sufficient moment order to ensure prevail of the loss term)**.** *For the single monatomic species, when the averaging is performed only in the scattering direction σ, it was sufficient to take the order of moment $k > 1$ to prove the dominance of the moment associated to the loss term with respect to the same moment* *of the gain term, with $k = 1$ corresponding to the energy. In the monatomic gas mixture setting, the value of $k = k_*$ depends on the ratio of mass species, and it is shown that k_* grows as this ratio deviates from $1/2$, where $1/2$ corresponds to the single specie case.*

In the current setting, corresponding to polyatomic gases, the averaging is performed over the compact manifold K, with respect to the angular scattering direction σ, as well as to the parameters r and R that are arguments of the partition functions. We seek for a sufficient order of moment $\bar{k}_*$ which secures (5.10), under the additional assumption of

$$b(\hat{u} \cdot \sigma) \in L^{\infty}(\mathrm{d}\sigma), \quad \textit{and} \quad d_{\gamma}^{ub}(r)\varphi_{\alpha}(r) \in L^{\infty}(\mathrm{d}r), \tag{5.24}$$

when we can explicitly compute the constant $\mathcal{C}_k$ from Lemma 5.3, as shown in (5.11). We focus on the three models for transition function $\mathcal{B}$ introduced in Section 3.2.

Note that for all the three models the condition of boundedness of the function $d_{\gamma}^{ub}\varphi_{\alpha}$ is fulfilled when $\alpha \geq 0$, in which case

$$\left\| d_{\gamma}^{ub}\varphi_{\alpha} \right\|_{L^{\infty}(\mathrm{d}r)} = 1.$$

Therefore, the condition (5.10) reduces to

$$\frac{1}{k+1} + \frac{2k}{(k+1)(k+2)} \left(1 - \left(\frac{1}{2} \right)^{k+2} \right) =: \mathcal{C}_k^{\infty} < \frac{1}{2} \frac{c_{\gamma,\alpha}^{lb}\, C_{\gamma,\alpha}^{lb}}{C_{\gamma,\alpha}^{ub}} := C_{\gamma,\alpha}^{*}. \tag{5.25}$$

To complete the study, it remains to calculate the constants $c_{\gamma,\alpha}^{lb}$, $C_{\gamma,\alpha}^{lb}$ and $C_{\gamma,\alpha}^{ub}$ for the three models. To that end, we need to determine multiplying functions $d_{\gamma}^{lb}(r)$, $e_{\gamma}^{lb}(R)$ and $e_{\gamma}^{ub}(R)$. For the Model 1 we use constants already calculated in (3.9), taking $m = 1$. The Model 2 takes the bounds from (3.11), while for the Model 3 we assume (3.13). The results are presented in the Figure 3.

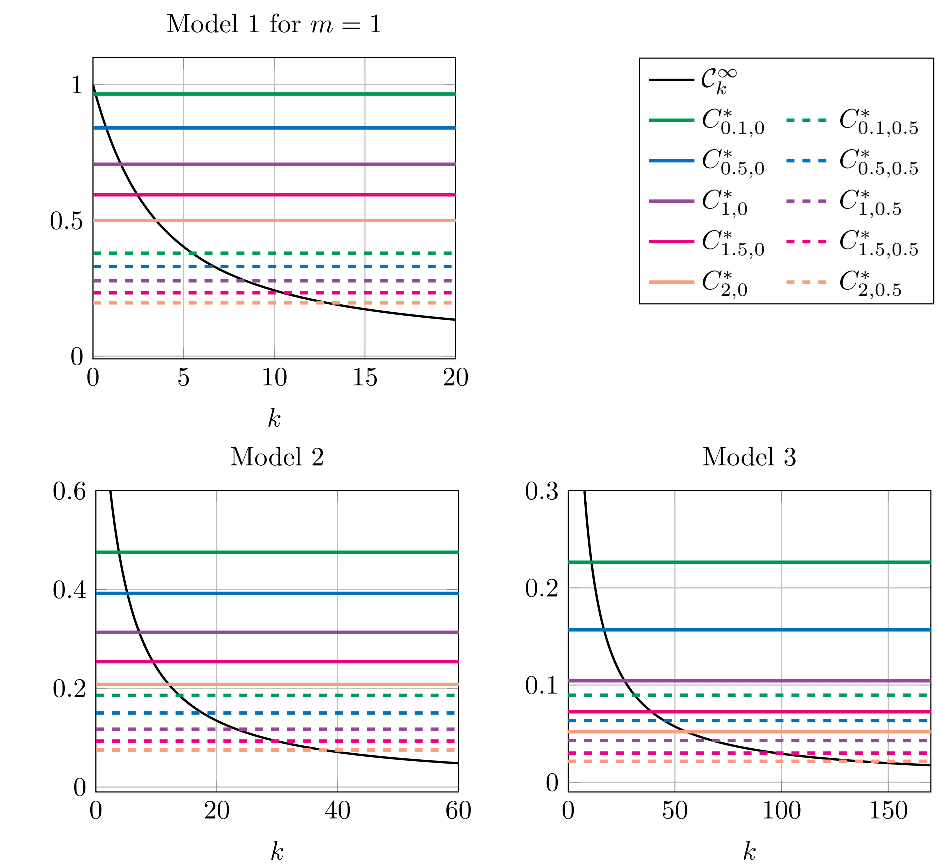

FIGURE 3. Study of the constant $C^*_{\gamma,\alpha}$ defined in (5.25), for the physical values of $\alpha = 0$ (——) and $\alpha = 0.5$ (- - -), while varying $\gamma \in (0, 2]$.

5.3. **The upper bound for the transition potential function.** The second important estimate enable us to control the positive contributions of the moments of the collisional polyatomic operator are simply derived from obtaining local upper bound estimates that establish the control of the potentials (3.1) by their associated Lebesgue weights (2.28). This is a much simpler argument that the one needed to obtain the coercive lower bounds calculated in the previous Section 4.

Lemma 5.4 (Upper bound)**.** *For any $\gamma \in (0, 2]$ the following inequality holds*

$$|v - v_*|^\gamma + \left(\frac{I + I_*}{m}\right)^{\gamma/2} \leq 2^{\frac{3\gamma}{2}-1} \left(\langle v, I\rangle^\gamma + \langle v_*, I_*\rangle^\gamma\right), \tag{5.26}$$

for $v, v_* \in \mathbb{R}^3$ and $I, I_* \in [0, \infty)$.

Proof. We first write

$$|v-v_*|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2} \le 2^\gamma \left(\left(\frac{1}{4}|v-v_*|^2\right)^{\gamma/2} + \left(\frac{I+I_*}{m}\right)^{\gamma/2} \right)$$
$$\le 2^{\frac{3\gamma}{2}-1} \left(\frac{1}{4}\left|v-v_*\right|^2 + \frac{I+I_*}{m} \right)^{\gamma/2}. \quad (5.27)$$

Then, using

$$\frac{1}{4}\left|v-v_*\right|^2 + \frac{I}{m} + \frac{I_*}{m} \le \frac{1}{2}\left|v\right|^2 + \frac{1}{2}\left|v_*\right|^2 + \frac{I}{m} + \frac{I_*}{m} \le \langle v, I\rangle^2 + \langle v_*, I_*\rangle^2,$$

and since $\gamma/2 \le 1$, we can estimate

$$\left(\frac{1}{4}\left|v-v_*\right|^2 + \frac{I}{m} + \frac{I_*}{m} \right)^{\gamma/2} \le \langle v, I\rangle^\gamma + \langle v_*, I_*\rangle^\gamma. \quad (5.28)$$

Combining this result with (5.27), we conclude the proof. □

6. L^1_k moments a priori estimates

The Boltzmann collisional model for polyatomic gases becomes particularly challenging when we gather a priori estimates for the propagation and generation of $k-$Lebesgue moment generating the Banach space $L^1_k(\mathbb{R}^{3+})$ defined in (2.29) and (2.30). More precisely, the biggest challenge is to find sufficient conditions for transition functions and their corresponding probability measures and bounds, as discussed in Section 3, in order to be able to generate an Ordinary Differential Inequality (ODI) that describes the times evolution of the solution norm of $\$f(\cdot,t\|_{L^1_k(\mathbb{R}^{3+})}$. In fact, the previous polyatomic compact manifold averaging Lemma 5.3 together with the requirement (5.10) are sufficient conditions to show that the evolution of the k-th polynomial moment of the collision operator will become negative for $k > k_*$ and $\gamma > 0$, due to the fact that the negative contribution of the collision operator moments are superlinear with respect to their positive contribution.

Thus, the following Lemma is a non-trivial adaptation form the one proven in [5] for the classical binary elastic Cauchy Theory of the Boltzmann equation in the Banach $k-$Lebesgue moment weighted space $L^1_k(\mathbb{R}^3)$ to the Cauchy initial value problem Boltzmann for polyatomic gases (7.1), whose solution posed as a probability density in now, for the polyatomic case the Banach $k-$Lebesgue moment space $L^1_k(\mathbb{R}^{3+})$ with transition probability functions described in Section 3.

Lemma 6.1 (Moments bound for the collision operator)**.** *Let $f \in L^1_1$ satisfying assumptions from Lower bound Lemma 4.1 and condition (4.5). Moreover, suppose that the transition function $\mathcal{B}$ satisfies Assumption 3.1. Then for any $\gamma \in (0, 2]$, the following inequality holds*

$$\int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\,\langle v, I\rangle^{2k} \mathrm{d}I\mathrm{d}v = \mathfrak{m}_k\left[Q(f,f)\right] \le -A_{k_*}\left\|f\right\|_{L^1_k}^{1+\frac{\gamma/2}{k}} + B_k\left\|f\right\|_{L^1_k}, \quad (6.1)$$

for large enough k such that

$$k > k_*, \qquad k_* = \max\left\{\bar{k}_*, 1+\gamma, 1+\delta/2\right\} \quad \textit{finite} \quad (6.2)$$

where $\bar{k}_*$, defined in (5.18), is such that inequality (5.10) holds, and $\delta > 0$ is from condition (4.5).

For such k_*, the constant $A_{k_*} \geq A_k > 0$ is independent of k, and $B_k > 0$,

$$A_{k_*} := \frac{c_{lb}}{2}\left(\kappa^{lb} - \mathcal{C}_{k_*}\right) \|f\|_{L^1_0}^{-\frac{\gamma/2}{k_*}}, \tag{6.3}$$
$$B_k = \mathcal{C}_k 2^{\frac{3\gamma+k}{2}} \max\left\{ \left(\frac{\kappa^{lb}\, 2^{\frac{3\gamma+k}{2}}}{A_{k_*}}\right)^{\frac{\theta_k}{1-\theta_k}} \frac{\eta_k^{\frac{1}{1-\theta_k}}}{\|f\|_{L^1_1}}, 1 \right\} \|f\|_{L^1_1},$$

with

$$\eta_k := \|f\|_{L^1_0}^{\frac{1+\gamma/2}{k+\gamma/2}} \|f\|_{L^1_1}^{\frac{k-1}{k-1+\gamma/2}}, \qquad \theta_k = \frac{\gamma/2}{k-1+\gamma/2} + \frac{k-1}{k+\gamma/2} < 1, \tag{6.4}$$

and c_{lb} is the lower bound constant for the convolution of f and power law function of order γ in terms of the γ-power of Lebesgue weight from (4.5) that enables the super linear behavior for moments of order $k > k_*$, $\mathcal{C}_k$, monotone decreasing in k, is the constant from Lemma 5.3 and κ^{lb} from (3.7).

Proof. For the test function

$$\chi(v, I) = \langle v, I\rangle^{2k},$$

the weak form (2.23) yields

$$\begin{aligned} \mathcal{W} &:= \int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\, \langle v, I\rangle^{2k} \mathrm{d}I \mathrm{d}v \\ &= \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} \frac{f f_*}{I^\alpha I_*^\alpha}\left(\langle v', I'\rangle^{2k} + \langle v'_*, I'_*\rangle^{2k} - \langle v, I\rangle^{2k} - \langle v_*, I_*\rangle^{2k}\right) \mathrm{d}A. \end{aligned} \tag{6.5}$$

We now use the form of transition probability rate (3.2). Because of the integrability properties of all multiplying functions involved, we can separate the integral $\mathcal{W}$ into the gain $\mathcal{W}^+$

$$\begin{aligned} \mathcal{W}^+ = \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} & f f_* \left(\langle v', I'\rangle^{2k} + \langle v'_*, I'_*\rangle^{2k}\right) \mathcal{B}(v, v_*, I, I_*, r, R, \sigma) \\ & \times \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R) R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma\, \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v, \end{aligned} \tag{6.6}$$

and loss part $\mathcal{W}^-$,

$$\begin{aligned} \mathcal{W}^- = \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} & f f_* \left(\langle v, I\rangle^{2k} + \langle v_*, I_*\rangle^{2k}\right) \mathcal{B}(v, v_*, I, I_*, r, R, \sigma) \\ & \times \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R) R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma\, \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v, \end{aligned} \tag{6.7}$$

so that

$$\mathcal{W} = \mathcal{W}^+ - \mathcal{W}^-. \tag{6.8}$$

We treat each term separately. For the gain part weak form, we use the bound from above stated in (3.2),

$$\begin{aligned} \mathcal{W}^+ \leq \frac{1}{2}\int_{(\mathbb{R}^{3+})^2} f f_* \tilde{\mathcal{B}}(v, v_*, I, I_*) \int_K & \left(\langle v', I'\rangle^{2k} + \langle v'_*, I'_*\rangle^{2k}\right) b(\hat{u}\cdot\sigma)\, d_\gamma^{ub}(r)\, e_\gamma^{ub}(R) \\ & \times \varphi_\alpha(r)\, \psi_\alpha(R)\, (1-R) R^{1/2}\, \mathrm{d}R\, \mathrm{d}r\, \mathrm{d}\sigma\, \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v. \end{aligned}$$

The Averaging Lemma 5.3 estimates the primed quantities averaged over the compact set K, and for the gain term it yields

$$\mathcal{W}^+ \leq \frac{\mathcal{C}_k}{2} \int_{(\mathbb{R}^{3+})^2} f f_* \left(|v - v_*|^\gamma + \left(\frac{I + I_*}{m} \right)^{\gamma/2} \right) \times \left(\left(\langle v, I \rangle^2 + \langle v_*, I_* \rangle^2 \right)^k \right) \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v, \quad (6.9)$$

Now the polynomial inequalities from Lemmas D.1 and D.2 yield

$$\begin{aligned} &\left(\langle v, I \rangle^2 + \langle v_*, I_* \rangle^2 \right)^k \\ &\leq \langle v, I \rangle^{2k} + \langle v_*, I_* \rangle^{2k} + \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} \left(\langle v, I \rangle^{2\ell} \langle v_*, I_* \rangle^{2(k-\ell)} + \langle v, I \rangle^{2(k-\ell)} \langle v_*, I_* \rangle^{2\ell} \right), \\ &\leq \langle v, I \rangle^{2k} + \langle v_*, I_* \rangle^{2k} + \tilde{c}_k \left(\langle v, I \rangle^2 \langle v_*, I_* \rangle^{2(k-1)} + \langle v, I \rangle^{2(k-1)} \langle v_*, I_* \rangle^2 \right), \quad (6.10) \end{aligned}$$

with

$$\sum_{\ell=1}^{\ell_k} \binom{k}{\ell} \leq 2^{\frac{k+2}{2}} - 1 =: \tilde{c}_k, \qquad \text{for } \ell_k = \lfloor \frac{k+1}{2} \rfloor. \quad (6.11)$$

Thus, the bound for $\mathcal{W}^+$ becomes

$$\begin{aligned} \mathcal{W}^+ \leq \frac{\mathcal{C}_k}{2} \int_{(\mathbb{R}^{3+})^2} f f_* \left(|v - v_*|^\gamma + \left(\frac{I + I_*}{m} \right)^{\gamma/2} \right) \Big(\langle v, I \rangle^{2k} + \langle v_*, I_* \rangle^{2k} \\ + \tilde{c}_k \left(\langle v, I \rangle^2 \langle v_*, I_* \rangle^{2(k-1)} + \langle v, I \rangle^{2(k-1)} \langle v_*, I_* \rangle^2 \right) \Big) \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v. \quad (6.12) \end{aligned}$$

Now we turn to the loss term $\mathcal{W}^-$ defined in (6.7). We first use the bound from below of the transition function $\mathcal{B}$ as stated in (3.2), and obtain

$$\mathcal{W}^- \geq \frac{\kappa^{lb}}{2} \int_{(\mathbb{R}^{3+})^2} f f_* \left(|v - v_*|^\gamma + \left(\frac{I + I_*}{m} \right)^{\gamma/2} \right) \times \left(\langle v, I \rangle^{2k} + \langle v_*, I_* \rangle^{2k} \right) \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v, \quad (6.13)$$

where the constant κ^{lb} is defined in (3.7). Gathering the estimates for the gain term (6.12) and for the loss term (6.13), the weak form $\mathcal{W}$ from (6.8) becomes

$$\begin{aligned} \mathcal{W} \leq \frac{1}{2} \int_{(\mathbb{R}^{3+})^2 \times K} f f_* \left(|v - v_*|^\gamma + \left(\frac{I + I_*}{m} \right)^{\gamma/2} \right) \Big\{ -\tilde{A}_{k_*} \left(\langle v, I \rangle^{2k} + \langle v_*, I_* \rangle^{2k} \right) \\ + \tilde{B}_k \left(\langle v, I \rangle^2 \langle v_*, I_* \rangle^{2(k-1)} + \langle v, I \rangle^{2(k-1)} \langle v_*, I_* \rangle^2 \right) \Big\} \mathrm{d}I_* \mathrm{d}v_* \mathrm{d}I \mathrm{d}v, \quad (6.14) \end{aligned}$$

with the uniform in k constant $\tilde{A}_{k_*}$, for k_* chosen in (6.2), defined by

$$\tilde{A}_{k_*} = \kappa^{lb} - \mathcal{C}_{k_*} > \tilde{A}_k > 0, \quad (6.15)$$

i.e. strictly positive, for large enough $k > k_*$, by virtue of (5.10). On the other hand, for $\tilde{c}_k$ as defined in (6.11), the constant $\tilde{B}_k$ bounded for each fixed k

$$\tilde{B}_k = \tilde{c}_k \, \mathcal{C}_k \geq 0. \quad (6.16)$$

In order to reach the conclusion of this Lemma, we remind on the moment notation

$$\mathfrak{m}_k(t) := \|f\|_{L^1_k}(t).$$

from the Definition 2.6.

Now for (6.14) we make of use the upper bound (5.26) and the lower bound c_{lb} from (4.5, 4.6) for the term

$$|v - v_*|^\gamma + \left(\frac{I + I_*}{m}\right)^{\gamma/2},$$

combined with the Coercive Lemma estimate (4.32), the right hand side in (6.14) is controlled by

$$\mathcal{W} \leq -c_{lb}\,\tilde{A}_{k_*}\,\mathfrak{m}_{k+\gamma/2} + 2^{\frac{3\gamma}{2}-1}\tilde{B}_k\left(\mathfrak{m}_{1+\gamma/2}\,\mathfrak{m}_{k-1} + \mathfrak{m}_{k-1+\gamma/2}\,\mathfrak{m}_1\right),\ k \geq k_*. \quad (6.17)$$

For the second term we use monotonicity of moments (2.31) implying

$$\mathfrak{m}_{k-1+\gamma/2} \leq \mathfrak{m}_k, \quad \text{since } 0 < \gamma \leq 2.$$

Then we invoke arguments of [5] that involve moment interpolation formulas

$$\mathfrak{m}_j \leq \mathfrak{m}_{j_1}^{\tau}\,\mathfrak{m}_{j_2}^{1-\tau}, \quad 0 \leq j_1 \leq j \leq j_2, \quad 0 < \tau < 1, \quad j = \tau j_1 + (1-\tau) j_2, \quad (6.18)$$

and Young's inequality estimate, for a given parameter ϵ to be chosen,

$$|a\,b| \leq \frac{1}{p\epsilon^{p/q}}\,|a|^p + \frac{\epsilon}{q}\,|b|^q, \quad \text{for } \epsilon > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1. \quad (6.19)$$

We first interpolate the moment $\mathfrak{m}_{1+\gamma/2}$ by applying (6.18) for the following choice $j = 1 + \gamma/2, \quad j_1 = 1, \quad j_2 = k + \gamma/2, \quad \tau = \frac{k-1}{k-1+\gamma/2}$, leading to the inequality

$$\mathfrak{m}_{1+\gamma/2} \leq \mathfrak{m}_1^{\frac{k-1}{k-1+\gamma/2}}\,\mathfrak{m}_{k+\gamma/2}^{\frac{\gamma/2}{k-1+\gamma/2}}. \quad (6.20)$$

Next, we interpolate the moment $\mathfrak{m}_{k-1}$, by taken $j = k-1, \quad j_1 = 0, \quad j_2 = k + \gamma/2, \quad \tau = \frac{1+\gamma/2}{k+\gamma/2}$, leading to

$$\mathfrak{m}_{k-1} \leq \mathfrak{m}_0^{\frac{1+\gamma/2}{k+\gamma/2}}\,\mathfrak{m}_{k+\gamma/2}^{\frac{k-1}{k+\gamma/2}}. \quad (6.21)$$

Therefore, (6.20) and (6.21) yield

$$\mathfrak{m}_{1+\gamma/2}\mathfrak{m}_{k-1} \leq \eta_k\,\mathfrak{m}_{k+\gamma/2}^{\theta_k}, \quad (6.22)$$

with the factor η_k depending solely on the zeroth and first moment of the initial data $f_0(v)$,

$$\eta_k := \eta_k(\mathfrak{m}_0, \mathfrak{m}_1) = \mathfrak{m}_0^{\frac{1+\gamma/2}{k+\gamma/2}}\,\mathfrak{m}_1^{\frac{k-1}{k-1+\gamma/2}}, \quad (6.23)$$

and

$$\theta_k = \frac{\gamma/2}{k-1+\gamma/2} + \frac{k-1}{k+\gamma/2} = 1 - \frac{k-1}{(k-1+\gamma/2)(k+\gamma/2)} < 1. \quad (6.24)$$

Taking estimates (6.22,6.23, 6.24) and recalling that $\tilde{B}_k = \tilde{c}_k\,\mathcal{C}_k \leq 2^{\frac{k+2}{2}}\mathcal{C}_k$ from (6.16) and (6.11), then inequality (6.17) can be estimated from above by

$$\mathcal{W} \leq -c_{lb}\,\tilde{A}_{k_*}\,\mathfrak{m}_{k+\gamma/2} + 2^{\frac{3\gamma+k}{2}}\mathcal{C}_k\left(\eta_k\,\mathfrak{m}_{k+\gamma/2}^{\theta_k}, +\mathfrak{m}_k\,\mathfrak{m}_1\right),\ k \geq k_*. \quad (6.25)$$

Next, in order to obtain a Bernoulli type of ODI for the time evolutions of the $k-$ Lebesgue moments associated to an a priori estimate to the solutions of the Cauchy problem (7.1), it is needed to absorb the positive contribution of $\mathfrak{m}_{k+\gamma/2}$ in the second term of the right hand side of (6.25) into the negative one. This is perform by means of the Young inequality (6.19). Indeed, recalling definition of the constant the estimate for the term $2^{\frac{3\gamma+k}{2}}\,\eta_k\,\mathfrak{m}_{k+\gamma/2}^{\theta_k}$, is performed by the following choice of parameters in the inequality (6.19)

$$a = 2^{\frac{3\gamma+k}{2}}\,\eta_k, \quad b = \mathfrak{m}_{k+\gamma/2}^{\theta_k}, \quad q = \frac{1}{\theta_k}, \text{ and } p = \frac{1}{1-\theta_k}\,,$$

yields

$$\begin{aligned} 2^{\frac{3\gamma}{2}-1}\tilde{c}_k\eta_k\,\mathfrak{m}_{k+\gamma/2}^{\theta_k} &\le \epsilon^{-\frac{\theta_k}{1-\theta_k}}(1-\theta_k)\left(2^{\frac{3\gamma+k}{2}}\eta_k\right)^{\frac{1}{1-\theta_k}} + \epsilon\,\theta_k\,\mathfrak{m}_{k+\gamma/2} \\ &\le \epsilon^{-\frac{\theta_k}{1-\theta_k}}\left(2^{\frac{3\gamma+k}{2}}\tilde{c}_k\eta_k\right)^{\frac{1}{1-\theta_k}} + \epsilon\,\mathfrak{m}_{k+\gamma/2}. \end{aligned} \tag{6.26}$$

Therefore, (6.25) becomes

$$\mathcal{W} \le -\left(c_{lb}\,\tilde{A}_{k_*} - \epsilon\,\mathcal{C}_k\right)\mathfrak{m}_{k+\gamma/2} + \tilde{D}_k\,\mathfrak{m}_1\,\mathfrak{m}_k,\ \ k \ge k_*, \tag{6.27}$$

where the constant $\tilde{D}_k$ is given by the upper bound of the following expression

$$\begin{aligned} &\max\left\{\epsilon^{-\frac{\theta_k}{1-\theta_k}}\left(2^{\frac{3\gamma}{2}-1}\tilde{c}_k\eta_k\right)^{\frac{1}{1-\theta_k}}\frac{1}{\mathfrak{m}_1}, 2^{\frac{3\gamma}{2}-1}\tilde{B}_k\right\} \\ &\qquad\qquad \le \mathcal{C}_k 2^{\frac{3\gamma+k}{2}}\max\left\{\left(\frac{2^{\frac{3\gamma+k}{2}}}{\epsilon}\right)^{\frac{\theta_k}{1-\theta_k}}\frac{\eta_k^{\frac{1}{1-\theta_k}}}{\mathfrak{m}_1}, 1\right\} =: \tilde{D}_k. \end{aligned} \tag{6.28}$$

By the inequality (5.18) that upper bounds the constant $\mathcal{C}_k$ for any $k > \bar{k}_*$, we obtain

$$\mathcal{W} \le -\left(c_{lb}\,\tilde{A}_{k_*} - \kappa^{lb}\,\epsilon\right)\mathfrak{m}_{k+\gamma/2} + \tilde{D}_k\,\mathfrak{m}_1\,\mathfrak{m}_k,\ \ k \ge k_*. \tag{6.29}$$

Next, choosing ϵ, a positive constant uniform in k, by setting

$$\epsilon := \frac{c_{lb}\,\tilde{A}_{k_*}}{2\,\kappa^{lb}} = \frac{c_{lb}}{2}\left(1 - \frac{\mathcal{C}_{k_*}}{\kappa^{lb}}\right) > 0, \tag{6.30}$$

where we have used a definition of $\tilde{A}_{k_*}$ from (6.15).

In particular, the $k-$moment of the collisional integral (6.31) is bounded by

$$\mathcal{W} \le -\frac{c_{lb}\,\tilde{A}_{k_*}}{2}\mathfrak{m}_{k+\gamma/2} + \tilde{D}_k\,\mathfrak{m}_1\,\mathfrak{m}_k, \quad k \ge k_*, \tag{6.31}$$

with $\tilde{D}_k$ as in (6.28) with specified ϵ from (6.30).

Even further, applying Jensen's inequality to each moment $\mathfrak{m}_{k+\gamma/2}$, $k > 1$,

$$\begin{aligned} &\int_{\mathbb{R}^{3+}} f(v,I)\,\langle v,I\rangle^{2k+\gamma}\,\mathrm{d}I\mathrm{d}v \\ &\qquad \ge \left(\int_{\mathbb{R}^{3+}} f(v,I)\mathrm{d}I\mathrm{d}v\right)^{-\frac{\gamma/2}{k}}\left(\int_{\mathbb{R}^{3+}} f(v,I)\,\langle v,I\rangle^{2k}\,\mathrm{d}I\mathrm{d}v\right)^{1+\frac{\gamma/2}{k}}, \end{aligned}$$

or in terms of moments,

$$\mathfrak{m}_{k+\gamma/2} \ge \mathfrak{m}_0^{-\frac{\gamma/2}{k}}\mathfrak{m}_k^{1+\frac{\gamma/2}{k}} = \|f\|_{L_0^1}^{-\frac{\gamma/2}{k}}\,\mathfrak{m}_k^{1+\frac{\gamma/2}{k}}.$$

Hence, In particular, the $k-$moment of the collisional integral (6.5) is bounded by (6.17) becomes

$$\mathcal{W} \le -\frac{c_{lb}\,\tilde{A}_{k_*}}{2}\,\mathfrak{m}_0^{-\frac{\gamma/2}{k}}\,\mathfrak{m}_k^{1+\frac{\gamma/2}{k}} + \tilde{D}_k\,\mathfrak{m}_1\,\mathfrak{m}_k,$$

where strictly positive the coercive constant c_{lb} is from Lemma 4.1 and also the strictly positive constant $\tilde{A}_{k_*}$ is from estimate (6.15) independent of k as k_* is fixed, depending on nonnegative factor calculated (5.10) from the Polyatomic Compact Manifold Averaging Lemma 5.3. Denoting

$$\begin{aligned} 0 < A_{k_*} &:= \frac{c_{lb}\,\tilde{A}_{k_*}}{2}\,\mathfrak{m}_0^{-\frac{\gamma/2}{k}} = \frac{c_{lb}}{2}\left(\kappa^{lb} - \mathcal{C}_{k_*}\right)\|f\|_{L^1_0}^{-\frac{\gamma/2}{k_*}}, \qquad \text{and} \\ B_k &:= \tilde{D}_k\,\mathfrak{m}_1 \qquad (6.32) \\ &= \mathcal{C}_k 2^{\frac{3\gamma+k}{2}} \max\left\{ \left(\frac{\kappa^{lb}\,2^{\frac{3\gamma}{2}+\frac{k}{2}}}{A_{k_*}}\right)^{\frac{\theta_k}{1-\theta_k}} \frac{\eta_k^{\frac{1}{1-\theta_k}}}{\|f\|_{L^1_1}}, 1 \right\} \|f\|_{L^1_1}\,, , \end{aligned}$$

with η_k as defined in (6.23) and θ_k as in (6.24), the last inequality concludes the proof of Lemma (6.1) and the constant identities (6.3).

□

When regularizing properties of the collision operator stated in Lemma 6.1 are combined with the Boltzmann equation (2.24), then we obtain ordinary differential inequality for L^1 polynomially weighted norms or polynomial moments $\mathfrak{m}_k$ in the sense of Definition 2.6.

Lemma 6.2 (Ordinary Differential Inequality for Polynomial Moment)**.** *If f is a solution of the Boltzmann equation for polyatomic gases (2.24) and $\|f\|_{L^1_k}$ its associated norm of order k, then, for any $k > k_*$, with k_* finite from (6.2), and $\gamma \in (0,2]$ the following estimate holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^1_k} = \mathfrak{m}_k\left[Q(f,f)\right] \le -A_{k_*}\|f\|_{L^1_k}^{1+\frac{\gamma/2}{k}} + B_k\|f\|_{L^1_k}, \qquad (6.33)$$

where both A_{k_*} and B_k are positive constants form Lemma 6.1, equations (6.15) and (6.16), respectively.

Remark 5. *The constant $A_{k_*} = \frac{c_{lb}}{2}\left(\kappa^{lb} - \mathcal{C}_{k_*}\right)\|f\|_{L^1_0}^{-\frac{\gamma/2}{k_*}}$, defined in (6.3) for k_* specify in (6.2), can be identified as the k independent* coercive factor. *This strictly positive factor, which controls the lower bound to the absorption term on the moments inequality (6.33), provides a sufficient condition to proceed next with Theorem 6.3 yielding the global in time propagation and generation of k^{th}-moments of any order $k > k_*$, provided that the initial data $f_0(v,I)$ has positive mass, positive energy, as much as satisfies conditions of the Lower Bound Lemma 4.1.*

While these estimates are obtained without the need of entropy estimates, yet, if the initial entropy is bounded, the constructed solutions will have well defined entropy (2.25) that remains bounded for all times by the initial one.

Proof. In order to get ODI for the evolution of $\|f\|_{L^1_k}(t) = \mathfrak{m}_k(t)$ defined in 2.6, we integrate the Boltzmann equation (2.24) over the space $(v,I) \in \mathbb{R}^{3+}$ against test function

$$\chi(v,I) = \langle v, I\rangle^{2k}.$$

Using the weak form (2.23), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^1_k} = \frac{\mathrm{d}\,\mathfrak{m}_k}{\mathrm{d}t} = \int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\,\langle v, I\rangle^{2k}\mathrm{d}I\mathrm{d}v.$$

Applying Lemma 6.1 on the right-hand side we get the desired estimate. □

This differential inequality by means of a comparison principle for ODEs implies the following Theorem.

Theorem 6.3 (Generation and propagation of polynomial moments)**.** *If f a solution of the Boltzmann equation (2.24) with the transition function from Assumption 3.1 , and the constants $A_{k_*} > 0$ and $0 \leq B_k$ bounded , for all $k > k_*$, as defined in (6.33), then the following properties hold.*

1. *(Generation) Let the initial data $f_0(v, I) \in L^1_{k_*}(\mathbb{R}^{3+})$, i.e. $\mathfrak{m}_0[f](0) < \infty$, then there is a constant $\mathfrak{C}^{\mathfrak{m}}$ uniformly in $k > k_*$, k_* from (6.2), such that for any $\gamma \in (0, 2]$,*

$$\mathfrak{m}_k[f](t) \leq \mathfrak{B}_k \ \max\{1, t^{-\frac{k}{\gamma/2}}\}, \quad \forall t > 0, \quad \textit{with} \tag{6.34}$$

$$\mathfrak{B}_k = \left(\frac{B_k}{A_{k_*}}\right)^{\frac{k}{\gamma/2}} \max\left\{\left(\frac{\gamma B_k}{2k}\right)^{-\frac{2k}{\gamma}} e^{\frac{B_k}{2}}, \left(1 - e^{-\frac{\gamma B_k}{2k}}\right)^{-\frac{2k}{\gamma}}\right\}, \quad \textit{for any } k > k_*.$$

2. *(Propagation) Moreover, if $\mathfrak{m}_k[f](0) < \infty$, then*

$$\mathfrak{m}_k[f](t) \leq \max\left\{\left(\frac{B_k}{A_{k_*}}\right)^{\frac{2k}{\gamma}}, \mathfrak{m}_k[f](0)\right\}, \quad \textit{for all } t \geq 0. \tag{6.35}$$

Proof. The aim of the proof is to associate an ODE of Bernoulli type to the derived ODI (6.33) from Lemma 6.2. A comparison or maximum principle argument for Ordinary Differential Equation argument enables us to establish that that solution to the initial value to the following family of ODEs, for each fixed $k \geq k_*$, and time $t > 0$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_k[f](t) = \mathfrak{m}_k[Q(f,f)](t), \\ \mathfrak{m}_k[f](0) = y_0, \end{cases} \tag{6.36}$$
$$\text{with} \ \ \mathfrak{m}_k[Q(f,f)](t) \leq -A_{k_*} \|f\|^{1+\frac{\gamma/2}{k}}_{L^1_k}(t) + B_k \|f\|_{L^1_k}(t).$$

Thus, we solve the auxiliary upper ODE with the same initial data as in (6.36),

$$\begin{cases} y'(t) = -a\, y(t)^{1+c} + b\, y(t)\,, \qquad \text{for } a, \ b > 0, \\ y(0) = y_0, \end{cases} \tag{6.37}$$
$$\text{with} \quad a := A_{k_*}, \ b := B_k, \ c := \gamma/(2k), \ \text{for each same } k > k_* \text{ fixed.}$$

That means any solution $y(t)$ of the initial value problem to (6.37) controls from above any solution $\mathfrak{m}_k[f](t)$, to the initial value problem for the ODI from (6.36), with the same initial data for both problems. That implies $y(0) \geq \mathfrak{m}_k[f](0) > 0$

In addition solutions to $y(t)$ of the super linear initial value problem given in (6.37), are of Bernoulli type, that is they have a unique explicit solution of the form

$$y(t) = \left(\frac{a}{b}\left(1 - e^{-c\,b\,t}\right) + y_0^{-c} e^{-c\,b\,t}\right)^{-\frac{1}{c}}. \tag{6.38}$$

Yet, a simple argument allow us to describe the solution analytical properties. Indeed, the ODE (6.37) has a unique stationary point at $(b/a)^{1/c}$, since the initial data y_0 positive and finite, will monotonically increase whenever $y_0 < (b/a)^{1/c}$, monotonically decrease whenever $y_0 > (b/a)^{1/c}$, and both cases will converge for larger time t, to the stationary value when the initial data $y_0 \equiv (b/a)^{1/c}$, making the uniform in time estimate

$$y(t) \leq \max\{y(0), (b/a)^{1/c}\}, \qquad \forall t \geq 0. \tag{6.39}$$

Then, by the maximum principle for ODE's the solution of (6.37), controls from above the solution of moments ODIs (6.33).

Therefore, if $0 < \mathfrak{m}_k[f](0) < y_0$, for $k > k_*$, with k_* from (6.2), estimate (6.39) implies

$$\mathfrak{m}_k[f](t) \leq \max\left\{ \left(\frac{B_k}{A_{k_*}}\right)^{\frac{2k}{\gamma}}, \mathfrak{m}_k[f](0) \right\}, \quad \text{for all } t > 0, \tag{6.40}$$

with which means the polynomial k-moments are bounded uniformly in time, if initially so. Thus, the propagation result (6.35) follows.

However, the moment's generation property is different since one needs to show $\mathfrak{m}_k[f](t) \leq y(t)$, with $k > k_*$, with an initial data corresponds to a lower order moment that k, that is $y_0 := \mathfrak{m}_{k_*}[f](0)$ finite, but not necessarily the value of $\mathfrak{m}_k[f](0)$ which could be infinity. In this case we use the exact from of the solution of the Bernoulli type ODE (6.38) in order to search for an upper bound to $y(t)$ independent of the value of y_0. In fact estimating from above the right hand side of the $y(t)$ solution (6.38) independently from the initial data y_0, yields

$$y(t) < \left(\frac{a}{b}\left(1 - e^{-c\,b\,t}\right)\right)^{-\frac{1}{c}} \leq \left(\frac{a}{b}\right)^{-\frac{1}{c}} \begin{cases} (c\,b)^{-\frac{1}{c}}\, e^{\frac{b}{2}}\, t^{-\frac{1}{c}}, & t < 1 \\ \left(1 - e^{-c\,b}\right)^{-\frac{1}{c}}, & t \geq 1. \end{cases}$$

Therefore, an upper bound to $\mathfrak{m}_k[f](t)$ uniformly in time readily follows, for any $k > k_*$ and initial finite data for just $\mathfrak{m}_{k_*}[f](0)$, clearly yields

$$\mathfrak{m}_k[f](t) \leq \mathfrak{B}_k \ \max\{1, t^{-\frac{k}{\gamma/2}}\}, \quad \forall t > 0,$$

where the constant is

$$\mathfrak{B}_k = \left(\frac{B_k}{A_{k_*}}\right)^{\frac{k}{\gamma/2}} \max\left\{ \left(\frac{\gamma B_k}{2k}\right)^{-\frac{2k}{\gamma}} e^{\frac{B_k}{2}}, \left(1 - e^{-\frac{\gamma B_k}{2k}}\right)^{-\frac{2k}{\gamma}} \right\}, \qquad \text{for any } k > k_*,$$

as stated in (6.34). Thus, Theorem's 6.3 proof is now complete.

□

7. Existence and Uniqueness Theory

It this Section, we establish an existence and uniqueness theorem that solves the Cauchy problem

$$\begin{cases} \partial_t f(t,v,I) = Q(f,f)(v,I) \\ f(0,v,I) = f_0(v,I), \end{cases} \tag{7.1}$$

under the Assumption 3.1 on the transition function $\mathcal{B}$.

The invariant region Ω to solve the Cauchy problem for Boltzmann equation for polyatomic gases. Our goal is to set conditions on initial data that ensures existence and uniqueness of the solution to the Cauchy problem (7.1) associated to the Boltzmann equation under conditions described in Section 3.

These conditions will include moments with physical interpretation of mass and energy of the gas, and the imposed restrictions will be physically relevant. For instance, we will necessitate bounded mass and energy both from below and above, thus excluding zero and infinitely large mass and energy. Moreover, we will require bounded moment of order

$$k_* := \max\left\{\bar{k}_*, 1+\gamma, 1+\delta/2\right\}, \tag{7.2}$$

for $\gamma \in (0,2]$ related to the potential of the transition function (3.1), $\delta > 0$ is from the lower bound (4.4) and $\bar{k}_*$ from (6.2) is sufficiently large to ensure the prevail of the polynomial moments of loss term with respect to those same moments of the gain term. Such $\bar{k}_*$ depends on γ, α and the model of transition function at hand. Now we define the bounded, convex and closed subset $\Omega \subseteq L^1_1$,

$$\begin{aligned}\Omega = \Big\{ f(v,I) \in L^1_1 : f \geq 0 \text{ in } (v,I), \quad \int_{\mathbb{R}^{3+}} v\, f \,\mathrm{d}I\,\mathrm{d}v = 0, \\ \exists\, C_0, C_1, \text{ such that } \forall t \geq 0,\ \mathfrak{m}_0[f](t) = C_0,\ \mathfrak{m}_1[f](t) = C_1, \\ \mathfrak{m}_{k_*}[f](t) \leq C_{k_*}, \text{ with } k_* \text{ from (7.2)} \Big\}.\end{aligned} \tag{7.3}$$

The value of k_* which determines how many moments need to be bounded initially in order to guarantee existence and uniqueness to the Cauchy problem (7.1) is strongly related to the evolution of of collision operator $Q(f,f)$, $\mathfrak{m}_k[Q(f,f)]$. More precisely, after the estimates obtained in Sections 4, 6 and 5, the a priori estimates applied in Lemma 6.2, enable us to study an upper bound for $\mathfrak{m}_k[Q(f,f)]$. That is, equivalent to study the map associated to the right hand side of Ordinary Differential Inequality (6.33),

More precisely, we set $x := \mathfrak{m}_k(t)$, and consider the map $\mathcal{L}_k(x) : [0,\infty) \to \mathbb{R}$,

$$\mathfrak{m}_k[Q(f,f)] = \int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\,\langle v,I\rangle^{2k} \mathrm{d}I\mathrm{d}v \leq\ \mathcal{L}_k(x) := -A_{k_*}\, x^{1+\frac{\gamma/2}{k}} + B_k x, \tag{7.4}$$

where A_{k_*} is strictly positive and independent of k, and B_k non-negative constants for any $\gamma > 0$ for $k > k_*$ with k_* from (7.2).

The next result follows.

Theorem 7.1 (Existence and Uniqueness)**.** *Assume that* $f(0,v,I) = f_0(v,I) \in \Omega$. *Then the Boltzmann equation* (7.1) *for the transition function* $\mathcal{B}$ *under the Assumption 3.1 has the unique solution in* $\mathcal{C}\left([0,\infty),\Omega\right) \cap \mathcal{C}^1\left((0,\infty), L^1_1\right)$.

Proof. The goal is apply general ODE theory from Appendix E, that is to study collision operator Q as mapping $Q : \Omega \to L^1_1$, and to show

(1) Hölder continuity condition

$$\|Q(f,f) - Q(g,g)\|_{L^1_1} \leq C_H \|f-g\|^{1/2}_{L^1_1}, \tag{7.5}$$

(2) Sub-tangent condition

$$\lim_{h\to 0+} \frac{\mathrm{dist}\left(f + hQ(f,f), \Omega\right)}{h} = 0,$$

where

$$\mathrm{dist}\left(\mathbb{H}, \Omega\right) = \inf_{\omega\in\Omega} \|h-\omega\|_{L^1_1},$$

(3) One-sided Lipschitz condition

$$[Q(f,f) - Q(g,g), f-g] \leq C_L \|f-g\|_{L^1_1}, \tag{7.6}$$

where brackets $[\cdot,\cdot]$ by Remark 6 become

$$\begin{aligned}
&[Q(f,f)-Q(g,g),f-g] \\
&\quad = \lim_{h\to 0^-} \frac{\left(\|(f-g)+h\,(Q(f,f)-Q(g,g))\|_{L^1_1}-\|(f-g)\|_{L^1_1}\right)}{h} \\
&\quad \leq \int_{\mathbb{R}^{3+}} (Q(f,f)(v,I)-Q(g,g)(v,I))\,\mathrm{sign}\,(f(v,I)-g(v,I))\,\langle v,I\rangle^2\,\mathrm{d}I\,\mathrm{d}v.
\end{aligned}$$

First, we check $Q:\Omega\to L^1_1$ is well defined. Indeed, for any $f\in\Omega$, using $|\cdot|=\cdot\,\mathrm{sign}(\cdot)$

$$\begin{aligned}
\|Q(f,f)\|_{L^1_1} &= \int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\,\mathrm{sign}\,(Q(f,f)(v,I))\,\langle v,I\rangle^2\,\mathrm{d}I\,\mathrm{d}v \\
&\leq \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} \frac{ff_*}{(II_*)^\alpha}\left(\langle v',I'\rangle^2+\langle v'_*,I'_*\rangle^2+\langle v,I\rangle^2+\langle v_*,I_*\rangle^2\right)\mathrm{d}A
\end{aligned}$$

by virtue of the weak form (2.23) for the test function

$$\chi(v,I)=\mathrm{sign}\,(Q(f,f)(v,I))\,\langle v,I\rangle^2.$$

Using microscopic energy law (2.2) and the form of transition function (3.2) with the multiplying functions from above together with the upper bound (5.26), as much as monotonicity of moments (2.31) we get

$$\begin{aligned}
\|Q(f,f)\|_{L^1_1} &\leq 2^{\frac{3\gamma}{2}-1}\kappa^{ub}\int_{(\mathbb{R}^{3+})^2} ff_*\left(\langle v,I\rangle^2+\langle v_*,I_*\rangle^2\right)\left(\langle v,I\rangle^\gamma+\langle v_*,I_*\rangle^\gamma\right)\mathrm{d}I_*\,\mathrm{d}v_*\mathrm{d}I\,\mathrm{d}v \\
&= 2^{\frac{3\gamma}{2}}\kappa^{ub}\left(\|f\|_{L^1_{1+\gamma/2}}\|f\|_{L^1_0}+\|f\|_{L^1_1}\|f\|_{L^1_{\gamma/2}}\right)\leq 2^{\frac{3\gamma}{2}+1}\kappa^{ub}\,\|f\|_{L^1_{1+\gamma/2}}\|f\|_{L^1_1},
\end{aligned}$$

with κ^{ub} from (3.7). Since $f\in\Omega$, the right hand side is bounded, and thus $Q(f,f)\in L^1_1$.

Then, the proof consists in three parts.

Part I: Hölder continuity condition. We first rewrite difference of the two collision operators acting on distribution functions f and g as collision operator on sums and differences of these two distribution functions,

$$Q(f,f)-Q(g,g)=\frac{1}{2}\left(Q(f+g,f-g)+Q(f-g,f+g)\right), \tag{7.7}$$

by virtue of the bilinear structure of the strong form of collision operator (2.13). Using this relation, we write the L^1_1 norm

$$\begin{aligned}
\mathcal{I}_H := \|Q(f,f)-Q(g,g)\|_{L^1_1} &= \int_{\mathbb{R}^{3+}} |Q(f,f)-Q(g,g)|\,\langle v,I\rangle^2\,\mathrm{d}I\,\mathrm{d}v \\
&\leq \frac{1}{2}\int_{\mathbb{R}^{3+}} \left(|Q(f+g,f-g)|+|Q(f-g,f+g)|\right)\langle v,I\rangle^2\,\mathrm{d}I\,\mathrm{d}v.
\end{aligned}$$

The absolute value of collision operators will be rewritten in terms of the sign function using $|\cdot|=\cdot\,\mathrm{sign}(\cdot)$, that will be understood as a test function. This implies

$$\mathcal{I}_H \leq \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} \left((f(v,I)+g(v,I))\,|f(v_*,I_*)-g(v_*,I_*)|\right.$$
$$+\,|f(v,I)-g(v,I)|\,(f(v_*,I_*)+g(v_*,I_*))\big)$$
$$\times\left(\langle v',I'\rangle^2+\langle v'_*,I'_*\rangle^2+\langle v,I\rangle^2+\langle v_*,I_*\rangle^2\right)$$
$$\times\,\mathcal{B}\,\varphi_\alpha(r)\,\psi_\alpha(R)\,(1-R)R^{1/2}\,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}I_*\,\mathrm{d}v_*\mathrm{d}I\,\mathrm{d}v$$
$$=\int_{(\mathbb{R}^{3+})^2\times K}\left((f(v,I)+g(v,I))\,|f(v_*,I_*)-g(v_*,I_*)|\right.$$
$$+\,|f(v,I)-g(v,I)|\,(f(v_*,I_*)+g(v_*,I_*))\big)\left(\langle v,I\rangle^2+\langle v_*,I_*\rangle^2\right)$$
$$\times\,\mathcal{B}\,\varphi_\alpha(r)\,\psi_\alpha(R)\,(1-R)R^{1/2}\,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\mathrm{d}I_*\,\mathrm{d}v_*\mathrm{d}I\,\mathrm{d}v,$$

and the last equality is by energy conservation law during collision (2.2). Now we make use of the transition function (3.2), its bound from above (5.26), and the upper bound constant κ^{ub} for the integration over the compact manifold K given in (3.7),

$$\mathcal{I}_H \leq 2^{\frac{3\gamma}{2}-1}\kappa^{ub}\int_{(\mathbb{R}^{3+})^2}\left((f(v,I)+g(v,I))\,|f(v_*,I_*)-g(v_*,I_*)|\right.$$
$$+\,|f(v,I)-g(v,I)|\,(f(v_*,I_*)+g(v_*,I_*))\big)$$
$$\times\left(\langle v,I\rangle^2+\langle v_*,I_*\rangle^2\right)\left(\langle v,I\rangle^\gamma+\langle v_*,I_*\rangle^\gamma\right)\mathrm{d}I_*\,\mathrm{d}v_*\mathrm{d}I\,\mathrm{d}v. \quad (7.8)$$

We rewrite (7.8) in moment notation,

$$\mathcal{I}_H \leq 2^{\frac{3\gamma}{2}}\kappa^{ub}\Big(\|f+g\|_{L^1_{1+\gamma/2}}\|f-g\|_{L^1_0}+\|f+g\|_{L^1_1}\|f-g\|_{L^1_{\gamma/2}}$$
$$+\|f+g\|_{L^1_{\gamma/2}}\|f-g\|_{L^1_1}+\|f+g\|_{L^1_0}\|f-g\|_{L^1_{1+\gamma/2}}\Big).$$

Furthermore, monotonicity of norms (2.31) yields

$$\mathcal{I}_H \leq 2^{\frac{3\gamma}{2}+1}\kappa^{ub}\,\|f-g\|_{L^1_{1+\gamma/2}}\left(\|f+g\|_{L^1_{1+\gamma/2}}+\|f+g\|_{L^1_1}\right).$$

Next, we use interpolation inequality (D.1) on $\|f-g\|_{L^1_{1+\gamma/2}}$ and get

$$\|f-g\|_{L^1_{1+\gamma/2}} \leq \|f-g\|_{L^1_1}^{1/2}\,\|f-g\|_{L^1_{1+\gamma}}^{1/2}.$$

Moreover, characterization of the set Ω gives the following bounds

$$\|f-g\|_{L^1_{1+\gamma}}^{1/2} \leq \|f\|_{L^1_{1+\gamma}}^{1/2}+\|g\|_{L^1_{1+\gamma}}^{1/2} \leq 2\,C_{1+\gamma}^{1/2},$$

and

$$\|f+g\|_{L^1_{1+\gamma/2}} \leq 2\,C_{1+\gamma/2}, \quad \|f+g\|_{L^1_1} \leq 2\,C_1.$$

Finally, denoting

$$C_H := 2^{\frac{3\gamma}{2}+3}\kappa^{ub}\,C_{1+\gamma}^{1/2}\left(C_{1+\gamma/2}+C_1\right),$$

we get desired estimate (7.5).

Part II: Sub-tangent condition. We first study the collision frequency

$$\nu(f)(v,I) := \int_{\mathbb{R}^{3+}\times K} f(v_*,I_*)\mathcal{B}\,\varphi_\alpha(r)\psi_\alpha(R)(1-R)R^{1/2}\,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}I_*\,\mathrm{d}v_*.$$

Using the form (3.2) of the transition function $\mathcal{B}$ together with its bound from above (5.26), we obtain

$$\begin{aligned}\nu(f)(v,I) &\leq 2^{\frac{3\gamma}{2}-1}\kappa^{ub}\int_{\mathbb{R}^{3+}} f(v_*,I_*)\left(\langle v,I\rangle^\gamma + \langle v_*,I_*\rangle^\gamma\right)\,\mathrm{d}I_*\,\mathrm{d}v_* \\ &\leq 2^{\frac{3\gamma}{2}-1}\kappa^{ub}\left(C_0\langle v,I\rangle^\gamma + C_{\gamma/2}\right) \\ &\leq 2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_0 + C_{\gamma/2}\right)\left(1+\left(\frac{1}{2}\left|v\right|^2 + \frac{I}{m}\right)^{\gamma/2}\right), \end{aligned} \tag{7.9}$$

with κ^{ub} from (3.7) and using characterization of the invariant region Ω as in (7.3).

The idea of the proof of sub-tangent condition is to prove that for $f \in \Omega$ and for any $\epsilon > 0$ there exists $h_1 > 0$ such that the interval $\mathcal{N}$ centered at $f + hQ(f,f)$ with radius $h\epsilon$ denoted by

$$\mathcal{N}(f + hQ(f,f), h\epsilon), \tag{7.10}$$

has non-empty intersection with Ω for any $0 < h < h_1$, as formulated in the Proposition 1 below. Then for such h_1 we have

$$h^{-1}\mathrm{dist}\left(f + hQ(f,f), \Omega\right) \leq \epsilon,$$

for all $0 < h < h_1$, which concludes the sub-tangent condition. Therefore, it remains to prove the following Proposition 1.

Proposition 1. *Fix $f \in \Omega$. Then for any $\epsilon > 0$ there exists $h_1 > 0$ such that*

$$\mathcal{N}(f + hQ(f,f), h\epsilon) \cap \Omega \neq \emptyset, \tag{7.11}$$

for any $0 < h < h_1$.

Proof. We recall the definition of the semi-sphere in the velocity-internal energy space (4.9),

$$B_\rho(0,0) := \left\{ (v,I) \in \mathbb{R}^3 \times [0,\infty) : \sqrt{\frac{1}{2}\left|v\right|^2 + \frac{I}{m}} \leq \rho \right\}.$$

Then with the help of the characteristic function of this sphere $B_\rho(0,0)$, we define truncated distribution function

$$f_\rho(t,v,I) := f(t,v,I)\,\mathbb{1}_{B_\rho(0,0)}(v,I), \tag{7.12}$$

and consider the following function

$$g_\rho = f + h\,Q(f_\rho,f_\rho), \quad \text{for } h > 0. \tag{7.13}$$

Our goal is to find ρ and h so that $g_\rho \in \mathcal{N}(f + hQ(f,f), h\epsilon) \cap \Omega$.

We first note that for any $f \in \Omega$, its truncation $f_\rho \in \Omega$ as well. Then using definition (2.19) yields

$$\begin{aligned} Q(f_\rho,f_\rho)(v,I) &\geq Q^-(f_\rho,f_\rho)(v,I) \\ &= -f_\rho \int_{\mathbb{R}^{3+}\times K} f_{\rho *}\,\mathcal{B}\,\varphi_\alpha(r)\,\psi_\alpha(R)\,(1-R)R^{1/2}\,\mathrm{d}R\,\mathrm{d}r\,\mathrm{d}\sigma\,\mathrm{d}I_*\,\mathrm{d}v_*, \end{aligned}$$

since the gain term is positive. Next, using an upper bound on the collision frequency (7.9), yields the following lower bound estimate on the collision operator acting on f_ρ,

$$Q(f_\rho,f_\rho)(v,I) \geq -2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_0+C_{\gamma/2}\right)\left(1+\left(\frac{1}{2}|v|^2+\frac{I}{m}\right)^{\gamma/2}\right)f_\rho$$
$$\geq -2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_0+C_{\gamma/2}\right)(1+\rho^\gamma)\,f.$$

Therefore, for g_ρ we can bound

$$g_\rho \geq f\left(1-2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_0+C_{\gamma/2}\right)(1+\rho^\gamma)\,h\right)\geq 0,$$

for any $h\in(0,\frac{1}{2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_0+C_{\gamma/2}\right)(1+\rho^\gamma)})$.

Next, weak form (2.23) implies

$$\int_{\mathbb{R}^{3+}} Q(f_\rho,f_\rho)(v,I)\,\mathrm{d}I\,\mathrm{d}v=0,\qquad \int_{\mathbb{R}^{3+}} Q(f_\rho,f_\rho)(v,I)\langle v,I\rangle^2\,\mathrm{d}I\,\mathrm{d}v=0,$$

which yields

$$\mathfrak{m}_0[g_\rho](t)=\mathfrak{m}_0[f](t),\quad \mathfrak{m}_1[g_\rho](t)=\mathfrak{m}_1[f](t),$$

independently of ρ and h. Then upper and lower bounds for these polynomial moments of f imply the same kind of estimates for g_ρ.

Finally, as anticipated at the opening of this section, we prove that $L^1_{k_*}$ norm of g_ρ is bounded.

To this end we study the the map $\mathcal{L}_k(x):[0,\infty)\to\mathbb{R}$ from (7.4),

$$\mathcal{L}_k(x):=-A_{k_*}x^{1+\frac{\gamma/2}{k}}+B_kx,$$

where A_{k_*} and B_k are positive constants for any $\gamma>0$, $k>k_*$, as described in (7.4). That means, this map has only one root, denoted by

$$\hat{x}_k:=\left(\frac{B_k}{A_{k_*}}\right)^{\frac{\gamma/2}{k+\gamma/2}},\quad k\geq k_*, \tag{7.14}$$

at which $\mathcal{L}_k(x)$ changes from positive to negative.

Thus, we set, for any $x\geq 0$,

$$\mathcal{L}_k(x)\leq \max_{0\leq x\leq \hat{x}_{\gamma,k}}\mathcal{L}_k(x)=:\hat{\mathcal{L}}_k,$$

since such maximum can be explicitly computed by

$$\hat{\mathcal{L}}_k\ :=\ \mathcal{L}_k\left(\left(\frac{B_kk}{A_{k_*}(k+\gamma/2)}\right)^{\frac{k}{\gamma/2}}\right)\ =\ \mathcal{L}_k\left(\hat{x}_k^{\frac{k+\gamma/2}{\gamma/2}}\frac{k}{k+\gamma/2}\right)^{\frac{k}{\gamma/2}}\ :=\ K_{\gamma,k} \tag{7.15}$$

and $K_{\gamma,k}$ also depends on the initial mass, energy and $\|f_0\|_{L^1_{k_*}}$.

In particular, from (6.1) it follows that, for any $k>1$

$$\int_{\mathbb{R}^{3+}} Q(f,f)(v,I)\,\langle v,I\rangle^k\mathrm{d}I\mathrm{d}v\leq \mathcal{L}_k\left(\mathfrak{m}_k[f]\right)\leq\hat{\mathcal{L}}_{\gamma,k}.$$

Next, define

$$C_k=\hat{x}_k+\hat{\mathcal{L}}_{\gamma,k}. \tag{7.16}$$

For any $f\in\Omega$, we have two possibilities: either (i) $\mathfrak{m}_k[f]\leq\hat{x}_k$ or (ii) $\mathfrak{m}_k[f]>\hat{x}_k$. In the first case, for the k-moment of g_ρ we get

$$\mathfrak{m}_k[g_\rho]\leq\hat{x}_k+h\int_{\mathbb{R}^{3+}}Q(f_\rho,f_\rho)(v,I)\,\langle v,I\rangle^{2k}\mathrm{d}I\mathrm{d}v\leq\hat{x}_k+h\hat{\mathcal{L}}_{\gamma,k}\leq C_k,$$

where we have used $h \leq 1$, without loss of generality.

In the second case, we take $\rho = \rho(f) > 0$ large enough to ensure $\mathfrak{m}_k[f_\rho] > \hat{x}_k$ as well. In that case, $\mathcal{L}_k$ is negative, i.e.

$$\mathcal{L}_k\left(\mathfrak{m}_k[f_\rho]\right) \leq 0.$$

Therefore,

$$\mathfrak{m}_k[g_\rho] \leq \hat{x}_k \leq C_k, \quad k \geq k_*.$$

Therefore, in either case $\mathfrak{m}_k[g_\rho]$ is bounded, and moreover we have constructed the constant of boundedness C_k for any $k \geq k_*$.

We conclude that $g_\rho \in \Omega$ provided that $0 < h < h_*$,

$$h_* = \min\left\{1, \frac{1}{2^{\frac{3\gamma}{2}} \kappa^{ub}\left(C_0 + C_{\gamma/2}\right)(1+\rho(f)^\gamma)}\right\}$$

On the other hand, let us show that $g_\rho \in B(f + hQ(f,f), h\epsilon)$. From the Hölder estimate (7.5) we get

$$h^{-1}\left\|f + hQ(f,f) - g_\rho\right\|_{L^1_1} = \left\|Q(f,f) - Q(f_\rho, f_\rho)\right\|_{L^1_1} \leq C_H \left\|f - f_\rho\right\|_{L^1_1}^{1/2} \leq \epsilon,$$

for $\rho = \rho(\epsilon)$ large enough. Thus, for this choice of ρ, we have $g_\rho \in B(f + hQ(f,f), h\epsilon)$.

Finally, we conclude the proof of the Proposition by choosing

$$\rho = \max\left\{\rho(f), \rho(\epsilon)\right\}, \text{ and } h_1 = \min\left\{1, \frac{1}{2\,\kappa\left(C_0 + C_{\gamma/2}\right)(1+\rho^\gamma)}\right\}.$$

□

Part III: One-sided Lipschitz condition. The left hand side of (7.6) after use of representation (7.7) becomes

$$\begin{aligned}
\mathcal{I}_L &:= [Q(f,f) - Q(g,g), f-g] \\
&\leq \int_{\mathbb{R}^{3+}} \left(Q(f,f)(v,I) - Q(g,g)(v,I)\right) \operatorname{sign}\left(f(v,I) - g(v,I)\right) \langle v, I\rangle^2 \,\mathrm{d}I\,\mathrm{d}v \\
&\qquad \leq \frac{1}{2}\int_{\mathbb{R}^{3+}} \left(Q(f+g, f-g)(v,I) + Q(f-g, f+g)(v,I)\right) \\
&\qquad\qquad \times \operatorname{sign}\left(f(v,I) - g(v,I)\right) \langle v, I\rangle^2 \,\mathrm{d}I\,\mathrm{d}v.
\end{aligned}$$

Using the weak form (2.23), we get

$$\begin{aligned}
\mathcal{I}_L \leq \frac{1}{4}\int_{(\mathbb{R}^{3+})^2\times K} &\left(\frac{(f+g)(f_*-g_*)}{(II_*)^\alpha} + \frac{(f-g)(f_*+g_*)}{(II_*)^\alpha}\right) \\
&\times \Big(\operatorname{sign}\left(f'-g'\right)\langle v', I'\rangle^2 + \operatorname{sign}\left(f'_*-g'_*\right)\langle v'_*, I'_*\rangle^2 \\
&\quad -\operatorname{sign}\left(f-g\right)\langle v, I\rangle^2 - \operatorname{sign}\left(f_*-g_*\right)\langle v_*, I_*\rangle^2\Big)\,\mathrm{d}A
\end{aligned}$$

We bound sign function by 1 for the first two terms, and from the last two terms we use $|\cdot| = \cdot\,\mathrm{sign}(\cdot)$ where applicable,

$$\begin{aligned}\mathcal{I}_L \leq \frac{1}{4}\int_{(\mathbb{R}^{3+})^2\times K} \Big\{ &\left((f+g)\left|f_* - g_*\right| + \left|f-g\right|(f_*+g_*)\right) \times \left(\langle v', I'\rangle^2 + \langle v'_*, I'_*\rangle^2\right) \\ &+ \left((f+g)\left|f_* - g_*\right| - \left|f-g\right|(f_*+g_*)\right)\langle v, I\rangle^2 \\ &+ \left(-(f+g)\left|f_* - g_*\right| + \left|f-g\right|(f_*+g_*)\right)\langle v_*, I_*\rangle^2 \Big\} \frac{\mathrm{d}A}{(II_*)^\alpha}.\end{aligned}$$

Using the energy collision law (2.2), after cancellations of some terms we get

$$\begin{aligned}\mathcal{I}_L &\leq \frac{1}{2}\int_{(\mathbb{R}^{3+})^2\times K} \left((f+g)\left|f_* - g_*\right|\langle v, I\rangle^2 + \left|f-g\right|(f_*+g_*)\langle v_*, I_*\rangle^2\right)\frac{\mathrm{d}A}{(II_*)^\alpha} \\ &= \int_{(\mathbb{R}^{3+})^2\times K} \left|f-g\right|(f_*+g_*)\langle v_*, I_*\rangle^2 \frac{\mathrm{d}A}{(II_*)^\alpha},\end{aligned}$$

the last equality is due to the change of variables (2.17) in the first integral. We make use of the transition function $\mathcal{B}$ assumption (3.2) and the upper bound (5.26),

$$\begin{aligned}\mathcal{I}_L &\leq C_K \int_{(\mathbb{R}^{3+})^2} \left|f-g\right|(f_*+g_*)\langle v_*, I_*\rangle^2 \left(\langle v, I\rangle^\gamma + \langle v_*, I_*\rangle^\gamma\right)\mathrm{d}I_*\,\mathrm{d}v_*\,\mathrm{d}I\,\mathrm{d}v \\ &= 2^{\frac{3\gamma}{2}-1}\kappa^{ub}\left(\left\|f-g\right\|_{L^1_{\gamma/2}}\left\|f+g\right\|_{L^1_1} + \left\|f-g\right\|_{L^1_0}\left\|f+g\right\|_{L^1_{1+\gamma/2}}\right) \\ &\leq 2^{\frac{3\gamma}{2}}\kappa^{ub}\left(C_1 + C_{1+\gamma/2}\right)\left\|f-g\right\|_{L^1_1}\end{aligned}$$

where κ^{ub} is from (3.7), and we have used monotonicity of norms (2.31) and definition of the set Ω from (7.3), which concludes the proof.

□

8. Generation and propagation of exponential moments

In the case of single monatomic gas [16], [33], [27], [4], [42], and more recently in [5] and monatomic gas mixtures [26], generation and propagation of polynomial moments implied the same properties of exponential moments, that polynomial moments estimates to solutions of the Boltzmann models allow for the summability of polynomial moments if the initial data has this summability property. The expert reader can easily observe that the analog results hold for the Boltzmann equation for polyatomic gases. However the expository proof we present in this section makes modifications that would clarify many points of these techniques to first time readers.

The notion of exponential moments as defined in (2.33) is associated to the finding conditions for the summability of polynomial moments (or only moments) to be a convergent series. More precisely, the conditions for the summability of moments propagation, if the initial data has that the same property, is related to show that there is an associated geometric convergent series of moments, whose radius of convergence is the rate of decay in such exponential moment form. The natural link between these two objects is provided by the Taylor series of an exponential form, formally written

$$\mathcal{E}_s[f](\beta,t) := \int_{\mathbb{R}^{3+}} f(t,v,I)\, e^{\beta\langle v,I\rangle^{2s}} \mathrm{d}I\,\mathrm{d}v \tag{8.1}$$
$$= \int_{\mathbb{R}^{3+}} f(t,v,I) \sum_{k=0}^{\infty} \langle v,I\rangle^{2sk} \frac{\beta^k}{k!} \mathrm{d}I\,\mathrm{d}v = \sum_{k=0}^{\infty} \frac{\beta^k}{k!}\, \mathfrak{m}_{sk}[f](t), \quad \text{for } t>0,$$

which we refer as to exponential moments with order $2s$, with $s \in (0,1]$, $s=1$ corresponding to Gaussians, and rate $\beta > 0$.

Indeed, our goal is to show that the constructed solutions in Section 7 can propagate or generate exponential moments depending on the integrability properties of the exponential moment of the initial data. Propagation of initial data in this context means given $f_0 \in \Omega$, a order factor $0 < s \leq 1$ and a rate $0 < \beta_0$ such that $\mathcal{E}_s[f](\beta_0, 0)$ is finite, then the solution of the Cauchy problem for $f(t,v,I)$ posed in (7.1) satisfies $\mathcal{E}_s[f](\beta,t)$ is finite for all time $t \geq 0$ and $0 < \beta \leq \beta_0$. However, generation of data means the following stronger property: given just $f_0 \in \Omega$, (not necessarily $\mathcal{E}_s[f](\beta_0,0)$ finite for any order $2s$, $0 < s \leq 1$, and $0 < \beta_0$) the solution of the corresponding Cauchy problem (7.1) satisfies that $\mathcal{E}_s[f](\beta_0,t)$ is finite for all time $t > 0$, for some order factor s and rate β to be found depending on the data.

The following Theorem proves the accuracy of these two statements. Their proofs consists in developing ordinary differential inequalities for the quantities $\mathcal{E}_s[f](\beta,t)$ valid for $t > 0$, whose initial data is referred as to $\mathcal{E}_s[f](\beta,t)\mid_{t=0}$.

Theorem 8.1 (Propagation and Generation of exponential moments). *Let f be the solution of the Cauchy problem (7.1). The following properties hold.*

(a) (Propagation) Let $0 < s \leq 1$. Suppose that there exists a constant $\beta_0 > 0$, such that the initial data $f_0(v,I)$ has a bounded exponential moment of order $2s$ and rate β_0

$$\mathcal{E}_s[f](\beta,t)\mid_{t=0} = \mathcal{E}_s[f_0](\beta_0,0) = \int_{\mathbb{R}^{3+}} f_0(v,I)\, e^{\beta_0\langle v,I\rangle^{2s}} \mathrm{d}I\,\mathrm{d}v =: M_P < \infty, \tag{8.2}$$

then there exist a constant $0 < \beta \leq \beta_0$ such that

$$\mathcal{E}_s[f](\beta,t) \leq 3\,\mathcal{E}_s[f_0](\beta_0,t) = 3M_P \qquad \forall t \geq 0. \tag{8.3}$$

(b) (Generation) If the prescribed initial data is in the solution set Ω defined in (7.3), that is

$$\mathcal{E}_s[f](\beta,t)\mid_{t=0} = \mathfrak{m}_{k_*}[f_0] = \int_{\mathbb{R}^{3+}} f_0(v,I)\ \langle v,I\rangle^{2k_*}\, \mathrm{d}I\,\mathrm{d}v =: M_G < \infty, \tag{8.4}$$

then, there exist a rate constant $\beta > 0$ and a positive order $2s$, with $0 < s \leq 1$, such that the exponential moment is generated to be bounded uniformly in time by the initial polynomial moment

$$\mathcal{E}_{\gamma/2}[f](\beta\min\{t,1\}, t) \leq \mathfrak{m}_{k_*}[f_0] = \int_{\mathbb{R}^{3+}} f_0(v,I)\,\langle v,I\rangle^{2k_*}\,\mathrm{d}I\,\mathrm{d}v = M_G, \quad \forall t > 0. \tag{8.5}$$

Proof. Let f be the solution of the Cauchy problem (7.1).

The proof of both items (a) and (b) in Theorem 8.1 strongly relies on estimates worked out in the previous sections to prove the generation and propagation of polynomial moments stated in Theorem 6.3. We will first present the sufficient estimates and the proof of each item it performed in the next two subsections.

In order to obtain information about the summability of moments, we need here to use the sharper upper bound of the binomial expansion (6.10), for any $0 < s \le 1$, associated to the Lebesgue bracket

$$\begin{aligned}&\Big(\langle v, I\rangle^2 + \langle v_*, I_*\rangle^2\Big)^{sk} \le \Big(\langle v, I\rangle^{2s} + \langle v_*, I_*\rangle^{2s}\Big)^{k}\\ &\le \langle v, I\rangle^{2sk} + \langle v_*, I_*\rangle^{2sk} + \sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\Big(\langle v, I\rangle^{2s\ell}\langle v_*, I_*\rangle^{2(sk-s\ell)} + \langle v, I\rangle^{2(sk-s\ell)}\langle v_*, I_*\rangle^{2s\ell}\Big),\end{aligned}$$

with $\ell_k = \lfloor \frac{k+1}{2} \rfloor$, which combined with estimate bounds from above (5.26) and below (4.5) for the transition function, $\tilde{\mathcal{B}} = |v - v_*|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}$, results in the following inequalities for polynomial moments

$$\mathfrak{m}_{sk}[f](t) =: \mathfrak{m}_{sk}(t), \quad 0 < s \le 1,\ k \ge 0, \quad \text{with } sk > k_*,$$

with k_* as defined in (6.2), where the only modification with respect to the proof of Lemma 6.2 is with the estimate on positive contribution. Then, after applying coercive estimate (4.5), we need the analog result as in (6.17) , now in a more accurate form suitable to study exponential moments propagation and generation properties, namely

$$\mathfrak{m}'_{sk}(t) \le -\bar{A}_{k_*}\mathfrak{m}_{sk+\gamma/2} + \bar{B}_{sk}\sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\left(\mathfrak{m}_{s\ell+\gamma/2}\mathfrak{m}_{sk-s\ell} + \mathfrak{m}_{sk-s\ell+\gamma/2}\mathfrak{m}_{s\ell}\right), \quad (8.6)$$

where $\bar{A}_{k_*}$ and $\bar{B}_{sk}$ are positive constants,

$$\begin{aligned}&\bar{A}_{k_*} = c_{lb}\tilde{A}_{k_*} = c_{lb}\left(\kappa^{lb} - \mathcal{C}_{k_*}\right), \text{ independent of } sk > k_*, \text{ and,}\\ &\bar{B}_{sk} := 2^{\frac{3\gamma}{2}-1}\mathcal{C}_{sk} \searrow 0, \text{ for } sk > k_*,\end{aligned} \quad (8.7)$$

$\mathcal{C}_{sk} \le \kappa^{lb}$ for $sk > k_*$, are from Lemma 5.3, estimate (5.9) and k_* from (6.2). The coercive constant satisfies $\bar{A}_{k_*} > A_{k_*}$, with A_{k_*} associated (6.32), c_{lb} is the one from Lemma 4.1, (4.6) and $\tilde{A}_{k_*}$ is from (6.15). Yet we note that $\bar{B}_{sk}$ is a much smaller constant than B_{sk}, as defined in (6.32), since the factor $2^{\frac{3\gamma}{2}-1} < 2^{\frac{3\gamma+sk}{2}} \le \max\left\{\left(\frac{\kappa^{lb}\, 2^{\frac{3\gamma}{2}+\frac{k}{2}}}{A_{k_*}}\right)^{\frac{\theta_k}{1-\theta_k}} \frac{\eta_k^{\frac{1}{1-\theta_k}}}{\|f\|_{L^1_1}}, 1\right\}\|f\|_{L^1_1}$, for any $sk > k_*$.

We are now in conditions to show both items in Theorem 8.1 of this section.

8.1. ***End of proof of Theorem 8.1, part (a): Propagation of exponential moments.*** Our goal is to find conditions for the summability of moments propagation, starting from an initial data $f_0 \in L^1_k(\mathbb{R}^{3+})$ finite for all $k \ge k_*$, that actually satisfies (8.2), that is, there is a rate $0 < \beta_0$, such that

$$\mathcal{E}_s[f](\beta, t)\mid_{t=0} = \sum_{k=0}^{\infty}\frac{\beta_0^k}{k!}\,\mathfrak{m}_{sk}[f_0] \quad \text{is finite}. \quad (8.8)$$

Next, the calculation of an ODI associated to $\mathcal{E}_s[f](\beta, t)$ is obtained as follows.

Starting from the Taylor series of an exponential function, one can represent exponential moment as presented in (8.1), set

$$\mathcal{E}_s[f](\beta, t) = \sum_{k=0}^{\infty}\frac{\beta^k}{k!}\,\mathfrak{m}_{sk}[f](t). \quad (8.9)$$

We consider its partial sum given above in (8.9), and another partial sum with a shift in the moment order $\gamma/2$, namely,

$$\mathcal{E}_s^n = \sum_{k=0}^{n} \frac{\beta^k}{k!}\, \mathfrak{m}_{sk}, \quad \mathcal{E}_{s;\gamma}^n = \sum_{k=0}^{n} \frac{\beta^k}{k!}\, \mathfrak{m}_{sk+\gamma/2}, \tag{8.10}$$

where we have omitted to highlight dependence on t and β, and relation to f, i.e. we have assumed

$$\mathcal{E}_s^n[f](\beta,t) =: \mathcal{E}_s^n, \qquad \mathcal{E}_{s;\gamma}^n[f](\beta,t) := \mathcal{E}_{s;\gamma}^n, \qquad \mathfrak{m}_{sk+\gamma/2}[f](t) =: \mathfrak{m}_{sk+\gamma/2}.$$

The goal if to prove that there is a β independent of time t such partial sum $\mathcal{E}_s^n$ is bounded uniformly in time t and n such that, $\lim_{n\to\infty} \mathcal{E}_s^n(\beta,t) = \mathcal{E}_s(\beta,t)$.

Taking derivative with respect to time t of (8.9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s^n = \sum_{k=0}^{n} \frac{\beta^k}{k!}\mathfrak{m}'_{sk} = \sum_{k=0}^{k_0-1} \frac{\beta^k}{k!}\mathfrak{m}'_{sk} + \sum_{k=k_0}^{n} \frac{\beta^k}{k!}\mathfrak{m}'_{sk}, \tag{8.11}$$

where k_0 is an index that will be determined later on. Since $\mathcal{E}_s^n$ is written in terms of $\mathfrak{m}_{sk}$ we derive ordinary differential inequality (ODI) for polynomial moment $\mathfrak{m}_{sk}$.

Indeed, from polynomial ODI, also (8.6), and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_{sk} \leq -\bar{A}_{k_*}\mathfrak{m}_{sk+\gamma/2} + \bar{B}_{sk}\sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\left(\mathfrak{m}_{s\ell+\gamma/2}\ \mathfrak{m}_{sk-s\ell} + \mathfrak{m}_{sk-s\ell+\gamma/2}\ \mathfrak{m}_{s\ell}\right). \tag{8.12}$$

Making use inequality (8.12) on the second term in (8.11), yields the new ODI

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s^n[f] = \mathcal{E}_s^n[Q(f,f)] \leq \sum_{k=0}^{k_0-1}\frac{\beta^k}{k!}\mathfrak{m}'_{sk} - \bar{A}_{k_*}\sum_{k=k_0}^{n}\frac{\beta^k}{k!}\,\mathfrak{m}_{sk+\gamma/2} \\ + \sum_{k=k_0}^{n}\bar{B}_{sk}\frac{\beta^k}{k!}\sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\left(\mathfrak{m}_{s\ell+\gamma/2}\mathfrak{m}_{sk-s\ell} + \mathfrak{m}_{sk-s\ell+\gamma/2}\mathfrak{m}_{s\ell}\right) \\ =: S_0 - \bar{A}_{k_*}S_1 + S_2.\end{aligned} \tag{8.13}$$

We estimate each sum S_0, S_1 and S_2 separately.

We first recall the k-polynomial moments global bounds in the case of propagation of the initial data from Theorem 6.3, (6.35), noting that the bounds for $\mathfrak{m}'_{sk}(t)$ easily follow from taking an upper bound to the upper ODE (6.37) to make the comparison argument to obtain global bounds for $\mathfrak{m}_{sk}(t)$, namely,

$$\mathfrak{m}_{sk} \leq \max\left\{\left(\frac{B_{sk}}{A_{k_*}}\right)^{\frac{2sk}{\gamma}}, \mathfrak{m}_{sk}[f](0)\right\}, \quad \text{and}$$
$$\mathfrak{m}'_{sk} \leq B_{sk}\max\left\{\left(\frac{B_{sk}}{A_{k_*}}\right)^{\frac{2sk}{\gamma}}, \mathfrak{m}_{sk}[f](0)\right\}.$$

It follows that, for any fixed $sk_0 > k_*$, the following constant is independent of time

$$c_{sk_0} := \max_{0\leq k\leq k_0-1}\left\{\mathfrak{m}_{sk}, \mathfrak{m}'_{sk}\right\}, \quad 0 < s \leq 1. \tag{8.14}$$

Note that this constant c_{sk_0} monotonically increases with respect to k_0, for $sk_0 > k_*$, by the same property of moments $\mathfrak{m}_{sk}$ in k. Therefore,

$$c_{sk_0} \leq c_{s(k_0+1)} \leq c_{sk_0+1}. \tag{8.15}$$

Estimate for the term S_0 from (8.13). The first term of (8.13) is estimated using the constant c_{sk_0} just defined in (8.14), to obtain an upper estimate to both

$$\mathfrak{m}_{sk}, \mathfrak{m}'_{sk} \leq c_{sk_0} \quad \text{for all } k \in \{0, 1, \ldots, k_0-1\}, \quad \text{for } sk_0 \geq k_*. \tag{8.16}$$

Hence, an upper bound for S_0 is controlled, with a good choice of the rate β, by

$$S_0 \leq \sum_{k=0}^{k_0-1} \frac{\beta^k}{k!} \mathfrak{m}'_{sk} \leq c_{sk_0} \sum_{k=0}^{k_0-1} \frac{\beta^k}{k!} \leq c_{sk_0} e^{\beta} \leq 2\, c_{sk_0}, \tag{8.17}$$

for some k_0 fixed to be chosen later, provided that β is small enough to satisfy

$$e^{\beta} \leq 2, \quad \text{or equivalently,} \quad \beta < \ln 2. \tag{8.18}$$

Estimate for the term S_1 from (8.13). The term containing S_1 is negative and so it needs to be bounded from below. Thus, recasting the partial sum $\mathcal{E}^n_{s;\gamma/2}$ minus the term similar to S_0 by

$$S_1 := \sum_{k=k_0}^{n} \frac{\beta^k}{k!} \mathfrak{m}_{sk+\gamma/2} = \mathcal{E}^n_{s;\gamma} - \sum_{k=0}^{k_0-1} \frac{\beta^k}{k!} \mathfrak{m}_{sk+\gamma/2},$$

and invoking the upper estimate to the second addend in terms of the moments upper bounds (8.39), namely

$$\mathfrak{m}_{sk+\gamma/2} \leq \mathfrak{m}_{sk+1} \leq c_{sk_0+1}, \quad \text{for all} \quad k = 1, \ldots k_0 - 1,$$

yields the following lower bound to the term S_1 in (8.13), as follows

$$S_1 \geq \mathcal{E}^n_{s;\gamma} - 2c_{sk_0+1}. \tag{8.19}$$

Estimate for the term S_2 from (8.13). This next term in (8.13) can be split into two parts

$$S_2 = \sum_{k=k_0}^{n} \bar{B}_{sk} \frac{\beta^k}{k!} \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} \left(\mathfrak{m}_{s\ell+\gamma/2} \mathfrak{m}_{sk-s\ell} + \mathfrak{m}_{sk-s\ell+\gamma/2} \mathfrak{m}_{s\ell} \right) =: S_{2_1} + S_{2_2}.$$

that are worked out in the same way. The sum S_{2_1} is rearrenged by

$$S_{2_1} = \sum_{k=k_0}^{n} \bar{B}_{sk} \sum_{\ell=1}^{\ell_k} \frac{\beta^\ell\, \mathfrak{m}_{s\ell+\gamma/2}}{\ell!} \frac{\beta^{k-\ell}\, \mathfrak{m}_{sk-s\ell}}{(k-\ell)!} \leq \bar{B}_{sk_0} \mathcal{E}^n_{s;\gamma} \mathcal{E}^n_s,$$

where the last inequality is due to the monotone decreasing property of $\bar{B}_{sk} \leq \bar{B}_{sk_0}$ as defined in (8.7).

Proceeding in a similar way, S_{2_2} can be estimated by

$$S_{2_2} \leq 2\, \bar{B}_{sk_0}\, \mathcal{E}^n_{s;\gamma}\, \mathcal{E}^n_s. \tag{8.20}$$

Finally, inserting all the estimates (8.17) with (8.15), (8.19) and (8.20) into the right hand side of inequality (8.13), the following upper ODI for the partial sum $\mathcal{E}^n_s$ follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}^n_s \leq -\bar{A}_{k_*} \mathcal{E}^n_{s;\gamma} + 2c_{sk_0+1}(1 + \bar{A}_{k_*}) + 2\, \bar{B}_{sk_0}\, \mathcal{E}^n_{s;\gamma}\, \mathcal{E}^n_s. \tag{8.21}$$

The next step consists in finding a positive rate constant rate β and an upper bound for $\mathcal{E}^n_s$ from this ODI (8.21). To this end, for each $n \in \mathbb{N}$, define

$$T_n := \sup\{t \geq 0 : \mathcal{E}^n_s(\beta, \tau) \leq 3\, M_P,\ \forall \tau \in [0, t]\}, \tag{8.22}$$

where M_P bounds the initial exponential moment of order $2s$ and rate β_0, from (8.2).

Since we already imposed the condition (8.18), our task is to show that we can find a $\beta \leq \min\{\beta_0, \ln 2\}$, such that $\mathcal{E}_s^n(\beta, t)$ is uniformly bounded in t and n, by proving that the time for which partial sums remain bounded, as defined in (8.22), is actually unbounded. That means $T_n = \infty$ for all $n \in \mathbb{N}$.

Indeed, since $0 < \beta \leq \beta_0$, then assumption by (8.2) at at $t = 0$, yields

$$\mathcal{E}_s^n(\beta, 0) = \sum_{k=0}^{n} \frac{\beta^k}{k!} \mathfrak{m}_{sk}(0) \leq \sum_{k=0}^{\infty} \frac{\beta_0^k}{k!} \mathfrak{m}_{sk}(0) = \mathcal{E}_s(\beta_0, 0) = M_P, \tag{8.23}$$

uniformly in n. Thus, as each term $\mathfrak{m}_{sk}(t)$ is continuous in t, then $\mathcal{E}_s^n(\beta, t)$ is continuous in t as well. Therefore, $\mathcal{E}_s^n(\beta, t) < M_P$ on some time interval $[0, t_n)$, with $t_n > 0$, which implies that the sequence T_n is well-defined and positive, for every $n \in \mathbb{N}$.

In addition, from the definition (8.22) of T_n, it follows that $\mathcal{E}_s^n(\beta, t) \leq 3\, M_P$, for $t \in [0, T_n]$, so right hand side of the ODI (8.21) is control from above by

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_s^n \leq -\mathcal{E}_{s;\gamma}^n \left(\bar{A}_{k_*} - 8\, \bar{B}_{sk_0}\, M_P \right) + 2c_{sk_0+1} \left(1 + \bar{A}_{k_*} \right). \tag{8.24}$$

Since, from (8.14), $\bar{B}_{sk_0}$ converges to zero as k_0 goes to infinity, allow us to conclude that there is a sufficiently large, fixed $k_0 > \frac{k_*}{s}$, such that

$$\bar{A}_{k_*} - 8\, \bar{B}_{sk_0} M_P > \frac{\bar{A}_{k_*}}{2}. \tag{8.25}$$

Therefore, the ODI in (8.24) is estimated by the following upper ODI for the partial sums $\mathcal{E}_s^n(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_s^n \leq -\frac{\bar{A}_{k_*}}{2}\, \mathcal{E}_{s;\gamma}^n + 2c_{sk_0+1} \left(1 + \bar{A}_{k_*} \right), \qquad \text{for} \quad sk_0 > k_*, \tag{8.26}$$

with k_0 defined in (8.25).

It remains to find a lower bound for the $\gamma/2$ shifted partial sum $\mathcal{E}_{s;\gamma}^n$ in terms of $\mathcal{E}_s^n$, for a suitable rate β to be chosen. We start by estimating by recalling that the conserved quantity of the Boltzmann flow satisfies $\mathfrak{m}_0(t) = \mathfrak{m}_0(0) < \mathcal{E}_s^n(\beta_0, 0)$. Second, we note that the rate β, to be chosen soon, needs to satisfy from (8.18) $\beta < \ln 2$. That means, since $0 < s \leq 1$, then $e^{\beta^{1-s}} < 2^{1-s} < 2$.

Therefore, these two observations pave the way to find a bound from below to the partial sum $\mathcal{E}_{s;\gamma}^n$, for any $n > sk_*$, as follows

$$\begin{aligned}
\mathcal{E}_{s;\gamma}^n &= \sum_{k=0}^{n} \frac{\beta^k}{k!} \mathfrak{m}_{sk+\gamma/2} \geq \sum_{k=0}^{n} \frac{\beta^k}{k!} \int_{\left\{\langle v,I\rangle \geq \beta^{-1/2}\right\}} f(t,v,I)\, \langle v,I\rangle^{2(sk+\gamma/2)}\, \mathrm{d}I\, \mathrm{d}v \\
&\geq \beta^{-\gamma/2} \left(\mathcal{E}_s^n - \sum_{k=0}^{n} \frac{\beta^k}{k!} \int_{\left\{\langle v,I\rangle < \beta^{-1/2}\right\}} f(t,v,I)\, \langle v,I\rangle^{2sk}\, \mathrm{d}I\, \mathrm{d}v \right) \\
&\geq \beta^{-\gamma/2} \left(\mathcal{E}_s^n - \sum_{k=0}^{n} \frac{\beta_0^{k(1-s)}}{k!} \mathfrak{m}_0(t) \right) \geq \beta^{-\gamma/2} \left(\mathcal{E}_s^n - \mathfrak{m}_0 e^{\beta^{1-s}} \right) \\
&\geq \beta^{-\gamma/2} \left(\mathcal{E}_s^n - 2\mathcal{E}_s^n(\beta_0, 0) \right) = \beta^{-\gamma/2} \left(\mathcal{E}_s^n - 2M_P \right).
\end{aligned} \tag{8.27}$$

Thus, using this lower estimate to control the negative term from (8.26) yields the Ordinary Differential Inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s^n \le -\frac{\bar{A}_{k_*}}{2}\,\beta^{-\gamma/2}\mathcal{E}_s^n + \bar{A}_{k_*}\beta^{-\gamma/2}M_P + 2c_{sk_0+1}\left(1+\bar{A}_{k_*}\right),$$

for $sk_0 > k_*$, or equivalently, the following absorption ODI,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}_s^n - 2M_P\right) \le -\frac{\bar{A}_{k_*}}{2}\,\beta^{-\gamma/2}\left(\mathcal{E}_s^n - 2M_P\right) + 2c_{sk_0+1}\left(1+\bar{A}_{k_*}\right), \text{for } \; sk_0 > k_*. \tag{8.28}$$

Therefore, invoking the maximum principle for ODI's majorized by an upper associated ODE to (8.1), we obtain

$$\begin{aligned}\mathcal{E}_s^n(\beta,t) - 2M_P &\le \max\left\{\left(\mathcal{E}_s^n(\beta_0,0) - 2M_P\right), \frac{4c_{sk_0+1}\left(1+\bar{A}_{k_*}\right)}{\bar{A}_{k_*}\,\beta^{-\gamma/2}}\right\} \\ &\le \beta^{\gamma/2}\,\frac{4c_{sk_0+1}\left(1+\bar{A}_{k_*}\right)}{\bar{A}_{k_*}},\end{aligned} \tag{8.29}$$

for any $t \in [0, T_n]$, since $\mathcal{E}_s^n(\beta_0, t) = M_p$, implying the term $\mathcal{E}_s^n(\beta_0, 0) - 2M_P = -M_P = -\mathcal{E}_s^n(\beta_0, t) < 0$. In particular, inequality (8.29) yields, by taking $\beta = \beta_1$ small enough such that

$$\beta_1^{\gamma/2}\,\frac{4c_{sk_0+1}\left(1+\bar{A}_{k_*}\right)}{\bar{A}_{k_*}} \;\le\; M_P$$

or equivalently,

$$\beta_1 \;\le\; \left(\frac{\bar{A}_{k_*}}{4c_{sk_0+1}\left(1+\bar{A}_{k_*}\right)}M_P\right)^{\gamma/2} \tag{8.30}$$

Therefore setting β to be the minimum satisfying conditions (8.2), (8.18), (8.25) and (8.30), namely,

$$\beta \le \min\{\beta_0, \ln 2, \beta_1\}, \tag{8.31}$$

the inequality (8.1) implies that the following strict inequality of the partial sums up to term n are strictly controlled by a $3M_P$, or equivalently,

$$\mathcal{E}_s^n(\beta,t) < 3\mathcal{E}_s[f](\beta_0, 0), \quad \text{for any } 0 < \beta \le \min\{\beta_0, \ln 2, \beta_1\},\; t \in [0, T_n], \tag{8.32}$$

with β a constant independent of time t, depending on a fixed $k_0 \ge k_*$ from (8.25).

Finally, due to the continuity of $\mathcal{E}_s^n(\beta,t)$ with respect to time t, this strict inequality actually holds on a slightly larger time interval $[0, T_n + \varepsilon)$, $\varepsilon > 0$. This contradicts the maximality of T_n unless $T_n = +\infty$. Therefore, $\mathcal{E}_s^n(\beta,t) \le 3M_P$ for all $t \ge 0$ and $n \in \mathbb{N}$. Thus, letting $n \to \infty$ we conclude

$$\mathcal{E}_s[f](\beta,t) = \lim_{n\to\infty}\mathcal{E}_s^n[f](\beta,t) \le 3M_P = 3\int_{\mathbb{R}^{3+}} f_0(v,I)\,e^{\beta_0\langle v,I\rangle^{2s}}\mathrm{d}I\,\mathrm{d}v, \quad \forall t \ge 0,$$

i.e. the solution $f(t,v)$ to Boltzmann equation with finite initial exponential moment of order $2s$ and propagates the exponential moments of order $2s$, but at a lower rate β than the initial rate β_0 , with β defined by

$$\beta = \min\left\{\beta_0,\, \ln 2,\, \left(\frac{\bar{A}_{k_*}}{4c_{sk_0+1}\left(1+\bar{A}_{k_*}\right)}\,\mathcal{E}_s(\beta_0,0)\right)^{\gamma/2}\right\}. \tag{8.33}$$

It is significant to notice that the exponential moment rate β is proportional to the coercive constant $\bar{A}_{k_*} = c_{lb}\tilde{A}_{k_*}$ as described in (8.7) at the top of this last section, with c_{lb} from the functional lower estimate calculated in Lemma 4.1, estimate (4.6), and positive factor $\tilde{A}_{k_*}$ defined in (6.15), estimating the moments negative contribution derived from the compact manifold averaging Lemma 5.3 crucial for moments estimates of the gain collisional operator, with k_* defined in (6.2).

8.2. ***End of proof of Theorem 8.1, part (b): Generation of exponential moments.*** The proof is more delicate since we seek to derive an ODI for the exponential moment $\mathcal{E}_s[f](\beta,t)$ where the order factor s and rate β need to be found, just from an initial data $f_0 \in L^1_{k_*}(\mathbb{R}^{3+})$, that is $\int_{\mathbb{R}^{3+}} f_0(v,I)\,\langle v,I\rangle^{2k_*}\,\mathrm{d}I\,\mathrm{d}v =: M_G$ as defined in (8.4).

We will show that the generated exponential moment associated to this polynomial moment initial data has order $\gamma \in (0,2]$, and a rate β that actually is proportional to time t, as it provides a continuous in time venue to pass from a data with a k_*-Lesbegue polynomial moment to be able to show the instantaneously in time generation of an exponential moment. This calculation clearly needs the developed estimates on the generation of moments Theorem 6.3, (6.34).

Following the ideas developed in [4], [42], and more recently in [5] and [26], we start by associating an exponential moment of order γ, with a rate linearly dependent on time, namely βt, with β depending on k_* from (6.2), to the solution $f(t,v,I) \in \Omega$ of the Boltzmann equation in the invariant region defined in (7.3),

$$\mathcal{E}_{\gamma/2}[f](\beta t,t) := \int_{\mathbb{R}^{3+}} f(t,v,I)\,e^{\beta t\langle v,I\rangle^{\gamma}}\,\mathrm{d}I\mathrm{d}v = \sum_{k=0}^{\infty}\frac{(\beta t)^k}{k!}\mathfrak{m}_{\gamma k/2}[f](t). \tag{8.34}$$

As in part (a), we also define partial sums and the associated shifted ones now with time dependent rate, that is,

$$\mathcal{E}^n_{\gamma/2}[f](\beta t,t)=\sum_{k=0}^{n}\frac{(\beta t)^k}{k!}\mathfrak{m}_{\gamma k/2}[f](t) \text{ and } \mathcal{E}^n_{\gamma/2;\gamma}[f](\beta t,t)=\sum_{k=0}^{n}\frac{(\beta t)^k}{k!}\mathfrak{m}_{\gamma k/2+\gamma/2}[f](t). \tag{8.35}$$

Proceeding in a similar way as in the previous Subsection for the propagation of exponential moments proof, we will also relieve notation by omitting explicit dependence on time t and relation to f by setting

$$\mathcal{E}^n_{\gamma/2}[f](\beta t,t) =: \mathcal{E}^n_{\gamma/2}, \quad \text{and} \quad \mathcal{E}^n_{\gamma/2;\gamma}[f](\beta t,t) := \mathcal{E}^n_{\gamma/2;\gamma}.$$

Next, taking the time derivative of $\mathcal{E}^n_{\gamma/2}$ yields the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_{\gamma/2} = \beta\sum_{k=1}^{n}\frac{(\beta t)^{k-1}}{(k-1)!}\mathfrak{m}_{\gamma k/2} + \sum_{k=0}^{k_0-1}\frac{(\beta t)^k}{k!}\mathfrak{m}'_{\gamma k/2} + \sum_{k=k_0}^{n}\frac{(\beta t)^k}{k!}\mathfrak{m}'_{\gamma k/2}, \tag{8.36}$$

which, making use of the ODIs for $\mathfrak{m}'_{\gamma k/2}$ (8.6), evaluated in $s := \gamma/2$, results into

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_{\gamma k/2} &\le -\bar{A}_{k_*}\mathfrak{m}_{\gamma k/2+\gamma/2} \\ &\quad + \bar{B}_{\gamma k}\sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\left(\mathfrak{m}_{\gamma\ell/2+\gamma/2}\,\mathfrak{m}_{\gamma k/2-\gamma\ell/2} + \mathfrak{m}_{\gamma k/2-\gamma\ell/2+\gamma/2}\,\mathfrak{m}_{\gamma\ell/2}\right),\end{aligned} \tag{8.37}$$

that allows identity (8.36) to be estimated by above as follows.

The first and second terms are obtained by simply re-indexing the sum and using the definition of the corresponding shifted partial sum; and the last one is control by above by the right hand side of the moment ODI (8.37), which together implies

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{E}^n_{\gamma/2} \le \beta\,\mathcal{E}^n_{\gamma/2;\gamma} + \sum_{k=0}^{k_0-1}\frac{(\beta t)^k}{k!}\mathfrak{m}'_{\gamma k/2} - \bar{A}_{k_*}\sum_{k=k_0}^{n}\frac{(\beta t)^k}{k!}\mathfrak{m}_{\gamma k/2+\gamma/2}$$
$$+\sum_{k=k_0}^{n}\frac{(\beta t)^k}{k!}\bar{B}_{\frac{\gamma k}{2}}\sum_{\ell=1}^{\ell_k}\binom{k}{\ell}\left(\mathfrak{m}_{\gamma\ell/2+\gamma/2}\mathfrak{m}_{\gamma k/2-\gamma\ell/2}+\mathfrak{m}_{\gamma k/2-\gamma\ell/2+\gamma/2}\mathfrak{m}_{\gamma\ell/2}\right)$$
$$=: \beta\,\mathcal{E}^n_{\gamma/2;\gamma} + S_0 - \bar{A}_{k_*}S_1 + (S_{2_1}+S_{2_2})\,, \quad (8.38)$$

for any $k_0 \ge k_*$, since $0<\gamma/2\le 1$.

We treat each of this terms term separately.

Clearly, as we wrote in part (a) of the proof of Theorem 8.1, we need to invoke the global in time estimates for moments of $\mathfrak{m}_{\gamma k/2}(t)$ under initial conditions for the generation property as defined in (6.34).

Hence, set

$$\mathfrak{m}_{k\gamma/2} \le \mathfrak{B}_{k\gamma/2}\ \max_{t>0}\{1,t^{-k}\},\quad \mathfrak{m}'_{k\gamma/2} \le B_{k\gamma/2}\mathfrak{B}_{k\gamma/2}\ \max_{t>0}\{1,t^{-k}\},$$

and denote, for any fixed finite $\gamma k_0/2 \ge k_*$, the positive constant

$$\bar{c}_{\gamma k_0/2} := \max_{0\le k\le k_0-1}\{\mathfrak{B}_{\gamma k/2}, B_{k\gamma/2}\mathfrak{B}_{k\gamma/2}\}\,, \quad (8.39)$$

where $\mathfrak{B}_{k\gamma/2}$ was defined in (6.34)

It is worth to notice that these moments upper bounds are time dependent. This is a crucial feature that enables the generation of exponential moments with polynomial moment initial data (8.4) in a short interval of time, and then bootstrap the argument to get global in time estimates.

Estimate for the term S_0 from (8.38). From polynomial moment generation estimates (6.34), one can bound polynomial moments of any order $k>k_*$, as well as their derivatives by means of (6.33) with bounds $\mathfrak{B}^k$. In particular, set the moments upper bounds to be

$$\mathfrak{m}_{\gamma k/2} \le \mathfrak{B}^{\gamma k/2}\ \max_{t>0}\{1,t^{-k}\},\quad \mathfrak{m}'_{\gamma k/2} \le B_{\gamma k/2}\mathfrak{B}^{\gamma k/2}\ \max_{t>0}\{1,t^{-k}\},$$

and in particular

$$\mathfrak{m}_{\gamma k/2} \le \mathfrak{B}_{\gamma k/2}t^{-k},\quad \text{and}\quad \mathfrak{m}'_{\gamma k/2} \le B_{\gamma k/2}\mathfrak{B}_{\gamma k/2}t^{-k},\qquad \text{for } 0<t\le 1. \quad (8.40)$$

Then, for any $t\in(0,1]$ and $k\le k_0-1$, and estimating S_0 by

$$S_0 := \sum_{k=0}^{k_0-1}\frac{(\beta t)^k}{k!}\mathfrak{m}'_{\gamma k/2}(t) \le \sum_{k=0}^{k_0-1}\frac{\beta^k}{k!}\mathfrak{B}_{\gamma k/2} \le \bar{c}_{\gamma k_0/2}e^{\beta} \le 2\bar{c}_{\gamma k_0/2}, \quad (8.41)$$

with $\bar{c}_{\gamma k_0/2}$ defined in (8.39) constant in time, for $t\in(0,1]$ and $\gamma k_0/2\ge k_*$, which holds provided that β satisfies, at most

$$\beta \le \ln 2 \quad (8.42)$$

exactly as it was required in the proof of Theorem 8.1, part (a), (8.18).

Estimate for the term S_1 from (8.38). This estimate is crucial for the control from below of the negative contribution associated to the ODI for exponential moments that yields their generation argument, with initial data depending on just $f_0(v,I) \in L^1_{k_*}(\mathbb{R}^{3+})$.

Indeed, using the boundedness of the $\gamma/2$-shifted moments, namely $\mathfrak{m}_{\gamma k/2+\gamma/2} \leq \mathfrak{B}^{\gamma k/2+\gamma/2}\ \max_{t>0}\{1, t^{-k-1}\}$, and that β has been chosen to satisfy $e^{\beta} < 2$, then

$$S_1 := \sum_{k=k_0}^{n} \frac{(\beta t)^k}{k!} \mathfrak{m}_{\gamma k/2+\gamma/2} = \mathcal{E}^n_{\gamma/2;\gamma} - \sum_{k=0}^{k_0-1} \frac{(\beta t)^k}{k!} \mathfrak{m}_{\gamma k/2+\gamma/2}$$
$$\geq \mathcal{E}^n_{\gamma/2;\gamma} - 2\bar{c}_{\gamma(k_0+1)/2} \frac{1}{t}, \tag{8.43}$$

for β chosen as in (8.42) and $\bar{c}_{\gamma(k_0+1)/2}$ as defined in (8.39).

Note that the appearance of the factor $1/t$ multiplying $2\bar{c}_{\gamma(k_0+1)/2}$ is a major deviation from the analog estimate of the term S_1 in the propagation argument of Theorem 8.1, part (b), as it will be use later on in this proof.

Estimate for the term S_2 from (8.38). Terms S_{2_1} and S_{2_2} are treated in the same way as follows. We will detail calculation for S_{2_1}. First, reorganize the terms in sum to obtain

$$S_{2_1} := \sum_{k=k_0}^{n} \frac{(\beta t)^k}{k!} \bar{B}_{\frac{\gamma k}{2}} \sum_{\ell=1}^{\ell_k} \binom{k}{\ell} \mathfrak{m}_{\gamma \ell/2+\gamma/2} \mathfrak{m}_{\gamma k/2-\gamma \ell/2} \tag{8.44}$$
$$= \sum_{k=k_0}^{n} \bar{B}_{\frac{\gamma k}{2}} \sum_{\ell=1}^{\ell_k} \frac{(\beta t)^{\ell} \mathfrak{m}_{\gamma \ell/2+\gamma/2}}{\ell!} \frac{(\beta t)^{k-\ell} \mathfrak{m}_{\gamma k/2-\gamma \ell/2}}{(k-\ell)!} \leq \bar{B}_{\frac{\gamma k_0}{2}} \mathcal{E}^n_{\gamma/2;\gamma}\, \mathcal{E}^n_{\gamma/2}.$$

where, as defined in (8.7), $\bar{B}_{\frac{\gamma k}{2}} = 2^{\frac{3\gamma}{2}-1} \mathcal{C}_{\frac{\gamma k}{2}}$ monotonically decays with respect to $k \geq k_0$, which implies $\bar{B}_{\frac{\gamma k}{2}} \leq \bar{B}_{\frac{\gamma k_0}{2}}$ for $\frac{2}{\gamma} k_* < k_0 \leq k \leq n$. The estimate for the term S_{2_2} in obtain in a similar way.

Gathering and inserting estimates (8.41), (8.43) and (8.44) into (8.38), with β satisfying (8.42), that is $\beta < \ln 2$, then all partial sums from (8.38) satisfy the following a priori ODIs, for any $\frac{2}{\gamma} k_* < k_0 \underline{\leq k} \leq n$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}^n_{\gamma/2} \leq \beta\, \mathcal{E}^n_{\gamma/2;\gamma} + 2c_{\gamma k_0/2} - \bar{A}_{k_*} \left(\mathcal{E}^n_{\gamma/2;\gamma} - 2\bar{c}_{\gamma(k_0+1)/2} \frac{1}{t} \right) + \bar{B}_{\frac{\gamma}{2} k_0}\, \mathcal{E}^n_{\gamma/2;\gamma}\, \mathcal{E}^n_{\gamma/2}$$
$$= \mathcal{E}^n_{\gamma/2;\gamma} \left(\beta - \bar{A}_{k_*} + \bar{B}_{\frac{\gamma k_0}{2}} \mathcal{E}^n_{\gamma/2} \right) + 2\bar{c}_{\gamma(k_0+1)/2} \left(\frac{t + \bar{A}_{k_*}}{t} \right). \tag{8.45}$$

Now, fix $\gamma \in (0,2]$, and $\beta < \ln 2$, while seting the time interval $[0, \bar{T}_n]$ with the upper end given by

$$\bar{T}_n := \sup \left\{ t \in [0,1] : \mathcal{E}^n_{\gamma/2}[f](\beta t, t) \leq 4 M_G \right\}.$$

Note that this $\bar{T}_n$ is well defined. Indeed, taking the conserved quantity

$$M_G := \|f\|_{L^1_1} = \int_{\mathbb{R}^{3+}} f_0(v,I) \langle v, I \rangle^2 \,\mathrm{d}I \mathrm{d}v, \qquad \text{with } f_0(v,I) \in \Omega, \tag{8.46}$$

and noting that just for $t=0$, $\mathcal{E}^n_{\gamma/2}(0,0) \leq \mathcal{E}_{\gamma/2}(0,0) = \mathfrak{m}_0(0) < \|f\|_{L^1_1}(0) < 4M_G$, then by the continuity of partial sums $\mathcal{E}^n_{\gamma/2}$ with respect to t imply that $\mathcal{E}^n_{\gamma/2}(\beta t, t) \leq 4M_G$ on a slightly larger time interval $t \in [0, t_n)$, $t_n > 0$, and thus $0 < \bar{T}_n \leq 1$.

In the next step, we need to show that there is a rate $0 < \beta < \ln 2$ such that $\mathcal{E}^n_{\gamma/2}(t)$ is bounded for $t \in [0, \bar{T}_n]$. It is clear that for any $t \in [0, \bar{T}_n]$, the evaluation in the partial sum $\mathcal{E}^n_{\gamma/2}(\beta t, t) \leq 4M_G$, since $\bar{T}_n \leq 1$ and so $t^{-1} \geq 1$. Therefore, from (8.45), the ODI for the partial sums with time dependent rates βt yields the following upper estimate for the time derivative of $\mathcal{E}^n_{\gamma/2}(t)$, namely, for any time t, such that $0 \leq t \leq \bar{T}_n \leq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_{\gamma/2}(t) \leq -\mathcal{E}^n_{\gamma/2;\gamma}(t)\left(-\beta + \bar{A}_{k_*} - \bar{B}_{\frac{\gamma}{2}k_0}\, 4M_G\right) + 2\bar{c}_{\gamma(k_0+1)/2}\frac{1+\bar{A}_{k_*}}{t}\,. \tag{8.47}$$

Now, since $\bar{B}_{\frac{\gamma k_0}{2}}$ converges to zero as $\frac{2}{\gamma}k_* < k_0$ grows, we can choose large enough number of moments k_0, such that

$$\frac{\bar{A}_{k_*}}{2} > 4\bar{B}_{\frac{\gamma}{2}k_0} M_G, \tag{8.48}$$

with $M_G + = \|f\|_{L^1_1}$ from (8.46), then just taking a small enough β satisfying

$$0 < \beta < \frac{\bar{A}_{k_*}}{2}\,, \tag{8.49}$$

yields the following upper bound from (8.47)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_{\gamma/2} \leq -\frac{\bar{A}_{k_*}}{2}\mathcal{E}^n_{\gamma/2;\gamma} + 2\bar{c}_{\gamma(k_0+1)/2}\left(\frac{1+\bar{A}_{k_*}}{t}\right). \tag{8.50}$$

Finally, the shifted moment $\mathcal{E}^n_{\gamma/2;\gamma}(\beta t, t)$ can be bounded as follows

$$\mathcal{E}^n_{\gamma/2;\gamma}(\beta t, t) = \sum_{k=1}^{n+1}\frac{(\beta t)^k \mathfrak{m}_{\gamma k/2}(t)}{k!}\frac{k}{\beta t} \geq \frac{1}{\beta t}\sum_{k=2}^{n}\frac{(\beta t)^k \mathfrak{m}_{\gamma k/2}(t)}{k!} \geq \frac{\mathcal{E}^n_{\gamma/2}(\beta t, t) - M_G}{\beta t},$$

hence, the ODI (8.47) yields the reduced one

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_{\gamma/2} \leq -\frac{\bar{A}_{k_*}}{2\beta t}\left(\mathcal{E}^n_{\gamma/2} - M_G - \frac{2\beta}{\bar{A}_{k_*}}2\bar{c}_{\gamma(k_0+1)/2}(1+\bar{A}_{k_*})\right). \tag{8.51}$$

Now, recalling the value of M_G from the initial data from (8.46), and that $\gamma/2 < 1$, and so $\bar{c}_{\gamma(k_0+1)/2} \leq \bar{c}_{k_0+1}$, one can choose β small enough such that

$$M_G + 4\beta\bar{c}_{k_0+1}\left(\frac{1+\bar{A}_{k_*}}{\bar{A}_{k_*}}\right) < 2\|f\|_{L^1_{k_*}} \tag{8.52}$$

or equivalently, choosing β the smallest, between $\ln 2$, (8.49) and (8.52), by

$$0 < \beta \ < \ \min\left\{\ln 2;\, \frac{\bar{A}_{k_*}}{4\bar{c}_{k_0+1}}\frac{\|f\|_{L^1_{k_*}}}{1+\bar{A}_{k_*}}\right\}, \tag{8.53}$$

for k_0 large enough to satisfy condition (8.48) and the corresponding constant $\bar{c}_{k_0+1}$ from (8.39) the upper bound for the right hand side in (8.51) yields the absorption ODI,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^n_{\gamma/2}(\beta t, t) \leq -\frac{\bar{A}_{k_*}}{2\beta t}\left(\mathcal{E}^n_{\gamma/2}(\beta t, t) - 2\|f\|_{L^1_{k_*}}\right). \tag{8.54}$$

or, since $\|f\|_{L^1_1}$ is constant in time, due to conservation of mass and energy, we can equivalently write for $X(t) := \mathcal{E}^n_{\gamma/2}(\beta t, t) - 2\|f\|_{L^1_1}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \leq -\frac{\bar{A}_{k_*}}{2\beta t}X(t), \qquad \text{that implies} \quad X(t) \leq X(0), \quad \forall t \in (0, \bar{T}_n],$$

by Gronwall inequality or equivalently, since $\mathcal{E}^n_{\gamma/2}(0,0) < \|f\|_{L^1_{k_*}}$ for $f \in \Omega$,

$$\mathcal{E}^n_{\gamma/2}(\beta t, t) - 2\|f\|_{L^1_{k_*}} \leq \mathcal{E}^n_{\gamma/2}(0,0) - 2\|f\|_{L^1_{k_*}} \leq -1\|f\|_{L^1_{k_*}}, \text{ that implies}$$

$$\mathcal{E}^n_{\gamma/2}(\beta t, t) \leq \ \|f\|_{L^1_{k_*}}, \qquad \forall t \in (0, \bar{T}_n]. \tag{8.55}$$

By the continuity of partial sums $\mathcal{E}^n_{\gamma/2}(\beta t, t)$ these inequalities hold on a slightly larger interval, which contradicts maximality of $\bar{T}_n$, unless $\bar{T}_n = 1$. Therefore, we can conclude $\bar{T}_n = 1$ uniformly in $n \in \mathbb{N}$.

Hence, letting $n \to \infty$, we conclude

$$\mathcal{E}_{\gamma/2}(\beta t, t) := \int_{\mathbb{R}^{3+}} f(t,v,I)\, e^{\beta t \langle v, I\rangle^\gamma} \mathrm{d}I \mathrm{d}v \leq \|f\|_{L^1_{k_*}(\mathbb{R}^{3+})} \qquad \forall t \in [0,1], \tag{8.56}$$

and thus the exponential moment of the order γ, and a rate $0 < \beta$, from (8.49) and (8.52), propagates in the time interval $(0,1]$, and stays uniformly bounded for all $t > 1$ to obtain a global in time as the argument bootstraps on time intervals $[l, l+1]$, for $l \in \mathbb{N}^+$.

It is also very significant to notice in the generation of exponential moments that the exponential moment rate β is proportional to the coercive constant $\bar{A}_{k_*}$, and inversely proportional to the moments generation bounds expressed in the constant $\bar{c}_{k_0+1}$ defined in (8.39) with $k_0\gamma/2 > k_*$, k_0 large enough to satisfy condition (8.48), The k_*-moment is characterized from (6.2) and depends on the assumptions on the three different transition probability rates associated to the polyatomic gas model presented in this manuscript. These conditions, while somehow different in substance, they are closely related to the coerciviness property of the Boltzmann collisional binary form for polyatomic gases, as much as described at the end of the previous Subsection of theorem 8.1, part (a).

□

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Appendix A. Proof of the Lemma 2.1 (Jacobian of the collision transformation)

Proof. Using ideas from [24], we decompose the mapping T from (2.9) into a sequence of mappings and calculate Jacobian of each of them. Then the Jacobian of T will be a product of those Jacobians. More precisely, T can be understood as a composition of the following transformations

$$T = T_9 \circ T_8 \circ T_7 \circ T_6 \circ T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1,$$

where composition is understood as $(f \circ g)(x) = f(g(x))$ and each T_i is described below.

(1) We first pass to the center-of-mass reference frame

$$T_1 : (v, v_*, I, I_*, r, R, \sigma) \mapsto (u, V, I, I_*, r, R, \sigma),$$

where u and V are relative velocity and velocity of center of mass from (2.3). It is clear that Jacobian of this transformation is 1,

$$J_{T_1} = 1.$$

(2) For the relative velocity u we pass to its spherical coordinates $\left(|u|, \frac{u}{|u|}\right)$, where $u/|u| \in S^2$ is the angular variable, with the transformation T_2,

$$(u, V, I, I_*, r, R, \sigma) \mapsto \left(|u|, \frac{u}{|u|}, V, I, I_*, r, R, \sigma\right),$$

whose Jacobian is

$$J_{T_2} = |u|^{-2}.$$

(3) In order to facilitate further calculation, we consider square of relative speed instead of relative speed itself,

$$T_3 : \left(|u|, \frac{u}{|u|}, V, I, I_*, r, R, \sigma\right) \mapsto \left(|u|^2, \frac{u}{|u|}, V, I, I_*, r, R, \sigma\right)$$

with the Jacobian

$$J_{T_3} = 2|u|.$$

(4) Instead of I_* we will use total energy E, linked with the equation (2.4),

$$T_4 : \left(|u|^2, \frac{u}{|u|}, V, I, I_*, r, R, \sigma\right) \mapsto \left(|u|^2, \frac{u}{|u|}, V, I, E, r, R, \sigma\right)$$

whose Jacobian is 1,

$$J_{T_4} = 1.$$

(5) Moreover, instead of R we want to have ER,

$$T_5 : \left(|u|^2, \frac{u}{|u|}, V, I, E, r, R, \sigma\right) \mapsto \left(|u|^2, \frac{u}{|u|}, V, I, E, r, ER, \sigma\right),$$

with Jacobian

$$J_{T_5} = E.$$

(6) Finally, we pass to pre-collisional quantities with the following mapping

$$T_6 : \left(|u|^2, \frac{u}{|u|}, V, I, E, r, ER, \sigma\right) \mapsto \left(|u'|^2, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma'\right).$$

Let us compute Jacobian of this central transformation. First, for V we are using conservation law (2.5). Change of the unit vectors $\frac{u}{|u|}$ and σ can be considered as a rotation. Thus we eliminate these variables and for the rest of variables, we use the following relations

$$|u'|^2 = \frac{4RE}{m}, \; I' = r(1-R)E, \; I'_* = (1-r)(1-R)E,$$

$$r' = \frac{I}{E - \frac{m}{4}|u|^2}, \; R' = \frac{m|u|^2}{4E},$$

and compute the corresponding Jacobian

$$J_{\left(|u|^2,I,E,r,ER\right)\mapsto\left(|u'|^2,I',I'_*,r',R'\right)}$$
$$= \begin{vmatrix} 0 & 0 & \frac{4R}{m} & 0 & \frac{4}{m} \\ 0 & 0 & r(1-R) & (1-R)E & -r \\ 0 & 0 & (1-r)(1-R) & -(1-R)E & 1-r \\ \frac{mI}{4(E-\frac{m}{4}|u|^2)^2} & \frac{1}{E-\frac{m}{4}|u|^2} & -\frac{I}{(E-\frac{m}{4}|u|^2)^2} & 0 & 0 \\ \frac{m}{4E} & 0 & -\frac{m|u|^2}{4E^2} & 0 & 0 \end{vmatrix}$$

$$= \frac{(-1)^{4+2}}{E-\frac{m}{4}|u|^2}\frac{(-1)^{4+1}m}{4E}\begin{vmatrix} \frac{4R}{m} & 0 & \frac{4}{m} \\ r(1-R) & (1-R)E & -r \\ (1-r)(1-R) & -(1-R)E & 1-r \end{vmatrix}$$

$$= \frac{-m(1-R)E}{(E-\frac{m}{4}|u|^2)4E}\begin{vmatrix} \frac{4R}{m} & 0 & \frac{4}{m} \\ r(1-R) & 1 & -r \\ (1-r)(1-R) & -1 & 1-r \end{vmatrix}$$

$$= \frac{-m(1-R)}{4(E-\frac{m}{4}|u|^2)}\begin{vmatrix} \frac{4R}{m} & 0 & \frac{4}{m} \\ 1-R & 0 & -1 \\ (1-r)(1-R) & -1 & 1-r \end{vmatrix}$$

$$= \frac{-m(1-R)}{4\left(E-\frac{m}{4}|u|\right)}\left(\frac{-4R}{m}+\frac{4(R-1)}{m}\right)$$

Finally,

$$J_{T_6} = \frac{1-R}{(E-\frac{m}{4}|u|^2)} = \frac{1-R}{I+I_*} = \frac{1-R}{(1-R')E}.$$

(7) Now we go back, first from squares to squares of relative speed to relative speed itself,

$$T_7 : \left(|u'|^2, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma'\right) \mapsto \left(|u'|, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma'\right).$$

with Jacobian

$$J_{T_7} = \frac{1}{2|u'|}.$$

(8) For u' we pass from spherical coordinates to Cartesian ones,

$$T_8 : \left(|u'|, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma'\right) \mapsto (u', V', I', I'_*, r', R', \sigma').$$

with Jacobian

$$J_{T_8} = |u'|^2.$$

(9) We go back from center-of-mass reference frame,

$$T_9 : (u', V', I', I'_*, r', R', \sigma') \mapsto (v', v'_*, I', I'_*, r', R', \sigma')$$

with unit Jacobian

$$J_{T_9} = 1.$$

Finally, we get the Jacobian of transformation T,

$$J_T = \prod_{i=1}^{9} J_{T_i} = \frac{(1-R)\,|u'|}{(1-R')\,|u|} = \frac{(1-R)R^{1/2}}{(1-R')R'^{1/2}},$$

where for the last inequality we have used $|u'| = \sqrt{\frac{4RE}{m}}$ and $|u| = \sqrt{\frac{4R'E}{m}}$. □

Appendix B. Explicit calculation of multiplicative factors to the transition function models

This appendix provides upper and lower estimates for the multiplicative factors $d_\gamma^{lb}(r)$, $d_\gamma^{ub}(r)$, $e_\gamma^{lb}(R)$, $e_\gamma^{ub}(R)$ for the three models of transition function $\mathcal{B} = \mathcal{B}(v, v_*, I, I_*, r, R, \sigma)$ introduced in section 3.1, (3.8), (3.10) and (3.12), namely

(Model 1) $\mathcal{B} = b(\hat{u}\cdot\sigma)\left(\frac{m}{4}|u|^2 + I + I_*\right)^{\gamma/2}$,

(Model 2) $\mathcal{B} = b(\hat{u}\cdot\sigma)\left(R^{\gamma/2}|u|^\gamma + (1-R)^{\gamma/2}\left(\frac{I+I_*}{m}\right)^{\gamma/2}\right)$,

(Model 3) $\mathcal{B} = b(\hat{u}\cdot\sigma)\left(R^{\gamma/2}|u|^\gamma + \left(r(1-R)\frac{I}{m}\right)^{\gamma/2} + \left((1-r)(1-R)\frac{I_*}{m}\right)^{\gamma/2}\right)$,

where $u := v - v_*$ and $\hat{u} = u/|u|$.

B.1. Calculation of the upper bounds.

For the three models (3.8), (3.10) and (3.12), we determine functions $d_\gamma^{ub}(r)$ and $e_\gamma^{ub}(R)$ that appear in bound from above for the transition function $\mathcal{B}$, as defined in (3.2).

B.1.1. *Model 1.* We write for the model 1, from (3.8),

$$\left(\frac{m}{4}|v-v_*|^2 + I + I_*\right)^{\gamma/2} = m^{\gamma/2}\left(\frac{1}{4}|v-v_*|^2 + \frac{I}{m} + \frac{I_*}{m}\right)^{\gamma/2}$$
$$\le m^{\gamma/2}\left(|v-v_*|^2 + \frac{I}{m} + \frac{I_*}{m}\right)^{\gamma/2} \le m^{\gamma/2}\left(|v-v_*|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right),$$

for any $\gamma \in (0, 2]$. Therefore, we can take $d_\gamma^{ub}(r) = 1.$ and $e_\gamma^{ub}(R) = m^{\gamma/2}$.

B.1.2. *Model 2.* We first estimate model 2 in (3.10) by

$$R^{\gamma/2}|u|^\gamma + (1-R)^{\gamma/2}\left(\frac{I+I_*}{m}\right)^{\gamma/2}$$
$$\le \max\{R, (1-R)\}^{\gamma/2}\left(|u|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right),$$

and thus one possible choice is

$$d_\gamma^{ub}(r) = 1, \quad e_\gamma^{ub}(R) = \max\{R, (1-R)\}^{\gamma/2}. \tag{B.1}$$

Another more course estimate can be obtained by using $R \le 1$, which leads to the choice

$$d_\gamma^{ub}(r) = e_\gamma^{ub}(R) = 1.$$

B.1.3. *Model 3.* Finally, model 3 from (3.12) is estimated as follows

$$R^{\gamma/2}|v-v_*|^\gamma + \left(r(1-R)\frac{I}{m}\right)^{\gamma/2} + \left((1-r)(1-R)\frac{I_*}{m}\right)^{\gamma/2}$$
$$\leq \max\{R,(1-R)\}^{\gamma/2}\left(|u|^\gamma + \left(\frac{I}{m}\right)^{\gamma/2} + \left(\frac{I_*}{m}\right)^{\gamma/2}\right)$$
$$\leq 2^{1-\gamma/2}\max\{R,(1-R)\}^{\gamma/2}\left(|u|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right).$$

since $\max\left\{1, r^{\gamma/2}, (1-r)^{\gamma/2}\right\} = 1$. Thus, one possible choice is

$$d_\gamma^{ub}(r) = 1, \quad e_\gamma^{ub}(R) = 2^{1-\gamma/2}\max\{R,(1-R)\}^{\gamma/2}. \tag{B.2}$$

Another possibility comes with the use of $R \leq 1$, which leads to

$$d_\gamma^{ub}(r) = 1, \quad e_\gamma^{ub}(R) = 2^{1-\gamma/2}.$$

B.2. **Calculation of the lower bounds.** In this Section we provide various choices for functions $d_\gamma^{lb}(r)$ and $e_\gamma^{lb}(R)$ that appear in the bound from below of the transition function $\mathcal{B}$, from (3.2), for the three models of transition functions introduced in (3.8), (3.10) and (3.12).

B.2.1. *Model 1.* For the model 1 from (3.8) we can estimate

$$\left(\frac{m}{4}|u|^2 + I + I_*\right)^{\gamma/2} \geq m^{\gamma/2}2^{-(\gamma/2-1)}\left(|u|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right),$$

which implies $d_\gamma^{lb}(r) = 1$, and $e_\gamma^{lb}(R) = m^{\gamma/2}2^{-(\gamma/2-1)}$.

B.2.2. *Model 2.* For the model 2 introduced in (3.10), we write

$$R^{\gamma/2}|u|^\gamma + (1-R)^{\gamma/2}\left(\frac{I+I_*}{m}\right)^{\gamma/2}$$
$$\geq \min\{R^{\gamma/2},(1-R)^{\gamma/2}\}\left(|u|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right)$$
$$\geq R^{\gamma/2}(1-R)^{\gamma/2}\left(|v-v_*|^\gamma + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right),$$

and therefore $d_\gamma^{lb}(r) = 1$ and for $e_\gamma^{lb}(R)$ we have two possible choices,

$$e_\gamma^{lb}(R) = \min\{R^{\gamma/2},(1-R)^{\gamma/2}\}, \tag{B.3}$$

or

$$e_\gamma^{lb}(R) = R^{\gamma/2}(1-R)^{\gamma/2}. \tag{B.4}$$

B.2.3. *Model 3.* For the model 3 from (3.12) we estimate

$$
\begin{aligned}
R^{\gamma/2}|u|^{\gamma} + \left(r(1-R)\frac{I}{m}\right)^{\gamma/2} + \left((1-r)(1-R)\frac{I_*}{m}\right)^{\gamma/2} \\
\geq \min\{R,(1-R)\}^{\gamma/2}\left(|u|^{\gamma} + \left(r\frac{I}{m}\right)^{\gamma/2} + \left((1-r)\frac{I_*}{m}\right)^{\gamma/2}\right) \\
\geq \min\{R,(1-R)\}^{\gamma/2}\min\{r,(1-r)\}^{\gamma/2}\left(|u|^{\gamma} + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right) \\
\geq r^{\gamma/2}(1-r)^{\gamma/2}R^{\gamma/2}(1-R)^{\gamma/2}\left(|u|^{\gamma} + \left(\frac{I+I_*}{m}\right)^{\gamma/2}\right),
\end{aligned}
$$

which allows two different choices

$$
d_{\gamma}^{lb}(r) = \min\{r,(1-r)\}^{\gamma/2}, \quad e_{\gamma}^{lb}(R) = \min\{R,(1-R)\}^{\gamma/2}, \tag{B.5}
$$

or

$$
d_{\gamma}^{lb}(r) = r^{\gamma/2}(1-r)^{\gamma/2}, \quad e_{\gamma}^{lb}(R) = R^{\gamma/2}(1-R)^{\gamma/2}. \tag{B.6}
$$

Appendix C. Computation of the constant $\mathcal{C}_n^1$ from Lemma 5.3(The Polyatomic Compact Manifold Averaging Lemma)

This Section is devoted to the computation of the constant $\mathcal{C}_n^1$ from (5.22), namely,

$$
\mathcal{C}_n^1 = \int_0^1\int_0^1\left(\max\left\{\frac{1+\mu}{2}, r\right\}\right)^n \mathrm{d}r\,\mathrm{d}\mu.
$$

We split integration domain of r as $[0,1] = [0, \frac{1+\mu}{2}] \cup [\frac{1+\mu}{2}, 1]$. Then we use

$$
\max\left\{\frac{1+\mu}{2}, r\right\} = \begin{cases} \frac{1+\mu}{2}, & r \in [0, \frac{1+\mu}{2}], \\ r, & r \in [\frac{1+\mu}{2}, 1] \end{cases}
$$

which yields

$$
\begin{aligned}
\mathcal{C}_n^1 &= \int_0^1\int_0^{\frac{1+\mu}{2}}\left(\frac{1+\mu}{2}\right)^n \mathrm{d}r\,\mathrm{d}\mu + \int_0^1\int_{\frac{1+\mu}{2}}^1 r^n \mathrm{d}r\,\mathrm{d}\mu \\
&= \int_0^1\left(\frac{1+\mu}{2}\right)^{n+1}\mathrm{d}\mu + \frac{1}{n+1}\int_0^1\left(1-\left(\frac{1+\mu}{2}\right)^{n+1}\right)\mathrm{d}\mu \\
&= \frac{1}{n+1} + \frac{2n}{(n+1)(n+2)}\left(1-\left(\frac{1}{2}\right)^{n+2}\right).
\end{aligned} \tag{C.1}
$$

that completes the computation of $\mathcal{C}_n^1$ as given in (5.23).

Appendix D. Some technical results

Lemma D.1 (Polynomial inequality I)**.** *Assume $p > 1$, and let $n_p = \lfloor \frac{p+1}{2} \rfloor$. Then for all $x, y > 0$, the following inequality holds*

$$
(x+y)^p - x^p - y^p \leq \sum_{n=1}^{n_p}\binom{p}{n}\left(x^n y^{p-n} + x^{p-n}y^n\right).
$$

Lemma D.2 (Polynomial inequality II)**.** *Let* $b+1 \leq a \leq \frac{p+1}{2}$*. Then for any* $x, y \geq 0$,

$$x^a y^{p-a} + x^{p-a} y^a \leq x^b y^{p-b} + x^{p-b} y^b.$$

Lemma D.3 (Interpolation inequality)**.** *Let* $k = \alpha k_1 + (1-\alpha) k_2$, $\alpha \in (0,1)$, $0 < k_1 \leq k \leq k_2$*. Then for any* $g \in L^1_k$

$$\|g\|_{L^1_k} \leq \|g\|^{\alpha}_{L^1_{k_1}} \|g\|^{1-\alpha}_{L^1_{k_2}}. \tag{D.1}$$

Appendix E. Existence and Uniqueness Theory for ODE in Banach spaces

Theorem E.1. *Let* $E := (E, \|\cdot\|)$ *be a Banach space,* $\mathcal{S}$ *be a bounded, convex and closed subset of* E*, and* $\mathcal{Q} : \mathcal{S} \to E$ *be an operator satisfying the following properties:*

(a) Hölder continuity condition

$$\|\mathcal{Q}[u] - \mathcal{Q}[v]\| \leq C \|u - v\|^{\beta}, \ \beta \in (0,1), \ \forall u, v \in \mathcal{S};$$

(b) Sub-tangent condition

$$\lim_{h \to 0+} \frac{dist\,(u + h\mathcal{Q}[u], \mathcal{S})}{h} = 0, \ \forall u \in \mathcal{S};$$

(c) One-sided Lipschitz condition

$$[\mathcal{Q}[u] - \mathcal{Q}[v], u - v] \leq C \|u - v\|, \ \forall u, v \in \mathcal{S},$$

where $[\varphi, \phi] = \lim_{h \to 0^-} h^{-1} \left(\|\phi + h\varphi\| - \|\phi\| \right)$.

Then the equation

$$\partial_t u = \mathcal{Q}[u], \ \text{for } t \in (0, \infty), \ \text{with initial data } u(0) = u_0 \text{ in } \mathcal{S},$$

has a unique solution in $C([0,\infty), \mathcal{S}) \cap C^1((0,\infty), E)$.

The proof of this Theorem on ODE flows on Banach spaces can be found in [5].

Remark 6. *In Section 7, we will concentrate on* $E := L^1_1$*. Therefore, for one-sided Lipschitz condition, we will use the following inequality,*

$$[\varphi, \phi] \leq \int_{\mathbb{R}^{3+}} \varphi(v, I) \, sign(\phi(v, I)) \, \langle v, I \rangle^2 \, \mathrm{d}I \, \mathrm{d}v,$$

as pointed out in [5].

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