Generalized kinetic Maxwell type models of granular gases

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Abstract In this chapter we consider generalizations of kinetic granular gas models given by Boltzmann equations of Maxwell type. These type of models for non-linear elastic or inelastic interactions, have many applications in physics, dynamics of granular gases, economy, etc. We present the problem and develop its form in the space of characteristic functions, i.e. Fourier transforms of probability measures, from a very general point of view, including those with arbitrary polynomial non-linearities and in any dimension space. We find a whole class of generalized Maxwell models that satisfy properties that characterize the existence and asymptotic of dynamically scaled or self-similar solutions, often referred as *homogeneous cooling states*. Of particular interest is a concept interpreted as an operator generalization of usual Lipschitz conditions which allows to describe the behavior of solutions to the corresponding initial value problem. In particular, we present, in the most general case, existence of self similar solutions and study, in the sense of probability measures, the convergence of dynamically scaled solutions associated with the Cauchy problem to those self-similar solutions, as time goes to infinity. In addition we show that the properties of these self-similar solutions lead to non classical equilibrium stable states exhibiting power tails. These results apply to different specific problems related to the Boltzmann equation (with elastic and inelastic interactions) and show that all physically relevant properties of solutions follow directly from the general theory developed in this presentation.

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1 Introduction

It has been noticed in recent years that a significant non-trivial physical phenomena in granular gases can be described mathematically by dissipative Boltzmann type equations, as can be seen in [17] for a review in the area. As motivated by this particular phenomena of energy dissipation at the kinetic level, we consider in this chapter the Boltzmann equation for non-linear interactions of Maxwell type and some generalizations of such models.

The classical conservative (elastic) Boltzmann equation with the Maxwell-type interactions is well-studied in the literature (see [5, 14] and references therein). Roughly speaking, this is a mathematical model of a rarefied gas with binary collisions such that the collision frequency is independent of the velocities of colliding particles, and even though the intermolecular potentials are not of those corresponding to hard sphere interactions, still these models provide a very rich inside to the understanding of kinetic evolution of gases.

Recently, Boltzmann equations of Maxwell type were introduced for models of granular gases were introduced in [7] in three dimensions, and a bit earlier in [3] for in one dimension case. Soon after that, these models became very popular among the community studying granular gases (see, for example, the book [13] and references therein). There are two obvious reasons for such studies The first one is that the inelastic Maxwell-Boltzmann equation can be essentially simplified by the Fourier transform similarly as done for the elastic case, where its study becomes more transparent [6, 7]. The second reason is motivated by the special phenomena associated with *homogeneous cooling* behavior, i.e. solutions to the spatially homogeneous inelastic Maxwell-Boltzmann equation have a non-trivial self-similar asymptotics, and in addition, the corresponding self-similar solution has a power-like tail for large velocities. The latter property was conjectured in [16] and later proved in [9, 11]. This is a rather surprising fact, since the Boltzmannn equation for hard spheres inelastic interactions has been shown to have self similar solutions with all moments bounded and large energy tails decaying exponentially. The conjecture of self-similar (or homogeneous cooling) states for such model of Maxwell type interactions was initially based on an exact $1d$ solution constructed in [1]. It is remarkable that such an asymptotics is absent in the elastic case (as the elastic Boltzmann equation has *too many conservation laws*). Later, the self-similar asymptotics was proved in the elastic case for initial data with infinite energy [8] by using another mathematical tools compared to [9] and [12].

Surprisingly, the recently published exact self-similar solutions [12] for elastic Maxwell type model for a slow down process, derived as a formal asymptotic limit of a mixture, also is shown to have power-like tails. This fact definitely suggests

that self-similar asymptotics are related to total energy dissipation rather than local dissipative interactions. As an illustration to this fact, we mention some recent publications [2, 15], where 1*d* Maxwell-type models were introduced for non-standard applications such as models in economics and social interactions, where also self-similar asymptotics and power-like tail asymptotic states were found.

Thus all the above discussed models describe qualitatively different processes in physics or even in economics, however their solutions have a lot in common from mathematical point of view. It is also clear that some further generalizations are possible: one can, for example, include in the model multiple (not just binary) interactions still assuming the constant (Maxwell-type) rate of interactions. Will the multi-linear models have similar properties? The answer is *yes*, as we shall see below.

Thus, it becomes clear that there must be some general mathematical properties of Maxwell models, which, in turn, can explain properties of any particular model. That is to say there must be just one *main theorem*, from which one can deduce all above discussed facts and their possible generalizations. Our goal is to consider Maxwell models from very general point of view and to establish their key properties that lead to the self-similar asymptotics.

All the results presented in this chapter are mathematically rigorous. Their full proofs can be found in [10].

After this introduction, we introduce in Section 2 three specific Maxwell models of the Boltzmann equation: **(A)** classical (elastic) Boltzmann equation; **(B)** the model (A) in the presence of thermostat; **(C)** inelastic Boltzmann equation for Maxwell type interactions. Then, in Section 3, we perform the Fourier transform and introduce an equation that includes all the three models as particular cases. A further generalization is done in Section 4, where the concept of generalized multi-linear Maxwell model (in the Fourier space) is introduced. Such models and their generalizations are studied in detail in Sections 5 and 6. The most important for our approach concept of L-Lipschitz nonlinear operator is explained in Section 4. It is shown (Theorem 4.2) that all multi-linear Maxwell models satisfy the L-Lipschitz condition. This property of the models constitutes a basis for the general theory.

The existence and uniqueness of solutions to the initial value problem is stated in Subsection 5.1 (Theorem 5.2). Then, in Subsection 5.2, we present and study the large time asymptotics under very general conditions that are fulfilled, in particular, for all our models. It is shown that *L-Lipschitz* condition leads to self-similar asymptotics, provided the corresponding self-similar solution does exist. The existence and uniqueness of such self-similar solutions is stated in Subsection 5.3 (Theorem 5.12). This theorem can be considered, to some extent, as the *main theorem* for general Maxwell-type models. Then, in Subsection 5.4, we go back to multi-linear models of Section 4 and study more specific properties of their self-similar solutions.

We explain in Section 6 how to use our theory for applications to any specific model: it is shown that the results can be expressed in terms of just one function $\mu(p), p > 0$, that depends on spectral properties of the specific model. General properties (positivity, power-like tails, etc.) self-similar solutions are studied in Subsec-

tions 6.1 and 6.2. It includes also the case of $1d$ models, where the Laplace (instead of Fourier) transform is used. In Subsection 6.3, we formulate, in the unified statement (Theorem 11.1), the main properties of Maxwell models **(A),(B)** and **(C)** of the Boltzmann equation. This result is, in particular, an essential improvement of earlier results of [7] for the model **(A)** and quite new for the model **(B).**

Applications to one dimensional models are also briefly discussed at the end of Subsection 6.3.

2 Maxwell models of the Boltzmann equation

We consider a spatially homogeneous rarefied d-dimensional gas $(d = 2, 3, \ldots)$ of particles having a unit mass. Let $f(v,t)$, where $v \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ denote respectively the velocity and time variables, be a one-particle distribution function with usual normalization

$$\int_{\mathbb{R}^d} dv \, f(v,t) = 1 \, . \tag{1}$$

Then $f(v,t)$ has an obvious meaning of a time-dependent probability density in $\mathbb{R}^d$. We assume that the collision frequency is independent of the velocities of the colliding particles (Maxwell-type interactions). We discuss three different physical models (A), (B) and (C).

(A) Classical Maxwell gas (elastic collisions). In this case $f(v,t)$ satisfies the usual Boltzmann equation

$$f_t = Q(f,f) = \int_{\mathbb{R}^d \times S^{d-1}} dw \, d\omega \, g(\frac{u \cdot \omega}{|u|})[f(v')f(w') - f(v)f(w)] \, , \tag{2}$$

where the exchange of velocities after a collision are given by

$$v' = \frac{1}{2}(v + w + |u|\omega) \, , \qquad \text{and} \qquad w' = \frac{1}{2}(v + w - |u|\omega)$$

where $u = v - w$ is the relative velocity and $\Omega \in S^{d-1}$. For the sake of brevity we shall consider below the model non-negative collision kernels $g(s)$ such that $g(s)$ is integrable on $[-1, 1]$. The argument t of $f(v,t)$ and similar functions is often omitted below (as in Eq. (2)).

(B) Elastic model with a thermostat This case corresponds to model **(A)** in the presence of a thermostat that consists of Maxwell particles with mass $m > 0$ having the Maxwellian distribution

$$M(v) = (\frac{2\pi T}{m})^{-d/2} \exp(-\frac{m|v|^2}{2T}) \tag{3}$$

with a constant temperature $T > 0$. Then the evolution equation for $f(x,t)$ becomes

$$f_t = Q(f,f) + \theta \int dw \, d\omega \, g(\frac{u \cdot \omega}{|u|})[f(v')M(w') - f(v)M(w)] , \qquad (4)$$

where $\theta > 0$ is a coupling constant, and the exchange of velocities is now

$$v' = \frac{v + m(w + |u|\omega)}{1 + m}, \qquad \text{and} \qquad w' = \frac{v + mw - |u|\omega}{1 + m},$$

with $u = v - w$ the relative velocity and , $\omega \in S^{d-1}$.

Equation (4) was derived in [12] as a certain limiting case of a binary mixture of weakly interacting Maxwell gases.

(C) Maxwell model for inelastic particles. We consider this model in the form given in [9]. Then the inelastic Boltzmann equation in the weak form reads

$$\frac{\partial}{\partial t}(f, \psi) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} dv \, dw \, d\omega \, f(v)f(w)\frac{|u \cdot \omega|}{|u|}[\psi(v') - \psi(v)] , \qquad (5)$$

where $\psi(v)$ is a bounded and continuous test function,

$$(f, \psi) = \int_{\mathbb{R}^d} dv \, f(v,t)\psi(v), \; u = v - w, \; \omega \in S^{d-1}, \; v' = v - \frac{1+e}{2}(u \cdot \omega)\omega , \qquad (6)$$

the constant parameter $0 < e \le 1$ denotes the restitution coefficient. Note that the model (**C** with $e = 1$ is equivalent to the model (**A**) with some kernel $g(s)$.

All three models can be simplified (in the mathematical sense) by taking the Fourier transform.

We denote

$$\hat{f}(k,t) = \mathcal{F}[f] = (f, e^{-ik \cdot v}), \qquad k \in \mathbb{R}^d , \qquad (7)$$

and obtain (by using the same trick as in [6] for the model (**A**)) for all three models the following equations:

(A)
$$\hat{f}_t = \widehat{Q}(\hat{f}, \hat{f}) = \int_{S^{d-1}} d\omega \, g(\frac{k \cdot \omega}{|k|})[\hat{f}(k_+)\hat{f}(k_-) - \hat{f}(k)\hat{f}(0)] ,$$
$$(8)$$

where $k_{\pm} = \frac{1}{2}(k \pm |k|\omega)$, $\omega \in S^{d-1}$, $\hat{f}(0) = 1$.

(B)
$$\hat{f}_t = \widehat{Q}(\hat{f}, \hat{f}) + \theta \int_{S^{d-1}} d\omega \, g(\frac{k \cdot \omega}{|k|})[\hat{f}(k_+)\widehat{M}(k_-) - \hat{f}(k)\widehat{M}(0)] ,$$
$$(9)$$

where $\widehat{M}(k) = e^{-\frac{T|k|^2}{2m}}$, $k_+ = \frac{k+m|k|\omega}{1+m}$, $k_- = k - k_+$, $\omega \in S^{d-1}$, $\hat{f}(0) = 1$.

(C)
$$\hat{f}_t = \int_{S^{d-1}} d\omega \frac{|k \cdot \omega|}{|k|}[\hat{f}(k_+)\hat{f}(k_-) - \hat{f}(k)\hat{f}(0)] ,$$
$$(10)$$

where $\hat{f}(0) = 1$, $k_+ = \frac{1+e}{2}(k \cdot \omega)\omega$, $k_- = k - k_+$, with $\omega \in S^{d-1}$ is the direction containing the two centers of the particles at the time of the interaction. Equivalently, one may alternative write $k_- = \frac{1+e}{4}(k - |k|n\tilde{\omega})$, and $k_+ = k - k_-$, where now $\tilde{\omega} \in$

S^{d-1} is the direction of the post collisional relative velocity, and the term $\frac{|k \cdot \omega|}{|k|} dw$ is replaced by a function $g(\frac{k \cdot \tilde{\omega}}{|k|}) d\tilde{\omega}$.

Case **(B)** can be simplified by the substitution

$$\hat{f}(k,t) = \widetilde{\hat{f}}(k,t) \exp[-\frac{T|k|^2}{2}] \, , \tag{11}$$

leading, omitting tildes, to the equation

(B$'$)
$$\hat{f}_t = \widehat{Q}(\hat{f},\hat{f}) + \theta \int_{S^{d-1}} d\omega \, g(\frac{k \cdot \omega}{|k|})[\hat{f}(\frac{k+m|k|\omega}{1+m}) - \hat{f}(k)] \, , \tag{12}$$

i.e., the model for **(B)** with $T = 0$, or equivalently a linear collisional term the background singular distribution. Therefore, we shall consider below just the case **(B$'$)**, assuming nevertheless that $\hat{f}(k,t)$ in Eq. (12) is the Fourier transform (7) of a probability density $f(v,t)$.

3 Isotropic Maxwell model in the Fourier representation

We shall see that these three models **(A), (B)** and **(C)** admit a class of isotropic solutions with distribution functions $f = f(|v|,t)$. Indeed, according to (7) we look for solutions $\hat{f} = \hat{f}(|k|,t)$ to the corresponding isotropic Fourier transformed problem, given by

$$x = |k|^2 \, , \quad \varphi(x,t) = \hat{f}(|k|,t) = \mathscr{F}[f(|v|,t)] \, , \tag{13}$$

where $\varphi(x,t)$ solves the following initial value problem

$$\varphi_t = \int_0^1 ds \, G(s) \{\varphi[a(s)x]\varphi[b(s)x] - \varphi(x)\} +$$
$$+ \int_0^1 ds H(s) \{\varphi[c(s)x] - \varphi(x)\} \, , \tag{14}$$
$$\varphi_{t=0} = \varphi_0(x) \, , \quad \varphi(0,t) = 1 \, ,$$

where $a(s), b(s), c(s)$ are non-negative continuous functions on $[0,1]$, whereas $G(s)$ and $H(s)$ are generalized non-negative functions such that

$$\int_0^1 ds \, G(s) < \infty \, , \qquad \int_0^1 ds H(s) < \infty \, . \tag{15}$$

Thus, we do not exclude such functions as $G = \delta(s - s_0)$, $0 < s_0 < 1$, etc. We shall see below that, for isotropic solutions (13), each of the three equations (8), (10), (12) is a particular case of Eq. (14).

Let us first consider Eq. (8) with $\hat{f}(k,t) = \varphi(x,t)$ in the notation (13). In that case

$$|k_\pm|^2 = |k|^2 \frac{1 \pm (\omega_0 \cdot \omega)}{2} \, , \qquad \omega_0 = \frac{k}{|k|} \in S^{d-1} \, , \, d = 2, \ldots,$$

and the integral in Eq. (8) reads

$$\int_{S^{d-1}} d\omega \, g(\omega_0 \cdot \omega) \varphi \left[x \frac{1 + \omega_0 \cdot \omega}{2} \right] \varphi \left[x \frac{1 - \omega_0 \cdot \omega}{2} \right] . \qquad (16)$$

It is easy to verify the identity

$$\int_{S^{d-1}} d\omega \, F(\omega \cdot \omega_0) = |S^{d-2}| \int_{-1}^{1} dz \, F(z)(1 - z^2)^{\frac{d-3}{2}} \, , \qquad (17)$$

where $|S^{d-2}|$ denotes the "area" of the unit sphere in $\mathbb{R}^{d-1}$ for $d \geq 3$ and $|S^0| = 2$. The identity (17) holds for any function $F(z)$ provided the integral as defined in the right hand side of (17) exists.

The integral (16) now reads

$$|S^{d-2}| \int_{-1}^{1} dz \, g(z)(1 - z^2)^{\frac{d-3}{2}} \varphi(x \frac{1+z}{2}) \varphi(x \frac{1-z}{2}) =$$
$$= \int_{0}^{1} ds \, G(s) \varphi(sx) \varphi[(1-s)x] \, ,$$

where
$$G(s) = 2^{d-2} |S^{d-2}| g(1 - 2s)[s(1-s)]^{\frac{d-3}{2}} \, , \qquad d = 2, 3, \ldots . \qquad (18)$$

Hence, in this case we obtain Eq. (14), where

(A) $\qquad\qquad a(s) = s \, , \qquad b(s) = 1 - s \, , \qquad H(s) = 0 \, ,$

$$(19)$$

$G(s)$ is given in Eq. (18).

Two other models **(B$'$)** and **(C)**, described by eqs. (12), (10) respectively, can be considered quite similarly. In both cases we obtain Eq. (14), where

(B$'$) $\qquad\qquad a(s) = s, \quad b(s) = 1 - s, \quad c(s) = 1 - \dfrac{4m}{(1 + m)^2} s,$

$$H(s) = \theta G(s) \, ,$$

$$(20)$$

$G(s)$ is given in Eq. (18):

(C) $\qquad\qquad a(s) = \dfrac{(1 + e)^2}{4} s, \quad b(s) = 1 - \dfrac{(1 + e)(3 - e)}{4} s,$

$$H(s) = 0, \quad G(s) = |S^{d-2}|(1 - s)^{\frac{d-3}{2}} \, .$$

$$(21)$$

Hence, all three models are described by Eq. (14) where $0 < a(s), b(s), c(s) \leq 1$ are non-negative linear functions. One can also find in recent publications some other useful equations that can be reduced after Fourier or Laplace transformations to Eq. (14) (see, for example, [2], [15] that correspond to the case $G = \delta(s - s_0)$, $H = 0$).

The equation (14) with $H(s) = 0$ first appeared in its general form in [9] in connection with models **(A)** and **(C)**. The consideration of the problem of self-similar asymptotics for Eq. (14) in that paper made it quite clear that the most important properties of "physical" solutions depend very weakly on the specific functions $G(s)$, $a(s)$ and $b(s)$.

4 Models with multiple interactions

We present now a general framework to study solutions to the type of problems introduced in the previous section.

We assume, without loss of generality, (scaling transformations $\tilde{t} = \alpha t$, $\alpha = $ const.) that

$$\int_0^1 ds\,[G(s) + H(s)] = 1 \tag{22}$$

in Eq. (14). Then Eq. (14) can be considered as a particular case of the following equation for a function $u(x,t)$

$$u_t + u = \Gamma(u)\,, \qquad x \geq 0,\, t \geq 0\,, \tag{23}$$

where

$$\Gamma(u) = \sum_{n=1}^{N} \alpha_n \Gamma^{(n)}(u)\,, \quad \sum_{n=1}^{N} \alpha_n = 1\,,\ \alpha_n \geq 0\,,$$

$$\Gamma^{(n)}(u) = \int_0^\infty da_1 \ldots \int_0^\infty da_n\, A_n(a_1, \ldots, a_n) \prod_{k=1}^{n} u(a_k x), \quad n = 1, \ldots, N\,. \tag{24}$$

We assume that

$$A_n(a) = A_n(a_1, \ldots, a_n) \geq 0\,, \qquad \int_0^\infty da_1 \ldots \int_0^\infty da_n\, A(a_1, \ldots, a_n) = 1\,, \tag{25}$$

where $A_n(a) = A_n(a_1, \ldots, a_n)$ is a generalized density of a probability measure in $\mathbb{R}_+^n$ for any $n = 1, \ldots, N$. We also assume that all $A_n(a)$ have a compact support, i.e.,

$$A_n(a_1, \ldots, a_n) \equiv 0 \ \text{ if } \ \sum_{k=1}^{n} a_k^2 > R^2\,, \qquad n = 1, \ldots, N\,, \tag{26}$$

for sufficiently large $0 < R < \infty$.

Eq. (14) is a particular case of Eq. (23) with

$$N = 2 , \quad \alpha_1 = \int_0^1 ds\, H(s) , \quad \alpha_2 = \int_0^1 ds\, G(s)$$

$$A_1(a_1) = \frac{1}{\alpha_1} \int_0^1 ds\, H(s) \delta[a_1 - c(s)] \tag{27}$$

$$A_2(a_1, a_2) = \frac{1}{\alpha_2} \int_0^1 ds\, G(s) \delta[a_1 - a(s)] \delta[a_2 - b(s)] .$$

It is clear that Eq. (23) can be considered as a generalized Fourier transformed isotropic Maxwell model with multiple interactions provided $u(0,t) = 1$, the case $N = \infty$ in Eqs. (24) can be treated in the same way.

4.1 Statement of the general problem

The general problem we consider below can be formulated in the following way. We study the initial value problem

$$u_t + u = \Gamma(u) , \quad u_{|t=0} = u_0(x) , \quad x \geq 0 , \quad t \geq 0 , \tag{28}$$

in the Banach space $B = C(\mathbb{R}_+)$ of continuous functions $u(x)$ with the norm

$$\|u\| = \sup_{x \geq 0} |u(x)| . \tag{29}$$

It is usually assumed that $\|u_0\| \leq 1$ and that the operator Γ is given by Eqs. (24). On the other hand, there are just a few properties of $\Gamma(u)$ that are essential for existence, uniqueness and large time asymptotics of the solution $u(x,t)$ of the problem (28). Therefore, in many cases the results can be applied to more general classes of operators Γ in Eqs. (28) and more general functional space, for example $B = C(\mathbb{R}^d)$ (anisotropic models). That is why we study below the class (24) of operators Γ as the most important example, but simultaneously indicate which properties of Γ are relevant in each case. In particular, most of the results of Section 4–6 do not use a specific form (24) of Γ and, in fact, are valid for a more general class of operators.

Following this way of study, we first consider the problem (28) with Γ given by Eqs. (24) and point out the most important properties of Γ.

We simplify notations and omit in most of the cases below the argument x of the function $u(x,t)$. The notation $u(t)$ (instead of $u(x,t)$) means then the function of the real variable $t \geq 0$ with values in the space $B = C(\mathbb{R}_+)$.

Remark 1. We shall omit below the argument $x \in \mathbb{R}_+$ of functions $u(x)$, $v(x)$, etc., in all cases when this does not cause a misunderstanding. In particular, inequalities of the kind $|u| \leq |v|$ should be understood as a point-wise control in absolute value, i.e. "$|u(x)| \leq |v(x)|$ for any $x \geq 0$" and so on.

We first start by giving the following general definition for operators acting on a unit ball of a Banach space B denoted by

$$U = \{u \in B : \|u\| \leq 1\} \tag{30}$$

Definition 1. The operator $\Gamma = \Gamma(u)$ is called an L-Lipschitz operator if there exists a linear bounded operator $L : B \to B$ such that the inequality

$$|\Gamma(u_1) - \Gamma(u_2)| \leq L(|u_1 - u_2|) \tag{31}$$

holds for any pair of functions $u_{1,2}$ in U.

Remark 2. Note that the L-Lipschitz condition (31) holds, by definition, at any point $x \in \mathbb{R}_+$ (or $x \in \mathbb{R}^d$ if $B = C(\mathbb{R}^d)$). Thus, condition (31) is much stronger than the classical Lipschitz condition

$$\|\Gamma(u_1) - \Gamma(u_2)\| < C\|u_1 - u_2\| \quad \text{if} \quad u_{1,2} \in U \tag{32}$$

which obviously follows from (31) with the constant $C = \|L\|_B$, the norm of the operator L in the space of bounded operators acting in B. In other words, the terminology "L-Lipschitz condition" means the point-wise Lipschitz condition with respect to an specific linear operator L.

We assume, without loss of generality, that the kernels $A_n(a_1, \ldots, a_n)$ in Eqs. (24) are symmetric with respect to any permutation of the arguments $(a_1, \ldots, a_n)$, $n = 2, 3, \ldots, N$.

The next lemma states that the operator $\Gamma(u)$ defined in Eqs. (24), which satisfies $\Gamma(1) = 1$ (mass conservation) and maps U into itself, satisfies an L-Lipschitz condition, where the linear operator L is the one given by the linearization of Γ near the unity. See [10] for its proof.

Theorem 1. *The operator $\Gamma(u)$ defined in Eqs.* (24) *maps U into itself and satisfies the L-Lipschitz condition* (31)*, where the linear operator L is given by*

$$Lu = \int_0^\infty da\, K(a) u(ax) , \tag{33}$$

with

$$K(a) = \sum_{n=1}^N n\alpha_n K_n(a) ,$$

$$\text{where} \quad K_n(a) = \int_0^\infty da_2 \ldots \int_0^\infty da_n\, A_n(a, a_2, \ldots, a_n) \quad \text{and} \quad \sum_{n=1}^N \alpha_n = 1 . \tag{34}$$

for symmetric kernels $A_n(a, a_2, \ldots, a_n), n = 2, \ldots N$.

And the following corollary holds.

Corollary 1. *The Lipschitz condition* (32) *is fulfilled for* $\Gamma(u)$ *given in Eqs.* (24) *with the constant*

$$C = \|L\| = \sum_{n=1}^{N} n\alpha_n \, , \qquad \sum_{n=1}^{\infty} \alpha_n = 1 \, , \tag{35}$$

where $\|L\|$ *is the norm of L in B.*

It can also be shown that the *L*-Lipschitz condition holds in $B = C(\mathbb{R}^d)$ for "gain-operators" in Fourier transformed Boltzmann equations (8), (9) and (10).

5 The general problem in Fourier representation

5.1 Existence and uniqueness of solutions

It is possible to show, with minimal requirements, the existence and uniqueness results associated with the initial value problem (28) in the Banach space $(B, \| \cdot \|)$, where the norm associated to B is defined in (29) In fact, this existence and uniqueness result is an application of the classical Picard iteration scheme and holds for any operator Γ which satisfies the usual Lipschitz condition (32) and transforms the unit ball U into itself. The proof of all statements below can be found in [10].

Lemma 1. *(Picard Iteration scheme) The operator* $\Gamma(u)$ *maps U into itself and satisfies the L-Lipschitz condition* (32)*, then the initial value problem* (28) *with arbitrary* $u_0 \in U$ *has a unique solution* $u(t)$ *such that* $u(t) \in U$ *for any* $t \geq 0$.

Next, we observe that the *L*-Lipschitz condition yields an estimate for the difference of any two solutions of the problems in terms of their initial states. This is a key fact in the development of the further studies of the self similar asymptotics.

Theorem 2. *Consider the Cauchy problem* (28) *with* $\|u_0\| \leq 1$ *and assume that the operator* $\Gamma : B \to B$

(a) *maps the closed unit ball* $U \subset B$ *to itself, and*
(b) *satisfies a L-Lipschitz condition* (31) *for some positive bounded linear operator* $L : B \to B$.

Then

i) *there exists a unique solution* $u(t)$ *of the problem* (28) *such that* $\|u(t)\| \leq 1$ *for any* $t \geq 0$;
ii) *any two solutions* $u(t)$ *and* $w(t)$ *of problem* (28) *with initial data in the unit ball U satisfy the inequality*

$$|u(t) - w(t)| \leq \exp\{t(L-1)\}(|u_0 - w_0|) \, . \tag{36}$$

Note that under the same conditions as in Theorem 1 the operator Γ given in Eqs. (24) satisfies necessary conditions for the Theorem 2.

We remind to the reader that the initial value problem (28) appeared as a generalization of the initial value problem (14) for a characteristic function $\varphi(x,t)$, i.e., for the Fourier transform of a probability measure (see Eqs. (13), (1)). It is important therefore to show that the solution $u(x,t)$ of the problem (28) is a characteristic function for any $t > 0$ provided this is so for $t = 0$, which is addressed in the following statement.

Lemma 2. *Let $U' \subset U \subset B$ be any closed convex subset of the unit ball U (i.e., $u = (1 - \theta)u_1 + \theta u_2 \in U'$ for any $u_{1,2} \in U'$ and $\theta \in [0,1]$). If $u_0 \in U'$ in Eq. (28) and U is replaced by U' in the condition (1) of Theorem 2, the theorem holds and $u(t) \in U'$ for any $t \geq 0$.*

Remark 3. It is well-known (see, for example, the textbook [18]) that the set $U' \subset U$ of Fourier transforms of probability measures in $\mathbb{R}^d$ (Laplace transforms in the case of $\mathbb{R}_+$) is convex and closed with respect to uniform convergence. On the other hand, it is easy to verify that the inclusion $\Gamma(U') \subset U'$, where Γ is given in Eqs. (24), holds in both cases of Fourier and Laplace transforms. Hence, all results obtained for Eqs. (23), (24) can be interpreted in terms of "physical" (positive and satisfying the condition (1)) solutions of corresponding Boltzmann-like equations with multilinear structure of any order.

We also point out that all results of this section remain valid for operators Γ satisfying conditions (24), with a more general condition such as

$$\sum_{n=1}^{N} \alpha_n \leq 1 \,, \qquad \alpha_n \geq 0 \,, \tag{37}$$

so that $\Gamma(1) < 1$ and so the mass may not be conserved. The only difference in this case is that the operator L satisfying conditions (33), (34) is not a linearization of $\Gamma(u)$ near the unity, but nevertheless Theorem 1 remains true. The inequality (37) is typical for Fourier (Laplace) transformed Smoluchowski-type equations where the total number of particles is decreasing in time (see [21, 22] for related work).

In the next three sections we study in more detail the solutions to the initial value problem (28)-(29) constructed in Theorem 2 and, in particular, their long time behavior, existence, uniqueness, and properties of the self-similar solutions.

5.2 Large time asymptotics

The long time asymptotics results are a consequence of some very general properties of operators Γ, namely, that Γ maps the unit ball U of the Banach space $B = C(\mathbb{R}_+)$

into itself, Γ is an L-Lipschitz operator (i.e. satisfies (31)) and that Γ is invariant under dilations.

These three properties are sufficient to study self-similar solutions and large time asymptotic behavior for the solution to the Cauchy problem (28) in the unit ball U of the Banach space $C(\mathbb{R}_+)$.

Main properties of the operator Γ:

a) Γ *maps the unit ball U of the Banach space $B = C(\mathbb{R}_+)$ into itself, that is*

$$\|\Gamma(u)\| \leq 1 \quad \text{for any} \quad u \in C(\mathbb{R}_+) \quad \text{such that} \quad \|u\| \leq 1. \tag{38}$$

b) Γ *is an L-Lipschitz operator (i.e. satisfies (31)) with L from (33), i.e.*

$$|\Gamma(u_1) - \Gamma(u_2)|(x) \leq L(|u_1 - u_2|)(x) = \int_0^\infty da K(a) |u_1(ax) - u_2(ax)|, \tag{39}$$

for $K(a) \geq 0$, for all $x \geq 0$ and for any two functions $u_{1,2} \in C(\mathbb{R}_+)$ such that $\|u_{1,2}\| \leq 1$.

c) Γ *is invariant under dilations:*

$$e^{\tau \mathscr{D}} \Gamma(u) = \Gamma(e^{\tau \mathscr{D}} u), \quad \mathscr{D} = x \frac{\partial}{\partial x}, \quad e^{\tau \mathscr{D}} u(x) = u(xe^\tau), \quad \tau \in \mathbb{R}. \tag{40}$$

No specific information about Γ beyond these three conditions will be used in this section.

It was already shown in Theorem 2 that the conditions **a)** and **b)** guarantee existence and uniqueness of the solution $u(x,t)$ to the initial value problem (28)-(29). The property **b)** yields the estimate (36) that is very important for large time asymptotics, as we shall see below. The property **c)** suggests a special class of self-similar solutions to Eq. (28).

We recall the usual meaning of the notation $y = O(x^p)$ (often used below): $y = O(x^p)$ if and only if there exists a positive constant C such that

$$|y(x)| \leq Cx^p \quad \text{for any} \quad x \geq 0. \tag{41}$$

In order to study long time stability properties to solutions whose initial data differs in terms of $O(x^p)$, we will need some spectral properties of the linear operator L.

Definition 2. Let L be the positive linear operator given in Eqs. (33), (34), then

$$Lx^p = \lambda(p)x^p, \qquad 0 < \lambda(p) = \int_0^\infty da K(a) a^p < \infty, \quad p \geq 0, \tag{42}$$

and the spectral function $\mu(p)$ is defined by

$$\mu(p) = \frac{\lambda(p) - 1}{p}. \tag{43}$$

An immediate consequence of properties **a)** and **b)**, as stated in (39), is that one can obtain a criterion for a point-wise in x estimate of the difference of two solutions to the initial value problem (28) yielding decay properties depending on the spectrum of L, as the following statement and its corollary assert.

Lemma 3. *Let* $u_{1,2}(x,t)$ *be any two classical solutions of the problem* (28) *with initial data satisfying the conditions*

$$|u_{1,2}(x,0)| \le 1, \quad |u_1(x,0) - u_2(x,0)| \le Cx^p, \quad x \ge 0 \tag{44}$$

for some positive constant C and p. Then

$$|u_1(x,t) - u_2(x,t)| \le Cx^p e^{-t(1-\lambda(p))}, \qquad \text{for all } t \ge 0 \tag{45}$$

Corollary 2. *The minimal constant C for which condition* (44) *is satisfied is*

$$C_0 = \sup_{x \ge 0} \frac{|u_1(x,0) - u_2(x,0)|}{x^p} = \left\| \frac{u_1(x,0) - u_2(x,0)|}{x^p} \right\|, \tag{46}$$

and the following estimate holds

$$\left\| \frac{u_1(x,t) - u_2(x,t)|}{x^p} \right\| \le e^{-t(1-\lambda(p))} \left\| \frac{u_1(x,0) - u_2(x,0)|}{x^p} \right\| \tag{47}$$

for any $p > 0$.

A result similar to Lemma 3 was first obtained in [9] for the inelastic Boltzmann equation whose Fourier transform is given in example **(C)**, Eq. (10). Its corollary in the form similar to (47) for equation (10) was stated later in [11] and was interpreted there as "the contraction property of the Boltzmann operator" (note that the left hand side of Eq.(47) can be understood as a distance between two solutions). Independently of the terminology. the key reason for estimates (45)-(47) is the Lipschitz property of the operator Γ. It is remarkable that the large time asymptotics of $u(x,t)$, satisfying the problem (28) with such Γ, can be explicitly expressed through spectral characteristics of the linear operator L.

In order to study the large time asymptotics of $u(x,t)$ in more detail we distinguish two different kinds of asymptotic behavior:

1) convergence to stationary solutions,
2) convergence to self-similar solutions provided the condition (**c**), of the main properties on Γ, is satisfied.

The case **1)** is relatively simple. Any stationary solution $\bar{u}(x)$ of the problem (28) satisfies the equation

$$\Gamma(\bar{u}) = \bar{u}, \qquad \bar{u} \in C(\mathbb{R}_+), \ \|\bar{u}\| \le 1. \tag{48}$$

If the stationary solution $\bar{u}(x)$ does exists (note, for example, that $\Gamma(0) = 0$ and $\Gamma(1) = 1$ for Γ given in Eqs. (24)) then the large time asymptotics of some classes

of initial data $u_0(x)$ in (28) can be studied directly on the basis of Lemma 3. It is enough to assume that $|u_0(x) - \bar{u}(x)|$ satisfies (44) with p such that $\lambda(p) < 1$. Then $u(x,t) \to \bar{u}(x)$ as $t \to \infty$, for any $x \geq 0$.

This simple consideration, however, does not answer at least two questions:

A) What happens with $u(x,t)$ if the inequality (44) for $|u_0(x) - \bar{u}(x)|$ is satisfied with such p that $\lambda(p) > 1$?

B) What happens with $u(x,t)$ for large x (note that the estimate (45) becomes trivial if $x \to \infty$).

In order to address these questions we consider a special class of solutions of Eq. (28), the so-called self-similar solutions. Indeed the property **c)** of Γ shows that Eq. (28) admits a class of formal solutions $u_s(x,t) = w(xe^{\mu_* t})$ with some real μ_*. It is convenient for our goals to use a terminology that slightly differs from the usual one.

Definition 3. The function $w(x)$ is called a self-similar solution associated with the initial value problem (28) if it satisfies the problem

$$\mu_* \mathscr{D}w + w = \Gamma(w) , \qquad \|w\| \leq 1 , \tag{49}$$

in the notation of Eqs. (40), (24).

The convergence of solutions $u(x,t)$ of the initial value problem (28) to a stationary solution $\bar{u}(x)$ can be considered as a special case of the self-similar asymptotics with $\mu_* = 0$.

Under the assumption that self-similar solutions exists (the existence is proved in the next section), we state the fundamental result on the convergence of solutions $u(x,t)$ of the initial value problem (28) to self-similar ones (sometimes called in the literature *self-similar stability*).

Lemma 4. *We assume that*

i) *for some $\mu_* \in \mathbb{R}$, there exists a classical (continuously differentiable if $\mu_* \neq 0$) solution $w(x)$ of Eq. (49) such that $\|w\| \leq 1$;*

ii) *the initial data $u(x,0) = u_0$ in the problem (28) satisfies*

$$u_0 = w + O(x^p) , \qquad \|u_0\| \leq 1 , \text{ for } p > 0 \text{ such that } \mu(p) < \mu_* , \tag{50}$$

where $\mu(p)$ defined in (43) is the spectral function associated to the operator L.

Then
$$|u(xe^{-\mu_* t}, t) - w(x)| = O(x^p)e^{-pt(\mu_* - \mu(p))} \tag{51}$$

and therefore
$$\lim_{t \to \infty} u(xe^{-\mu_* t}, t) = w(x) , \qquad x \geq 0 . \tag{52}$$

Remark 4. Lemma 4 shows how to find a domain of attraction of any self-similar solution provided the self-similar solution is itself known. It is remarkable that

the domain of attraction can be expressed in terms of just the *spectral function* $\mu(p)$, $p > 0$, defined in (43), associated with the linear operator L for which the operator Γ satisfies the L-Lipschitz condition.

Generally speaking, the equality (52) can be also fulfilled for some other values of p with $\mu(p) > \mu_*$ in Eq. (50), but, at least, it always holds if $\mu(p) < \mu_*$.

We shall need some properties of the *spectral function* $\mu(p)$. Having in mind further applications, we formulate these properties in terms of the operator Γ given in Eqs. (24), though they depend only on $K(a)$ in Eqs. (42)

Lemma 5. *The spectral function $\mu(p)$ has the following properties:*

i) *It is positive and unbounded as $p \to 0^+$, with asymptotic behavior given by*

$$\mu(p) \approx \frac{\lambda(0) - 1}{p}, \qquad p \to 0,   \tag{53}$$

where, for Γ from (24)

$$\lambda(0) = \int_0^\infty da\, K(a) = \sum_{n=1}^N \alpha_n n \geq 1, \qquad \sum_{n=1}^N \alpha_n = 1, \ \alpha_n \geq 0,   \tag{54}$$

and therefore $\lambda(0) = 1$ if and only if the operator Γ (24) is linear ($N = 1$);

ii) *In the case of a multi-linear Γ operator, there is not more than one point $0 < p_0 < \infty$, where the spectral function $\mu(p)$ achieves its minimum, that is, $\mu'(p_0) = \frac{d\mu}{dp}(p_0) = 0$, with $\mu(p_0) \leq \mu(p)$ for any $p > 0$, provided $N \geq 2$ and $\alpha_N > 0$.*

Remark 5. From now on, we shall always assume below that the operator Γ from (24) is multi-linear. Otherwise it is easy to see that the problem (49) has no solutions (the condition $\|w\| \leq 1$ is important!) except for the trivial ones $w = 0, 1$.

The following corollaries are readily obtained from lemma 5 part **ii)** and its proof.

Corollary 3. *For the case of a non-linear Γ operator, i.e. $N \geq 2$, the spectral function $\mu(p)$ is always monotone decreasing in the interval $(0, p_0)$, and $\mu(p) \geq \mu(p_0)$ for $0 < p < p_0$. This implies that there exists a unique inverse function $p(\mu)$: $(\mu(p_0), +\infty) \to (0, p_0)$, monotone decreasing in its domain of definition.*

Corollary 4. *There are precisely four different kinds of qualitative behavior of $\mu(p)$ shown on Fig.1.*

Proof. There are two options: $\mu(p)$ is monotone decreasing function (Fig.1 (a)) or $\mu(p)$ has a minimum at $p = p_0$ (Fig.1 (b,c,d)). In case Fig.1 (a) $\mu(p) > 0$ for all $p > 0$ since $\mu(p) > 1/p$. The asymptotics of $\lambda(p)$ (42) is clear:

$$\textbf{(1)} \qquad \lambda(p) \xrightarrow[p \to \infty]{} \lambda_\infty \in \mathbb{R}_+ \ \text{ if } \ \int_{1^+}^\infty da\, K(a) = 0;   \tag{55}$$

$$\textbf{(2)} \qquad \lambda(p) \xrightarrow[p \to \infty]{} \infty \ \text{if } \ \int_{1^+}^\infty da\, K(a) > 0.   \tag{56}$$

In the case **(1)** when $\mu(p) \to \infty$ as $p \to 0$, two possible pictures (with and without minimum) are shown on Fig.1 (b) and Fig.1 (a) respectively. In case **(2)**, from Eq. (42) it is clear that $\lambda(p)$ grows exponentially for large p, therefore $\mu(p) \to \infty$ as $p \to \infty$. Then the minimum always exists and we can distinguish two cases: $\mu(p_0) < 0$ (Fig.1 (d) and $\mu(p_0) > 0$ (Fig.1 (c))
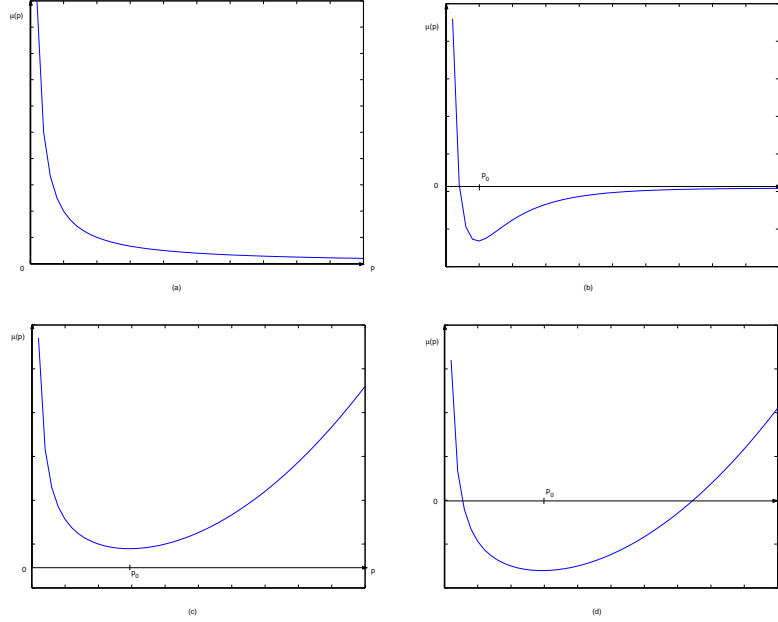

Fig. 1 Possible profiles of the spectral function $\mu(p)$

We note that, for Maxwell models **(A)**, **(B)**, **(C)** of Boltzmann equation (Sections 2 and 3), only cases (a) and (b) of Fig.1 can be possible (actually this is the case (b)) since the condition (55) holds. Fig.1 gives a clear graphic representation of the domains of attraction of self-similar solutions (Lemma 4): it is sufficient to draw the line $\mu(p) = \mu_* = constant$, and to consider a p such that the graph of $\mu(p)$ lies below this line.

Therefore, the following corollary follows directly from the properties of the spectral function $\mu(p)$, as characterized by the behaviors in Fig.1, where we assume that $\mu(p_0) = 0$ for $p_0 = \infty$, for the case shown on Fig.1 (a).

Corollary 5. *Any self-similar solution* $u_s(x,t) = w(xe^{\mu_* t})$ *with* $\mu(p_0) < \mu_* < \infty$ *has a non-empty domain of attraction, where* p_0 *is the unique (minimum) critical point of the spectral function* $\mu(p)$.

Proof. We use Lemma 4 part **ii)** on any initial state $u_0 = w + O(x^p)$ with $p > 0$ such that $\mu(p_0) \leq \mu(p) < \mu_*$. In particular, Eqs. (51) and (52) show that the domain of

attraction of $w(xe^{\mu_* t})$ contains any solution to the initial value problem (28) with the initial state as above.

It is clear that the inequalities of the kind $u_1 - u_2 = O(x^p)$ for any $p > 0$ such that $\mu(p) < \mu_*$, for any fixed $\mu_* \geq \mu(p_0)$ play an important role. We can use specific properties of $\mu(p)$ in order to express such inequalities in more convenient form.

Lemma 6. *For any given* $\mu_* \in (\mu(p_0), \infty)$ *and* $u_{1,2}(x)$ *such that* $\|u_{1,2}\| < \infty$, *the following two statements are equivalent:*

i) *there exists* $p > 0$ *such that*

$$u_1 - u_2 = O(x^p) , \qquad \text{with } \mu(p) < \mu_* . \tag{57}$$

ii) *There exists* $\varepsilon > 0$ *such that*

$$u_1 - u_2 = O(x^{\mathrm{p}(\mu_*) + \varepsilon}) , \qquad \text{with } \mathrm{p}(\mu_*) < p_0 , \tag{58}$$

where $\mathrm{p}(\mu)$ *is the inverse to* $\mu(p)$ *function, as defined in* **Corollary 3**.

Finally, to conclude this section, we show a general property of the initial value problem (28) for any non-linear Γ operator satisfying conditions **a)** and **b)** given in Eqs. (38) and (39) respectively. This property gives the control to the point-wise difference of any two rescaled solutions to (28) in the unit sphere of B, whose initial states differ by $O(x^p)$. It is formulated as follows.

Lemma 7. *Consider the problem* (28), *where* Γ *satisfies the conditions* **(a)** *and* **(b)**. *Let* $u_{1,2}(x,t)$ *are two solutions satisfying the initial conditions* $u_{1,2}(x,0) = u_0^{1,2}(x)$ *such that*

$$\|u_0^{1,2}\| \leq 1 , \quad u_0^1 - u_0^2 = O(x^p) , \quad p > 0 . \tag{59}$$

then, for any real μ_*,

$$\Delta_{\mu_*}(x,t) = u_1(xe^{-\mu_* t}, t) - u_2(xe^{-\mu_* t}, t) = O(x^p)e^{-pt[\mu_* - \mu(p)]} \tag{60}$$

and therefore

$$\lim_{t \to \infty} \Delta_{\mu_*}(x,t) = 0 , \qquad x \geq 0 , \tag{61}$$

for any $\mu_* > \mu(p)$.

Remark 6. There is an important point to understand here. Lemmas 3 and 4 hold for any operator Γ that satisfies just the two properties **(a)** and **(b)** stated in (38) and (39). It says that, in some sense, a distance between any two solutions with initial conditions satisfying Eqs. (59) tends to zero as $t \to \infty$. Such terminology and corresponding distances were introduced for specific forms of Maxwell-Boltzmann models in [4, 19]. It should be pointed out, however, that this *contraction property* may not say much about large time asymptotics of $u(x,t)$, unless the corresponding self-similar solutions are known, for which the operator Γ must be invariant under dilations (so it satisfies also property **(c)** as well, as stated in (40)). In such case one can use estimate (61) to deduce the convergence in the form (52), (53).

Therefore one must study the problem of existence of self-similar solutions, which is considered in the next Section.

5.3 Existence of self-similar solutions

We develop now a criteria for existence, uniqueness and self-similar asymptotics to the problem Eq. (49) for any operator Γ that satisfies conditions **(a)**, **(b)** and **(c)** from Section 6, with the corresponding spectral function $\mu(p)$ defined in (43).

Theorem 3 below shows the criteria for existence and uniqueness of self-similar solutions for any operator Γ that satisfies just conditions **(a)** and **(b)**. Then Theorem 4 follows, showing a general criteria to self-similar asymptotics for the problem (28) for any operator Γ that satisfying conditions **(a)** **(b)** and **(c)**.

We consider Eq. (49) written in the form

$$\mu x w'(x) + w(x) = g(x) , \qquad g = \Gamma(w) , \ \mu \in \mathbb{R} , \tag{62}$$

and, assuming that $\|w\| < \infty$, transform this equation to the integral form. It is easy to verify that the resulting integral equation reads

$$w(x) = \int_0^1 d\tau \, g(x\tau^\mu) \tag{63}$$

By means of an iteration scheme, the following result can be proved.

Theorem 3. *Consider Eq. (62) with arbitrary $\mu \in \mathbb{R}$ and the operator Γ satisfying the conditions* **(a)** *and* **(b)** *from Section 6. Assume that there exists a continuous function $w_0(x)$, $x \geq 0$, such that*

i) $\qquad \|w_0\| \leq 1$ *and*
ii)

$$\int_0^1 d\tau \, g_0(x\tau^\mu) = w_0(x) + O(x^p) , \qquad g_0 = \Gamma(w_0) , \tag{64}$$

with some $p > 0$ satisfying the inequality

$$\mu(p) = \frac{1}{p} \Big[\int_0^\infty da \, K(a) a^p - 1 \Big] < \mu . \tag{65}$$

Then there exists a classical solution $w(x)$ of Eq. (62). The solution is unique in the class of continuous functions satisfying conditions

$$\|w\| \leq 1 , \qquad w(x) = w_0(x) + O(x^{p_1}) , \tag{66}$$

with any p_1 such that $\mu(p_1) < \mu$.

Combining Lemma 1 and Lemma 4, the following general statement related to the self-similar asymptotics for the problem (28) is obtained.

Theorem 4. *Let $u(x,t)$ be a solution of the problem* (28) *with $\|u_0\| \leq 1$ and Γ satisfying the conditions* **(a)**, **(b)**, **(c)** *from Section 6. Let $\mu(p)$ denote the spectral function* (65) *having its minimum (infimum) at $p = p_0$ (see Fig.1), the case $p_0 = \infty$ is also included. We assume that there exists $p \in (0, p_0)$ and $0 < \varepsilon < p_0 - p$ such that*

$$\int_0^1 d\tau\, g_0(x\tau^{\mu(p)}) = u_0(x) + 0(x^{p+\varepsilon})\,, \qquad g_0 = \Gamma(u_0), \quad \varepsilon > 0\,. \tag{67}$$

Then

i) *there exists a unique solution $w(x)$ of the equation* (62) *with $\mu = \mu(p)$ such that*

$$\|w\| \leq 1\,, \qquad w(x) = u_0(x) + O(x^{p+\varepsilon})\,, \tag{68}$$

ii)

$$\lim_{t \to \infty} u(x e^{-\mu(p)t}, t) = w(x)\,, \qquad x \geq 0\,, \tag{69}$$

where the convergence is uniform on any bounded interval in $\mathbb{R}_+$ and

$$u(x e^{-\mu(p)t}, t) - w(x) = O(x^{p+\varepsilon} e^{-\beta(p,\varepsilon)t})\,, \tag{70}$$

with $\beta(p,\varepsilon) = (p+\varepsilon)(\mu(p) - \mu(p+\varepsilon)) > 0$.

Hence, a general criterion (68) is obtained for the self-similar asymptotics of $u(x,t)$ with a given initial condition $u_0(x)$. The criterion can be applied to the problem (28) with any operator Γ satisfying conditions **(a)**, **(b)**, **(c)** from Section 6. The specific class (24) of operators Γ is studied in Section 8. We shall see below that the condition (68) can be essentially simplified for such operators.

5.4 Properties of self-similar solutions

We now apply the general theory (in particular, Theorem 4) to the particular case of the multi-linear operators Γ considered in Section 4, where their corresponding spectral function $\mu(p)$ satisfies (65), (34) whose behavior corresponds to Fig.1. We also show that $p_0 = \min_{p>0} \mu(p) > 1$ is a necessary condition for self-similar asymptotics.

In addition, Theorem 5 establish sufficient conditions for which self-similar solutions of problem (62) will lead to well defined self-similar solutions (distribution functions) of the original problem after taking the inverse Fourier transform.

We consider the integral equation (63) written as

$$w = \Gamma_\mu(w) = \int_0^1 dt\, g(xt^\mu)\,, \qquad g = \Gamma(w)\,, \ \mu \in \mathbb{R}\,. \tag{71}$$

The following two properties of $w(x)$ that are independent of the specific form (24) of Γ.

Lemma 8.

i. *If there exist a closed subset $U' \subset U$ of the unit ball U in B, such that $\Gamma_\mu(U') \subset U'$ for any $\mu \in \mathbb{R}$, and for some function $w_0 \in U'$ the conditions of Theorem 3 are satisfied, then $w \in U'$.*

ii. *If the conditions of Theorem 3 for Γ are satisfied and, in addition, $\Gamma(1) = 1$,*

then the solution $w_* = 1$ of Eq. (71) is unique in the class of functions $w(x)$ satisfying the condition

$$w(x) = 1 + O(x^p) , \qquad \mu(p) < \mu . \tag{72}$$

We observe that the statement [ii] can be interpreted as a necessary condition for existence of non-trivial ($w \neq$ const.) solutions of Eq. (71): if there exists a non-trivial solution $w(x)$ of Eq. (71), where $\Gamma(1) = 1$, such that

$$\|w\| = 1 , \qquad w = 1 + O(x^p) , \quad p > 0 , \quad \text{then} \quad \mu \leq \mu(p) . \tag{73}$$

We recall that $\mu(p)$ satisfies the inequality $\mu(p) \geq \mu(p_0)$ (Fig.1).

If $p \geq p_0$ (provided $p_0 < \infty$) in Eqs. (73), then there are no non-trivial solutions with $\mu > \mu(p_0)$.

On the other hand, possible solutions with $\mu \leq \mu(p_0)$ (even if they exist) are irrelevant for the problem (28) since they have an empty domain of attraction (Lemma 4).

Therefore we always assume below that $\mu > \mu(p_0)$ and, consequently, $p \in (0, p_0)$ in Eq. (73).

Let us consider now the specific class (24)–(25) of operators Γ, with functions $u(x)$ satisfying the condition $u(0) = 1$. Then, $u(0,t) = 1$ for the solution $u(x,t)$ of the problem (28).

In addition, the operators (24) are invariant under dilation transformations (40) (property **(c)**, Section 6). Therefore, the problem (28) with the initial condition $u_0(x)$ satisfying

$$u(0) = 1 , \quad \|u_0\| = 1 ; \qquad u_0(x) = 1 - \beta x^p + \cdots , \quad x \to 0 , \tag{74}$$

can be always reduced to the case $\beta = 1$ by transformation $x' = x\beta^{1/p}$.

Moreover, the whole class of operators (24) with different kernels $A_n(a_1, \ldots, a_n)$, $n = 1, 2, \ldots$, is invariant under transformations $\tilde{x} = x^p$, $p > 0$. The result of such transformation of Γ is another operator $\tilde{\Gamma}$ of the same class (24) with kernels $\tilde{A}_n(a_1, \ldots, a_n)$.

Therefore we fix the initial condition (74) with $\beta = 1$ and transform the function (74) and the equation (28) to new variables $\tilde{x} = x^p$. Then we omit the tildes and reduce the problem (28), with initial condition (74) to the case $\beta = 1$, $p = 1$. We study this case in detail and formulate afterward the results in terms of initial variables.

Next, we assume a bit more about the asymptotics of the initial data $u_0(x)$ for small x, namely

$$\|u_0\| = 1 , \quad u_0(x) = 1 - x + O(x^{1+\varepsilon}) , \quad x \to 0 , \tag{75}$$

with some $\varepsilon > 0$.

Then, our goal now is to apply the general theory (in particular, Theorem 4 and criterion (67)) to this particular case. We assume that the spectral function $\mu(p)$ given by (65), (34), corresponds to one of the four cases shown on Fig.1 with $p_0 > 1$.

Let us take a typical function $u_0 = e^{-x}$ satifying (75) and apply the criterion (67), from Theorem 4 or, equivalently, look for such $p > 0$ that (67) is satisfied. That is, find possibles values of $p > 0$ such that

$$\Gamma_{\mu(p)}(e^{-x}) - e^{-x} = 0(x^{p+\varepsilon}) , \tag{76}$$

in the notation of Eq. (71).

It is important to observe that now the spectral function $\mu(p)$ is closely connected with the operator Γ (see Eqs. (65) and (34)), since this was not assumed in the general theory of Sections 4–7. This connection leads to much more specific results, than, for example, the general Theorems 3, 4.

The properties of self-similar solutions to problem (62) for $p_0 > 1$, and consequently, for

$$\mu(p) \geq \mu(p_0) > -\frac{1}{p_0} > -1 , \tag{77}$$

can be obtained from the structure of $\Gamma_\mu(e^{-x})$ for any $\mu > -1$ using its explicit formula (Eqs. (65) and (34)). In particular, for

$$\Gamma_\mu(e^{-x}) = \sum_{n=1}^{N} \alpha_n \int_{\mathbb{R}_+^n} da_1 \ldots da_n A_n(a_1, \ldots, a_n) I_\mu \left[x \sum_{k=1}^{n} a_k \right] , \tag{78}$$

where

$$I_\mu(y) = \int_0^1 dt \, e^{-yt^\mu} , \quad \mu \in \mathbb{R} , \, y > 0 , \qquad \sum_{n=1}^{N} \alpha_n = 1 . \tag{79}$$

the following two statements can be proven.

Lemma 9. *The condition* (76) *is fulfilled if and only if* $p \leq 1$, *and therefore* $\mu(p) \geq \mu(1)$ *whenever* $p_0 > 1$ *with* $\mu(p_0) = \min_{p>0} \mu(p)$.

Theorem 5. *The limiting function* $w(x)$ *constructed by the iteration scheme to prove existence in the solution* w *in Theorem 3 satisfies Eq.* (71) *with* $\mu = \mu(1)$, *where* Γ *is given in Eqs.* (24), $\mu(p)$ *is defined in Eqs.* (65), (34). *Then, the following conditions are fulfilled for* $w(x)$:

1. *It satisfies*

$$0 \leq< w(x) \leq 1 , \qquad with w(0) = 1 \ and \ w'(0) = -1 , \tag{80}$$

$$w'(x) \le 0 , \qquad |w'(x)| \le 1 , \qquad and \quad w(x) = e^{-x} + O(x^{\pi(\mu)}) , \qquad (81)$$

with

$$\pi(\mu) = \begin{cases} 2 & \text{if } \mu > -\frac{1}{2}, \\ 2 - \varepsilon & \text{for any } \varepsilon > 0 & \text{if } \mu = \frac{1}{2}, \\ \frac{1}{|\mu|} & \text{if } -1 < \mu < -\frac{1}{2}. \end{cases}$$

2. *Further*

$$e^{-x} \le w(x) \le 1 , \qquad \lim_{x \to \infty} w(x) = 0 , \qquad and \qquad (82)$$

3. *there exists a generalized non-negative function $R(\tau)$, $\tau \ge 0$, such that*

$$w(x) = \int_0^\infty d\tau R(\tau) e^{-\tau x} , \qquad \int_0^\infty d\tau R(\tau) = \int_0^\infty d\tau R(\tau)\tau = 1 . \qquad (83)$$

The integral representation (83) is important for properties of corresponding distribution functions satisfying Boltzmann-type equations. Now it is easy to return to initial variables with u_0 given in Eq. (74) and to describe the complete picture of the self-similar relaxation for the problem (28).

6 Main results for Maxwell models with multiple interactions

6.1 Self-similar asymptotics

We apply now the results of Section 6 to the specific case when the Cauchy problem (28) with a fixed operator Γ (24) corresponds to the Fourier transform problem for Maxwell models with multiple interactions. In particular we study the time evolution of $u_0(x)$ satisfying the conditions

$$\|u_0\| = 1 ; \quad u_0 = 1 - x^p + O(x^{p+\varepsilon}) , \ x \to 0 , \qquad (84)$$

with some positive p and ε. Then, from Theorems 3 and 4, there exists a unique classical solution $u(x,t)$ of the problem (28), (84) such that, for all $t \ge 0$,

$$\|u(\cdot,t)\| = 1 ; \qquad u(x,t) = 1 + O(x^p) , \ x \to 0 . \qquad (85)$$

First, consider the linearized operator L given in Eqs. (33)–(34) and construct the spectral function $\mu(p)$ given in Eq. (65) which will be of one of four kinds described qualitatively on Fig.1.

Second, find the value $p_0 > 0$ that corresponds to minimum (infimum) of $\mu(p)$. Note that $p_0 = \infty$ just for the case described on Fig.1 (a), otherwise $0 < p_0 < \infty$. Compare p_0 with the value p from Eqs. (84). If $p < p_0$ then the problem (28), (84) has a self-similar asymptotics (see below).

In particular, two different cases are possible: (1) $p \geq p_0$ provided $p_0 < \infty$; (2) $0 < p < p_0$. In the first case a behavior of $u(x,t)$ for large t may depend strictly on initial conditions.

Depending on how p compares with p_0, we can obtain We again use Lemma 7 with $u_1 = u$ and $u_2 = u_s = \psi(xe^{\mu(p)t})$ and obtain for the solution $u(x,t)$ of the problem (28), (84):

$$
\lim_{t\to\infty} u(xe^{-\mu t}, t) = \begin{cases} 1 & \text{if } \mu > \mu(p) \\ \psi(x) & \text{if } \mu = \mu(p) \\ 0 & \text{if } \mu(p) > \mu > \mu(p+\delta), \end{cases} \tag{86}
$$

with sufficiently small $\delta > 0$.

We see that $\psi(x) = w(x^p)$, where $w(x)$ has all properties described in Theorem 5. The equalities (86) explain the exact meaning of the approximate identity,

$$
u(x,t) \approx \psi(xe^{\mu(p)t}), \qquad t \to \infty, \ xe^{\mu(p)t} = \text{const.}, \tag{87}
$$

that we call self-similar asymptotics.

In particular, the following statement holds.

Proposition 1. *The solution $u(x,t)$ of the problem (28), (84), with Γ given in Eqs. (24), satisfies either one of the following limiting identities:*

(1)
$$
\lim_{t\to\infty} u(xe^{-\mu t}, t) = 1, \qquad x \geq 0, \tag{88}
$$

for any $\mu > \mu(p_0)$. if $p \geq p_0$ for the initial data (84),
(2) *Eqs. (86) provided $0 < p < p_0$.*

The convergence in Eqs. (88), (86) is uniform on any bounded interval $0 \leq x \leq R$, and

$$
u(xe^{\mu(p)t}, t) - \psi(x) = O(x^{p+\varepsilon})e^{-\beta(p,\varepsilon)t}, \qquad \beta(p,\varepsilon) = (p+\varepsilon)(\mu(p)-\mu(p+\varepsilon)),
$$

for $0 < p < p_0$ and $0 < \varepsilon < p_0 - p$.

Remark 7. There is a connection between self-similar asymptotics and non-linear wave propagation. It is easy to see that self-similar asymptotics becomes more transparent in logaritheoremic variables

$$
y = \ln x, \quad u(x,t) = \hat{u}(y,t), \quad \psi(x,t) = \hat{\psi}(y,t)
$$

Thus, Eq. (87) becomes

$$
\hat{u}(y,t) \approx \hat{\psi}(y+\mu(p)t), \qquad t \to \infty, \quad y+\mu(p)t = \text{const.}, \tag{89}
$$

hence, the self-similar solutions are simply nonlinear waves (note that $\psi(-\infty) = 1$, $\psi(+\infty) = 0$) propagating with constant velocities $c_p = -\mu(p)$ to the right if $c_p > 0$

or to the left if $c_p < 0$. If $c_p > 0$ then the value $u(-\infty,t) = 1$ is transported to any given point $y \in \mathbb{R}$ when $t \to \infty$. If $c_p < 0$ then the profile of the wave looks more naturally for the functions $\tilde{u} = 1 - \hat{u}$, $\tilde{\psi} = 1 - \psi$.

We conclude that Eq. (28) can be considered in some sense as the equation for nonlinear waves. The self-similar asymptotics (89) means a formation of the traveling wave with a universal profile for a broad class of initial conditions. This is a purely non-linear phenomenon, it is easy to see that such asymptotics cannot occur in the particular case ($N = 1$ in Eqs. (24)) of the linear operator Γ.

6.2 Distribution functions, moments and power-like tails

We have described above the general picture of behavior of the solutions $u(x,t)$ to the problem (28), (84). On the other hand, Eq. (28) (in particular, its special case (14)) was obtained as the Fourier transform of the kinetic equation. Therefore we need to study in more detail the corresponding distribution functions.

Set $u_0(x)$ in the problem (28) to be an isotropic characteristic function of a probability measure in $\mathbb{R}^d$, i.e.,

$$u_0(x) = \mathscr{F}[f_0] = \int_{\mathbb{R}^d} dv\, f_0(|v|) e^{-ik \cdot v} , \qquad k \in \mathbb{R}^d , \; x = |k|^2 , \qquad (90)$$

where f_0 is a generalized positive function normalized such that $u_0(0) = 1$ (distribution function). Let U be a closed unit ball in the $B = C(\mathbb{R}_+)$ as defined in (30).

Then, we can apply all results of Section 5, and conclude that there exists a distribution function $f(v,t)$, $v \in \mathbb{R}^d$, satisfying Eq. (1), such that

$$u(x,t) = \mathscr{F}[f(\cdot,t)] , \qquad x = |k|^2 , \qquad (91)$$

for any $t \geq 0$, and a similar conclusion can be obtain if we assume the Laplace (instead of Fourier) transform in Eqs. (90).

Then there exists a distribution function $f(v,t)$, $v > 0$, such that

$$u(x,t) = \mathscr{L}[f(\cdot,t)] = \int_0^\infty dv\, f(v,t) e^{-xv} , \qquad u(0,t) = 1 , \; x \geq 0, \; t \geq 0 , \quad (92)$$

where $u(x,t)$ is the solution of the problem (28) constructed in Theorem 2 and Lemma 2.

The approximate equation (87) in terms of distribution functions (91) reads

$$f(|v|,t) \simeq e^{-\frac{d}{2}\mu(p)t} F_p(|v|e^{-\frac{1}{2}\mu(p)t}) , \quad t \to \infty , \; |v|e^{-\frac{1}{2}\mu(p)t} = \text{const.} , \qquad (93)$$

where $F_p(|v|)$ is a distribution function such that

$$\psi_p(x) = \mathscr{F}[F_p] , \qquad x = |k|^2 , \qquad (94)$$

with ψ_p given by

$$u_s(x,t) = \psi(xe^{\mu(p)t}) \tag{95}$$

(the notation ψ_p is used in order to stress that ψ defined in (95), depends on p). The factor $1/2$ in Eqs. (93) is due to the notation $x = |k|^2$. Similarly, for the Laplace transform, we obtain

$$f(v,t) \simeq e^{-\mu(p)t}\Phi_p(ve^{-\mu(p)t}) , \qquad t \to \infty , \ ve^{-\mu(p)t} = \text{const.} , \tag{96}$$

where

$$\psi_p(x) = \mathscr{L}[\Phi_p] . \tag{97}$$

The positivity and some other properties of $F_p(|v|)$ follow from the fact that $\psi_p(x) = w_p(x^p)$, where $w_p(x)$ satisfies Theorem 5. Hence

$$\psi_p(x) = \int_0^\infty d\tau R_p(\tau)e^{-\tau x^p} , \qquad \int_0^\infty d\tau R_p(\tau) = \int_0^\infty d\tau R_p(\tau)\tau = 1 , \tag{98}$$

where $R_p(\tau)$, $\tau \geq 0$, is a non-negative generalized function (of course, both ψ_p and R_p depend on p).

In particular we can conclude that the self-similar asymptotics (93) for any initial data $f_0 \geq 0$ occurs if $p_0 > 1$, otherwise it occurs for $p \in (0, p_0) \subset (0, 1)$. Therefore, for any spectral function $\mu(p)$ (Fig.1), the approximate equality (93) holds for sufficiently small $0 < p \leq 1$. In addition,

$$m_2 = \int_{\mathbb{R}^d} dv f_0(|v|)|v|^2 < \infty \ \text{ if } \ p = 1$$

and $m_2 = \infty$ if $p < 1$. Similar conclusions can be made for the Laplace transforms.

The positivity of $F(|v|)$ in Eqs. (94)–(97) follows from the integral representation (98) with $p \leq 1$, since it is well-known that

$$\mathscr{F}^{-1}(e^{-|k|^{2p}}) > 0 , \qquad \mathscr{L}^{-1}(e^{-x^{2p}}) > 0$$

for any $0 < p \leq 1$ (the so-called infinitely divisible distributions [18]). Thus, Eqs. (98) explains the connection of the self-similar solutions of generalized Maxwell models with infinitely divisible distributions.

Using standard formulas for the inverse Fourier (Laplace) transforms, denote by ($d = 1, 2, \ldots$ is fixed)

$$M_p(|v|) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{-|k|^{2p}+ik\cdot v} ,$$

$$N_p(v) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dx e^{-x^p+xv} , \qquad 0 < p \leq 1 , \tag{99}$$

we obtain the self-similar solutions (distribution functions) are given in Eqs. (96), (98) (right hand sides), by

$$F_p(|v|) = \int_0^\infty d\tau\, R_p(\tau)\tau^{-\frac{d}{2p}} M_p(|v|\tau^{-\frac{1}{2p}}) \,,$$

$$\Phi_p(v) = \int_0^\infty d\tau\, R_p(\tau)\tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}}) \,, \qquad v \geq 0\,, \ 0 < p \leq 1 \,. \tag{100}$$

Note that $M_1(|v|)$ is the standard Maxwellian in $\mathbb{R}^d$. The functions $N_p(v)$ (99) are studied in detail in the literature [18, 20]. Thus, for given $0 < p \leq 1$, the kernel $R(\tau)$, $\tau \geq 0$, is the only unknown function that is needed to describe the distribution functions $F(|v|)$ and $\Phi(v)$. See [10] for a study of $R(\tau)$ in more detail.

Now, from (34), (42), and recalling $\mu = \mu(1)$, we can show the moments equation can be written in the form

$$(s\mu(1) - \lambda(s) + 1)m_s = \sum_{n=2}^{N} \alpha_n I_n(s) \,, \tag{101}$$

where

$$I_n(s) = \int_{\mathbb{R}_+^n} da_1 \ldots da_n A(a_1, \ldots, a_n) \int_{\mathbb{R}_+^n} d\tau_1 \ldots, d\tau_n\, g_n^{(s)}(a_1\tau_1, \ldots, a_n\tau_n) \prod_{j=1}^{n} R(\tau_j)$$

$$g_n^{(s)}(y_1, \ldots, y_n) = \left(\sum_{k=1}^{n} y_k \right)^s - \sum_{k=1}^{n} y_k^s \,, \qquad n = 1, 2, \ldots \,. \tag{102}$$

and, due to the properties of $R(\tau)$, one gets $g_1^{(s)} = 0$ for any $s \geq 0$ and $m_0 = m_1 = 1$.

Our aim is to study the moments m_s defined in Eq.(92), for $s > 1$, on the basis of Eq. (101). The approach is similar to the one used in [15] for a simplified version of Eq. (101) with $N = 2$. The main results are formulated below in terms of the spectral function $\mu(p)$ (see Fig.1) under assumption that $p_0 > 1$.

Proposition 2.

[i] *If the equation $\mu(s) = \mu(1)$ has the only solution $s = 1$, then $m_s < \infty$ for any $s > 0$.*
[ii] *If this equation has two solutions $s = 1$ and $s = s_* > 1$, then $m_s < \infty$ for $s < s_*$ and $m_s = \infty$ for $s > s_*$.*
[iii] *$m_{s_*} < \infty$ only if $I_n(s_*) = 0$ in Eq. (101) for all $n = 2, \ldots, N$.*

Now we can draw some conclusions concerning the moments of the distribution functions (100) as follows. Denote by

$$m_s(\Phi_p) = \int_0^\infty dv\, \Phi_p(v)v^s \,, \qquad m_s(R_p) = \int_0^\infty d\tau\, R_p(\tau)\tau^s \,,$$

$$m_{2s}(F_p) = \int_{\mathbb{R}^d} dv\, F_p(|v|)|v|^{2s} \,, \qquad s > 0\,, \ 0 < p \leq 1 \,,$$

and use similar notations for $N_p(v)$ and $M_p(|v|)$ in Eqs. (100). Then, by formal integration of Eqs. (100), we obtain

$$m_s(\Phi_p) = m_s(N_p)m_{s/p}(R_p)$$

$$m_{2s}(F_p) = m_{2s}(M_p)m_{s/p}(R_p) \,,$$

where M_p and N_p are given in Eqs. (99) and we show that finite only for $s < p < 1$.

In the remaining case $p = 1$ *all* moments of functions

$$M_1(|v|) = (4\pi)^{-d/2}\exp\left[-\frac{|v|^2}{4}\right] , \qquad v \in \mathbb{R}^d \,;$$

$$N_1(v) = \delta(v-1) \,, \qquad v \in \mathbb{R}_+ \,,$$

are finite. Therefore, everything depends on moments of R_1 in with $p = 1$, so it only needs to apply Proposition 2.

In particular, the following statement holds for the moments of the distribution functions (93), (96).

Proposition 3.

[i] *If $0 < p < 1$, then $m_{2s}(F_p)$ and $m_s(\Phi_p)$ are finite if and only if $0 < s < p$.*
[ii] *If $p = 1$, then Proposition 2 holds for $m_s = m_{2s}(F_1)$ and for $m_s = m_s(\Phi_1)$.*

Remark 8. Proposition 3 can be interpreted in other words: the distribution functions $F_p(|v|)$ and $\Phi_p(v)$, $0 < p \leq 1$, can have finite moments of all orders in the only case when two conditions are satisfied

(1) $p = 1$, and
(2) the equation $\mu(s) = \mu(1)$ (see Fig.1) has the unique solution $s = 1$.

In all other cases, the maximal order s of finite moments $m_{2s}(F_p)$ and $m_s(\Phi_p)$ is bounded.

This fact means that the distribution functions F_p and Φ_p have power-like tails.

6.3 Applications to the conservative or dissipative Boltzmann equation

We recall the three specific Maxwell models **(A)**, **(B)**, **(C)** of the Boltzmann equation from section 2. Our goal in this section is to study isotropic solutions $f(|v|,t), v \in \mathbb{R}^d$, of Eqs. (2), (4), and (5) respectively. All three cases are considered below from a unified point of view. First we perform the Fourier transform and denote

$$u(x,t) = \mathscr{F}[f(|v|,t)] = \int_{\mathbb{R}^d} dv\, f(|v|,t)e^{-ik\cdot v}, \quad x = |k|^2, \; u(0,t) = 1. \qquad (103)$$

It was already said at the beginning of section 4 that $u(x,t)$ satisfies (in all three cases) Eq. (23), where $N = 2$ and all notations are given in Eqs.(27), (14), (18)-(21).

Hence, all results of our general theory are applicable to these specific models. In all three cases **(A)**, **(B)**, **(C)** we assume that the initial distribution function

$$f(|v|,0) = f_0(|v|) \geq 0, \qquad \int_{\mathbb{R}^d} dv \, f_0(|v|) = 1, \tag{104}$$

and the corresponding characteristic function

$$u(0,t) = u_0(x) = \mathscr{F}[f_0(|v|)], \quad x = |k|^2, \tag{105}$$

are given. Moreover, let $u_0(x)$ be such that

$$u_0(x) = 1 - \alpha x^p + O(x^{p+\varepsilon}), \qquad x \to 0, \qquad 0 < p \leq 1, \tag{106}$$

with some $\alpha > 0$ and $\varepsilon > 0$. We distinguish below the initial data with finite energy (second moment)

$$E_0 = \int_{\mathbb{R}^d} dv \, |v|^2 f_0(|v|) < \infty \tag{107}$$

implies $p = 1$ in Eqs. (106) and the in-data with infinite energy $E_0 = \infty$. If $p < 1$ in Eqs. (106) then

$$m_q^{(0)} = \int_{\mathbb{R}^d} dv \, f_0(|v|) |v|^{2q} < \infty \tag{108}$$

only for $q \leq p < 1$ (see [18, 20]).

Also, the case $p > 1$ in Eqs. (106) is not possible for $f_0(|v|) \geq 0$.

In addition, note that the coefficient $\alpha > 0$ in Eqs. (106) can always be changed to $\alpha = 1$ by the scaling transformation $\tilde{x} = \alpha^{1/p} x$. Then, without loss of generality, we set $\alpha = 1$ in (106).

Since it is known that the operator $\Gamma(u)$ in all three cases belongs to the class (24), we can apply Theorem 5 and state that self-similar solutions of Eq. (23) are given by

$$u_s(x,t) = \Psi(x e^{\mu(p)t}), \quad \Psi(x) = w(x^p), \tag{109}$$

where $w(x)$ is given in theorem 5 and $0 < p < p_0$ (the spectral function $\mu(p)$, defined in (43), and its critical point p_0 depends on the specific model.)

According to Sections 9-10, we just need to find the spectral function $\mu(p)$. In order to do this we first define the linearized operator $L = \Gamma'(1)$ for $\Gamma(u)$ given in Eqs. (24), (27). One should be careful at this point since $A_2(a_1, a_2)$ in Eqs. (27) is not symmetric and therefore Eqs. (32) cannot be used. A straight-forward computation leads to

$$Lu(x) = \int_0^1 ds \, G(s)(u(a(s)x) + u(b(s)x)) + \int_0^1 ds H(s)u(c(s)x), \tag{110}$$

in the notation (18) - (21). Then, the eigenvalue $\lambda(p)$ is given by

$$Lx^p = \lambda(p)x^p \quad \text{which implies}$$

$$\lambda(p) = \int_0^1 ds\, G(s)\{(a(s))^p + (b(s))^p\} + \int_0^1 ds H(s)(c(s))^p \,, \tag{111}$$

and the spectral function (43) reads

$$\mu(p) = \frac{\lambda(p) - 1}{p}. \tag{112}$$

Note that the normalization (22) is assumed.

At that point we consider the three models **(A), (B), (C)** separately and apply Eqs. (111) and (112) to each case.

(A) Elastic Boltzmann Equation (2) in $\mathbb{R}^d, d \geq 2$. By using Eqs. (18), (19), and (22) we obtain

$$\lambda(p) = \int_0^1 ds\, G(s)(s^p + (1-s)^p)\,, \qquad G(s) = A_d\, g(1-2s)[s(1-s)]^{\frac{d-3}{2}}\,, \tag{113}$$

where the normalization constant A_d is such that Eq. (22) is satisfied with $H = 0$. Then

$$\mu(p) = \frac{1}{p}\int_0^1 ds\, G(s)(s^p + (1-s)^p - 1)\,, p > 0. \tag{114}$$

It is easy to verify that $p\mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, and

$$\mu(p) > 0 \ \text{if } p < 1; \qquad \mu(p) < 0 \ \text{if } p > 1;$$
$$\mu(1) = 0, \qquad \mu(2) = \mu(3) = -\int_0^1 ds\, G(s)\, s(1-s)\,. \tag{115}$$

Hence, $\mu(p)$ in this case is similar to the function shown on Fig.1 **b)** with $2 < p_0 < 3$ and such that $\mu(1) = 0$. Then the self-similar asymptotics hold all $0 < p < 1$.

(B) Elastic Boltzmann Equation in the presence of a thermostat (4) **in $\mathbb{R}^d, d \geq 2$.** We consider just the case of a cold thermostat with $T = 0$ in Eq. (14), since the general case $T > 0$ can be considered after that with the help of (11). Again, by using Eqs. (18), (19), and (22) we obtain

$$\lambda(p) = \int_0^1 ds\, G(s)(s^p + (1-s)^p) + \theta \int_0^1 ds\, G(s)(1 - \frac{4m}{(1+m)^2})^p\,,$$
$$G(s) = \frac{1}{1+\theta} A_d\, g(1-2s)[s(1-s)]^{\frac{d-3}{2}}\,, \tag{116}$$

with the same constant A_d as in Eq. (113). Then

$$\mu(p) = \frac{1}{p}\int_0^1 ds\, G(s)(s^p + (1-s)^p - \theta(1-\beta s)^p - (1+\theta))\,,$$
$$\beta = \frac{4m}{(1+m)^2}\,, \quad p > 0, \tag{117}$$

and therefore, as in the previous case **A)**, $p\,\mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, with

$$\mu(1) = -\theta\,\beta \int_0^1 ds\,G(s)\,s. \tag{118}$$

which again verifies that $\mu(p)$ is of the same kind as in the elastic case **A)** and shown on Fig.1 (b). A position of the critical point p_0 such that $\mu'(p_0) = 0$ (see Fig.1 (b)) depends on θ. It is important to distinguish two cases: **1)** $p_0 > 1$ and **2)** $p_0 < 1$. In case **1)** any non-negative initial data (104) has the self-similar asymptotics. In case **2)** such asymptotics holds just for in-data with infinity energy satisfying Eqs. (106) with some $p < p_0 < 1$. A simple criterion to separate the two cases follows directly from Fig.1 (b): it is enough to check the sign of $\mu'(1)$. If

$$\mu'(1) = \lambda'(1) - \lambda(1) + 1 < 0 \tag{119}$$

in the notation of Eqs. (116), then $p_0 > 1$ and the self-similar asymptotics hold for any non-negative initial data.

The inequality (119) is equivalent to the following condition on the positive coupling constant θ

$$0 < \theta < \theta_* = -\frac{\int_0^1 ds\,G(s)\,(s\log s + (1-s)\log(1-s))}{\int_0^1 ds\,G(s)\,(\beta\,s + (1-\beta s)\log(1-\beta s))}. \tag{120}$$

The right hand side of this inequality is positive and independent on the normalization of $G(s)$, therefore it does not depend on θ (see Eq. (117)). We note that a new class of exact self-similar solutions to Eq. (4) with finite energy was recently found in [12] for $\beta = 1, \theta = 4/3$ and $G(s) = const$. A simple calculation of the integrals in (120) shows that $\theta_* = 2$ in that case, therefore the criterion (119) is fulfilled for the exact solutions from [12] and they are asymptotic for a wide class of initial data with finite energy. Similar conclusions can be made in the same way about exact positive self-similar solutions with infinite energy constructed in [12]. Note that the inequality (119) shows the non-linear character of the self-similar asymptotics: it holds unless the linear term in Eq. (4) is 'too large'.

(C) Inelastic Boltzmann Equation (5) in $\mathbb{R}^d$. Equations (21) and (22) lead to

$$\lambda(p) = \int_0^1 ds\,G(s)((as)^p + (1-bs)^p),$$

where

$$G(s) = C_d\,(1-s)^{\frac{d-3}{2}}, \qquad a = \frac{(1+e)^2}{4}, \quad b = \frac{(1+e)(3-e)}{4},$$

with such constant C_d that Eq. (22) with $H = 0$ is fulfilled. Hence

$$\mu(p) = \frac{1}{p} \int_0^1 ds\,G(s)((as)^p + (1-bs)^p - 1), \qquad p > 0, \tag{121}$$

and once more as in the previous two cases, $p\mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, with now

$$\mu(1) = -\frac{1-e^2}{4} \int_0^1 ds\, G(s)\, s\,.$$

Thus, the same considerations lead to the shape of $\mu(p)$ shown in Fig.1 (b). The inequality (119) with $\lambda(p)$ given in Eqs. (121) was proved in [11] (see Eqs. (4.26) of [11], where the notation is slightly different from ours). Hence, the inelastic Boltzmann equation (5) has self-similar asymptotics for any restitution coefficient $0 < e < 1$ and any non-negative initial data.

Hence, the spectral function $\mu(p)$ in all three cases above is such that $p_0 > 1$ provided the inequality (120) holds for the model (**B**).

Therefore, according to our general theory, all 'physical' initial conditions (104) satisfying Eqs. (106) with any $0 < p \le 1$ lead to self-similar asymptotics. Hence, the main properties of the solutions $f(v,t)$ are qualitatively similar for all three models (**A**), (**B**) and (**C**), and can be described in one unified statement: Theorem 6 below.

Before we formulate such general statement, it is worth to clarify one point related to a special value $0 < p_1 \le 1$ such that $\mu(p_1) = 0$. The reader can see that on Fig.1 (b) that the unique root of this equation exits for all models (**A**), (**B**),(**C**) since $\mu(1) = 0$ in the case (**A**) (energy conservation) , and $\mu(1) < 0$ in cases (**B**) and (**C**) (energy dissipation). If $p = p_1$ in Eqs. (106) then the self-similar solution (109) is simply a stationary solution of Eq. (23). Thus, the time relaxation to the non-trivial $(u \ne 0, 1)$ stationary solution is automatically included in Theorem 6 as a particular case of self-similar asymptotics.

Thus we consider simultaneously Eqs. (2), (4), (5), with the initial condition (104) such that Eq. (106) is satisfied with some $0 < p \le 1$, $\varepsilon > 0$ and $\alpha = 1$. We also assume that $T = 0$ in Eq. (4) and the coupling parameter $\theta > 0$ satisfies the condition (120).

In the following Theorem 6, the solution $f(|v|,t)$ is understood in each case as a generalized density of probability measure in $\mathbb{R}^d$ and the convergence $f_n \to f$ in the sense of weak convergence of probability measures.

Theorem 6. *The following two statements hold*

[i] *There exists a unique (in the class of probability measures) solution $f(|v|,t)$ to each of Eqs. (2), (4), (5) satisfying the initial condition (104). The solution $f(|v|,t)$ has self-similar asymptotics in the following sense:*
For any given $0 < p \le 1$ in Eqs. (106) there exits a unique non-negative self-similar solution

$$f_s^{(p)}(|v|,t) = e^{-\frac{d}{2}\mu(p)t} F_p(|v|e^{-\frac{1}{2}\mu(p)t})\,, \tag{122}$$

such that

$$e^{\frac{d}{2}\mu(p)t} f(|v|e^{-\frac{1}{2}\mu(p)t},t) \xrightarrow{t\to\infty} F_p(|v|)\,, \tag{123}$$

where $\mu(p)$ is given in Eqs. (114), (117), (121), respectively, for each of the three models.

[ii]*Except for the special case of the Maxwellian*

$$F_1(|v|) = M(|v|) = (4\pi)^{-d/2} e^{-\frac{|v|^2}{4}} \tag{124}$$

for Eq. (2) with $p = 1$ in Eq. (106) (note that $\mu(1) = 0$ in this case), the function $F_p(|v|)$ does not have finite moments of all orders. If $0 < p < 1$, then

$$m_q = \int_{\mathbb{R}^d} dv \, F_p(|v|)|v|^{2q} < \infty \qquad only\ for\ 0 < q < p. \tag{125}$$

If $p = 1$ in the case of Eqs. (4), (5), then $m_q < \infty$ only for $0 < q < p_*$, where $p_* > 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$, with $\mu(p)$ given in Eqs. (106), (121) respectively.

In addition, we also we obtain the following corollary.

Corollary 6. *Under the same conditions of Theorem 6, the following two statements hold.*

[i] *The rate of convergence in Eq. (123) is characterized in terms of the corresponding characteristic functions in Proposition 1.*

[ii]*The function $F_p(|v|)$ admits the integral representation (100) through infinitely divisible distributions (99).*

Proof. It is enough to note that all results of section 9 and 10 are valid, in particular, for Eqs. (2), (4), (5).

Finally, we mention that the statement similar to theorem 6, can be easily derived from general results of sections 7.1 and 7.2 in the case of $1 - d$ Maxwell models introduced in [15], [2] for applications to economy models (Pareto tails, etc.). The only difference is that the 'kinetic' equation can be transformed to its canonical form (23)-(24) by the Laplace transform as discussed in section 10, and that the spectral function $\mu(p)$ can have in this case any of the four kind of behaviors shown in Fig.1. The only remaining problem for any such $1-$dimensional models is to study them for their specific function $\mu(p)$, and then to apply Propositions 1, 2 and 3.

Thus, the general theory developed in this chapter is applicable to all existing multi-dimensional isotropic Maxwell models and to $1d$ models as well.

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