Abstract

In this paper we present the first numerical method for a kinetic description of the Vicsek swarming model. The kinetic model poses a unique challenge, as there is a distribution dependent collision invariant to satisfy when computing the interaction term. We use a spectral representation linked with a discrete constrained optimization to compute these interactions. To test the numerical scheme we investigate the kinetic model at different scales and compare the solution with the microscopic and macroscopic descriptions of the Vicsek model. We observe that the kinetic model captures key features such as vortex formation and traveling waves.

Keywords: Spectral method; Finite Volume method; Kinetic equation; Vicsek model; Hyperbolic systems

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1. Introduction

Swarming behavior is a perfect illustration of multiscale phenomena. In a flock of birds for instance one can either decide to model each individual separately \cite{1,2,3,4,5}, or one can model the whole flock as a single entity.
These two points-of-view have led to two different types of models for swarming: microscopic models (a.k.a agent-based-models) and macroscopic models involving macroscopic quantities (e.g. mass, flux). In this work, we use an intermediate approach studying a swarming model at the kinetic scale (also called the mesoscopic scale).

Kinetic models offer the same possibilities as microscopic models to model phenomena. For instance, one can model complex interactions among agents or introduce various boundary conditions in a kinetic model. Macroscopic models are less flexible from this point of view. Meanwhile, kinetic models allow for analytic study, which is more scarce in microscopic models. Rigorous derivation of kinetic models from a microscopic model can be achieved in some cases, however, one needs to have the number of particles $N$ to tend to infinity. In gas dynamics, this would mean that the mean free path between particle interactions becomes small enough. Unfortunately, there is no convenient mean free path concept in swarming models. For this reason, it is also crucial to numerically connect kinetic models with their corresponding microscopic models.

Several works have already studied kinetic models for swarming [10, 11, 12], but few have done a numerical investigation. In this work, we first introduce a numerical scheme for a kinetic swarming model based on the Vicsek model. We then investigate numerically the model at different scales using the corresponding microscopic, kinetic, and macroscopic models. In particular, we emphasize how the kinetic model is able to capture typical solutions of both microscopic and macroscopic models.

Numerical solution of kinetic equations has long been a computational challenge due to the typically integro-differential nature of the interactions between particles, as well as the higher dimensionality of phase space. In particular, the kinetic model that has been most studied is the Boltzmann transport equation of rarefied gas dynamics. Stochastic methods, most notably Direct Simulation Monte Carlo [13], have long been the primary method of solution for these problems due to the reduction of dimensionality. However, these methods suffer from the presence of noise in their solutions owing to the stochastic nature of
their solution, and can become very expensive in problems that are far from equilibrium or problems with transients.

Since the creation of the DSMC method, the considerable increase of computational power has made deterministic computation of kinetic equations within the realm of possibility, despite the expense of dimensionality. Discrete velocity methods \cite{14, 15, 16} simulate particle interactions on a mesh in velocity space, but can suffer from low accuracy and lack of conservation \cite{15, 17, 18, 19}. Spectral methods exploit the weighted convolution structure of the Fourier transform \cite{20} of the interaction terms for high accuracy. Spectral approximations for Boltzmann collision operators were first proposed by Pareschi and Perthame \cite{21}, and many other authors developed numerical methods in this area \cite{22, 23, 24, 25, 26, 27}.

In this paper we present a novel spectral method for computing the kinetic form of the Vicsek model, in fact the first numerical method for this kinetic formulation. This model presents several new challenges for a numerical method. The alignment interaction between particles gives a nonlinear integro-differential operator that needs to be handled carefully. By reformulation in terms of the mean direction of motion of the particles, we obtain a nonlinear diffusion-like operator. We then take the Fourier transform and use orthogonality to obtain a coupled set of equations.

This nonlinear interaction gives rise to collision invariants, in the spirit of the Boltzmann collision operator, which are needed to obtain a hydrodynamic limit for the model. These invariant properties must also be preserved at the discrete level, and we perform a constrained optimization in a suitable norm of the solution obtained by the spectral method to ensure that the invariants are correctly preserved. We show that this preservation is crucial by comparing the solution with and without preservation of the collision invariants.

Numerical tests are performed comparing the solution of the kinetic equation to both the macroscopic and microscopic models. We first investigate the kinetic model in a 'hydrodynamic limit' by performing a change of scales. We observe the emergence of traveling waves (e.g. rarefaction and shock waves) that match
perfectly with the solutions of the macroscopic model. Those numerical results are remarkable as there is no analytic theory to handle traveling waves for the macroscopic model (the model being non-conservative). We then compare the kinetic model with the 'microscopic model'. For that, we take advantage that boundary conditions are easily implemented at the microscopic and kinetic level. We perform simulations in a closed domain with reflexive boundary conditions. We observe the emergence of vortex formation for the two models and compare the solutions.

The paper is organized as follows: in Section 2, we introduce the swarming model referred to as the Vicsek model at different scales (microscopic, kinetic, and macroscopic). In Section 3, we develop a numerical scheme (spectral method) for the kinetic model that preserves the so-called generalized collisional invariant. Numerical investigations comparing the model at different scales are presented in Section 4. We close with a discussion of future work in this area.

2. Self-organized dynamics at different scales

In this section, we introduce the model of self-organized dynamics at different scales. Based on the Vicsek model \cite{28, 29}, we first introduce a particle system describing alignment behavior. Next, we give a short review on the kinetic equation associated with these dynamics and discuss the collisional invariants, whose properties play a central role in the development of the numerical scheme \cite{30}. Finally, we introduce the macroscopic limit of the dynamics.

2.1. Microscopic model

At the microscopic level, the Vicsek model describes the evolution of \( N \) particles which tend to align with their neighbors. Each particle is represented by a position, \( x_k \in \mathbb{R}^d \), and a unit velocity vector, \( \omega_k \in \mathbb{S}^{d-1} \). The evolution of the particles is governed by the following dynamical system:

\[
\frac{dx_k}{dt} = \omega_k \quad \text{and} \quad d\omega_k = P_{\omega} \left( \Omega_k dt + \sqrt{2\sigma} \, dB^k \right),
\]  

(2.1)
Here, $\Omega_k$ denotes the mean velocity:

$$\Omega_k = \frac{\sum_{|x_j - x_k| < R \omega_j}}{\sum_{|x_j - x_k| < R \omega_j}}, \quad (2.2)$$

with $R$ the radius of interaction (see figure 1), $\sigma > 0$ is the intensity of the noise with $B_k^t$ the Brownian motion. The matrix $P_{\omega \perp}$ is a projector:

$$P_{\omega \perp} = \text{Id} - \omega \otimes \omega, \quad (2.3)$$

which enforces the velocity $\omega_k$ to remain of norm 1.

![Figure 1: The Vicsek model at the microscopic level. The particle $k$ aligns its velocity $\omega_k$ to $\Omega_k$ the average direction of its neighbors in the ball $B(x_k, R)$.
](image)
with \( K \) the characteristic function of the ball \( B(0, R) \), i.e. \( K(r) = 1_{\{|r|<R\}} \).

In the large scale limit in space and time, the vector velocity \( F(f) \) becomes local in space meaning that \( J \) is given by:

\[
J(x) = \int_{\omega^*} \omega^* f(x, \omega^*) d\omega^*.
\]

(2.7)

In the following, we only consider \( J(x) \) within this approximation. We rewrite the kinetic equation (2.4) in the following form:

\[
\partial_t f + \omega \cdot \nabla_x f = Q(f),
\]

(2.8)

with \( Q \) the collisional operator given by:

\[
Q(f) = -\nabla_\omega \cdot (F[f]f) + \sigma \Delta_\omega f,
\]

(2.9)

with \( F(f) \) given by (2.5), (2.7).

2.3. Macroscopic model (hydrodynamic limit)

In order to derive a macroscopic model associated with the kinetic model (2.4), one has to introduce an hydrodynamic scaling [29] introducing the macroscopic variables:

\[
t' = \varepsilon t, \quad x' = \varepsilon x,
\]

where \( \varepsilon \) is the ratio between micro and macro variables. In these new macro variables, the evolution of \( f^\varepsilon \) is given by:

\[
\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon).
\]

(2.10)

As \( \varepsilon \to 0 \), one can show that the evolution \( f^\varepsilon \) converges locally in space toward an equilibrium (see subsection 3.1):

\[
f^\varepsilon \rightharpoonup f^0(x, \omega) = \rho^0(x)M_u(x)(\omega).
\]

(2.11)

The evolution of the system is solely described by two macroscopic quantities: the density of particles \( \rho \) and the macroscopic velocity \( u \). Their evolutions are
governed by the following system:

\[ \partial_t \rho + \nabla_x \cdot (c_1 \rho u) = 0, \]  
\[ \rho \left( \partial_t u + c_2 (u \cdot \nabla_x) u \right) + \lambda P_{u \perp} \nabla_x \rho = 0, \]  
\[ |u| = 1. \]

Here, \( c_1, c_2 \) and \( \lambda \) are constants depending on the noise parameter \( \sigma \). \( P_{u \perp} \) is a projection operator given by \( P_{u \perp} = (\text{Id} - u \otimes u) \). It ensures the constraint that \( |u| = 1 \). The macroscopic model is a hyperbolic system but it is also non-conservative. Thus, few analytic results are known about this system. Local existence and uniqueness have been studied in [12] and we refer to [30] for the implementation of an accurate numerical scheme.

3. Numerical scheme for the kinetic model

We turn our attention on building a numerical scheme for the kinetic model (2.8) in 2D. With this aim, we propose a splitting method between the collision and the transport part of the equation. In other words, we solve separately:

\[ \partial_t f = Q(f), \]  
\[ \partial_t f + \omega \cdot \nabla_x f = 0. \]

The difficulty is essentially in the collisional part (3.1), it requires to use wisely the properties of \( Q \) (Subsection 3.1). We propose a spectral method (subsection 3.2) that preserves the invariants of \( Q \) (Subsection 3.3). Next, we use a finite volume method to solve the transport term (3.2) (Subsection 3.4). A summary of the proposed numerical scheme is given in Appendix A.

3.1. Properties of the collisional operator

The collisional operator (2.9) can be written as a Fokker-Planck operator. To do so, we introduce the equilibrium function \( M_\Omega(\omega) \), also known as the Von Mises distribution,

\[ M_\Omega(\omega) = C_0 \exp \left( \frac{\omega \cdot \Omega}{\sigma} \right), \]  

\[ 7 \]
where $C_0$ is a constant of normalization. In 2D, this gives the formula:

$$M_\theta(\theta) = C_0 e^{\cos(\theta - \bar{\theta})}$$

with $\Omega = (\cos \bar{\theta}, \sin \bar{\theta})$.

**Proposition 3.1.** Let $\Omega_f$ be the direction of the average velocity of $f$ (i.e. $\Omega_f = \int f \omega \, d\omega \int f \omega \, d\omega$). We have

$$Q(f) = \sigma \nabla \omega \cdot \left( M_{\Omega_f} \nabla \omega \left( \frac{f}{M_{\Omega_f}} \right) \right).$$

(3.5)

In particular,

$$\int \omega Q(f) f \frac{d\omega}{M_{\Omega_f}} = -\sigma \int \omega M_{\Omega_f} \left| \nabla \omega \left( \frac{f}{M_{\Omega_f}} \right) \right|^2 d\omega \leq 0.$$

In 2D, equation (3.5) reads:

$$Q(f) = \sigma \partial_\theta \left( M_{\bar{\theta}} \partial_\theta \left( \frac{f}{M_{\bar{\theta}}} \right) \right) = -\partial_\theta \left( \sin(\bar{\theta} - \theta) f \right) + \sigma \partial^2 \theta f.$$

(3.6)

**Corollary 3.2.** The equilibria of the operator $Q$ are given by the set:

$$\mathcal{E} = \{ \rho M_{\Omega} \mid \rho \in \mathbb{R}, |\Omega| = 1 \}.$$

(3.7)

Although the equilibria of the operator $Q$ forms a set of dimension $d$ (1 for $\rho$ and $d - 1$ for $\Omega$), the collisional invariants of $Q$ are only of dimension 1. In particular, $Q$ preserves only the mass and not the flux. In other words, for a general $f$, we have:

$$\int \omega Q(f) d\omega = 0 \quad \text{and} \quad \int \omega Q(f) \omega \, d\omega \neq 0.$$

To overcome the lack of conservations of $Q$, we generalize the notion of collisional invariant.

**Definition 1. (GCI)** Fix a unit vector $\Omega$. A function $\psi_{\Omega}$ is called a generalized collisional invariant (GCI) if it satisfies:

$$\int \omega Q(f) \psi_{\Omega}(\omega) \, d\omega = 0,$$

(3.8)

for any $f$ satisfying $\Omega_f = \Omega$. 

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Once we fix the direction $\Omega_f$, the operator $Q$ becomes linear in $f$. We denote by $Q_\Omega$ the linear operator defined by:

$$Q_\Omega(f) = \sigma \nabla_\omega \cdot \left( M_\Omega \nabla_\omega \left( \frac{f}{M_\Omega} \right) \right).$$

Thus, we can define the adjoint of $Q_\Omega$ in $L^2$:

$$Q^*_\Omega(\varphi) = \sigma M_\Omega^{-1} \nabla_\omega \cdot (M_\Omega \nabla_\omega \varphi).$$

Expressing the constraint $\Omega_f = \Omega$ as a Lagrange multiplier, we find that $\psi_\Omega$ is a collisional invariant for $\Omega$ if and only if it satisfies:

$$Q^*_\Omega(\psi_\Omega) = \beta \omega \times \Omega.$$

We deduce an explicit expression for $\psi_\Omega$ in 2D.

**Proposition 3.3.** In 2D, suppose that $\Omega = (1, 0)^T$. The collisional invariant $\psi$ satisfies:

$$\sigma \partial_\theta (M \partial_\theta \psi) = \beta \sin \theta M,$$

with $M(\theta) = C_0 \exp \left( \frac{\cos \theta}{\sigma} \right)$. Thus, a solution is given by:

$$\psi(\theta) = \sigma \theta - \sigma \pi \int_0^\theta e^{-\frac{\cos s}{\sigma}} ds - \int_0^\theta e^{-\frac{\cos s}{\sigma}} ds.$$

(3.10)

For $\Omega$ given by $\Omega = (\cos \bar{\theta} , \sin \bar{\theta})^T$, a solution is written as

$$\psi_\Omega(\theta) = \psi(\theta - \bar{\theta}).$$

(3.11)

3.2. Spectral method

In the section, we introduce a Hilbert space to decompose the collisional operator. In the following, we are looking at the equation (3.1) locally in $x$.

For clarity, we suppose that $\Omega(x)$ is given by the vector $(1, 0)^T$, the result for more general $\Omega$ can be found through rotation of this solution. We introduce the subspace of periodic functions $H$ defined as:

$$H := \{ f(\theta) / \int_0^\pi |f|^2 d\bar{\theta} < \infty \},$$

(3.12)
along with the scalar product $\langle \cdot, \cdot \rangle_H$:

$$\langle f, g \rangle_H := \int_\theta f(\theta)g(\theta)\frac{d\theta}{M},$$

The relevance of this scalar product comes from the symmetric property satisfied by $Q$: suppose $f, g$ are smooth functions of $H$, then

$$\langle Q(f), g \rangle_H = -\sigma \int_\theta M \partial_\theta \left( \frac{f}{M} \right) \partial_\theta \left( \frac{g}{M} \right) d\theta = \langle f, Q(g) \rangle_H.$$ 

As a Hilbert basis on $H$, we use the following functions:

$$P_k(\theta) = e^{ik\theta} \sqrt{\frac{M}{2\pi}}, \quad k \in \mathbb{Z}. \quad (3.13)$$

Using the formulation (3.6), we deduce that:

$$Q(P_k) = \left( -\frac{\sigma k^2}{2} + \frac{\cos \theta}{2} - \frac{\sin^2 \theta}{4\sigma} \right) P_k. \quad (3.14)$$

Thus,

$$Q(P_k) = \frac{1}{16\sigma} P_{k-2} + \frac{1}{4} P_{k-1} + \left( -\sigma k^2 - \frac{1}{8\sigma} \right) P_k + \frac{1}{4} P_{k+1} + \frac{1}{16\sigma} P_{k+2}. $$

For any function $f$ in $H$, we can decompose:

$$f(\theta) = \sum_{k \in \mathbb{Z}} c_k P_k(\theta), \quad \text{with } c_k = \langle f, P_k \rangle_H.$$ 

and deduce that $Q(f) = \sum_{k \in \mathbb{Z}} \tilde{c}_k P_k(\theta)$ with

$$\tilde{c}_k = \frac{1}{16\sigma} c_{k-2} + \frac{1}{4} c_{k-1} + \left( -\sigma k^2 - \frac{1}{8\sigma} \right) c_k + \frac{1}{4} c_{k+1} + \frac{1}{16\sigma} c_{k+2}. $$

Numerically, we use a uniform grid to divide the domain $[0, 2\pi)$ in $2N$ points: $\theta_s = s\Delta\theta$ with $\Delta\theta = \frac{2\pi}{2N}$. We approximate the Hilbert space $H$ (3.12) by a subspace of finite dimension:

$$V_N = \{ f_N(\theta) = \sum_{k=-N}^{N} c_k P_k(\theta), \quad \text{with } c_{-N}, \ldots, c_N \in \mathbb{C} \}. \quad (3.15)$$

Notice that on the grid point $\theta_s$, we have $P_k(\theta_s) = P_{k+2N}(\theta_s)$. Thus, only $2N+1$ polynomials $P_K$ are relevant. For a given function $f$ in $H$, we define its approximation $f_N$ in $V$ with coefficients $c_k$ given by:

$$c_k = \sum_{s=0}^{2N-1} f(\theta_s) P_k(\theta_s) \frac{\Delta\theta}{M(\theta_s)}, \quad (3.16)$$
Similarly to the Fast Fourier Transform, the function $f_N$ interpolates $f$ at the grid points $\theta_s$. By periodicity and using that $f$ is a real function, we deduce that (see figure 2):

$$c_k = c_{k+2N}, \quad c_{-k} = \overline{c_k}.$$  \hfill (3.17)

Therefore, only the coefficients $c_0, \ldots, c_N$ are required to describe $f_N$.

![Diagram showing periodicity and coefficients]

Figure 2: The coefficients \{c_k\} (3.16) satisfy the properties $c_k = c_{k+2N}$ and $c_{-k} = \overline{c_k}$. Thus, only the coefficients $c_k$ from 0 to $N$ are needed.

Applying the operator $Q$ to the approximation $f_N$ gives:

$$Q(f_N) = \sum_{k=-N}^{N} c_k \left[ \alpha_2 P_{k-2} + \alpha_1 P_{k-1} + (\sigma^2 - \alpha_0) P_k + \alpha_1 P_{k+1} + \alpha_2 P_{k+2} \right],$$

with $\alpha_0 = \frac{1}{8\sigma}$, $\alpha_1 = \frac{1}{4\sigma}$, $\alpha_2 = \frac{1}{16\sigma}$. Since $P_{k+2N}(\theta_s) = P_k(\theta_s)$, we deduce that $Q$ has the following matrix representation in the basis $\mathcal{B} = \{P_k\}_{-N \leq k \leq N}$ of $V_N$:

$$[Q]_{\mathcal{B}} = \begin{bmatrix}
-\sigma N^2 - \alpha_0 & \alpha_1 & \alpha_2 & 0 & \alpha_2 & \alpha_1 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \alpha_2 & \alpha_1 & -\sigma k^2 - \alpha_0 & \alpha_1 & \alpha_2 & 0 \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \alpha_1 & \ddots & \ddots & \ddots \\
& & & & \alpha_2 & \alpha_1 & 0 & \alpha_2 & \alpha_1 & -\sigma N^2 - \alpha_0
\end{bmatrix}.$$  \hfill (3.18)
3.3. Time discretization

We now propose a numerical scheme to solve the collision operator (3.1). We denote by \( c = (c_{-N}, \ldots, c_N)^T \) the coefficients (3.16). In the subspace \( V_N \), the equation (3.1) reduces to:

\[
\partial_t c = [Q] g c. \tag{3.19}
\]

We would like to find a discretization of this system that preserves the collisional invariants of \( f \) (see 3.1). In other words, if \( f^{n+1} \) is the update of \( f^n \), we should have:

\[
\int_\theta f^n \left( \begin{array}{c} 1 \\ \psi \end{array} \right) \, d\theta = \int_\theta f^{n+1} \left( \begin{array}{c} 1 \\ \psi \end{array} \right) \, d\theta, \tag{3.20}
\]

which leads to:

\[
Cc^{n+1} = Cc^n \tag{3.21}
\]

where \( c^n \) and \( c^{n+1} \) are (resp.) the coefficients of \( f^n \) and \( f^{n+1} \), and \( C \) is a \( 2 \times (2N + 1) \) matrix defined by:

\[
C_{0,k} = \int_\theta P_k(\theta) \, d\theta , \quad C_{1,k} = \int_\theta P_k(\theta) \psi(\theta) \, d\theta \quad \text{for all } k \in [-N, N].
\]

The algorithm we propose consists in solving (3.19) using an Euler scheme and to project the solution obtained to satisfy the constraint (3.21). In other words, we write:

\[
c^* = c^n + \Delta t [Q] g c^n. \tag{3.22}
\]

and then compute \( c^{n+1} \) such that the constraint (3.21) is satisfied with \( \|c^{n+1} - c^n\|_H \) minimized (see figure 3).

Following [25], we obtain the following algorithm:

**Proposition 3.4.** The coefficients \( c^{n+1} \) are given by:

\[
c^{n+1} = c^n + \Delta t \Lambda_N(C) [Q] g c^n, \tag{3.23}
\]

with \( \Lambda_N(C) \) the square matrix defined as:

\[
\Lambda_N(C) = \text{Id} - C^T (CC^T)^{-1} C. \tag{3.24}
\]
Numerically, the *explicit* Euler scheme (3.22) could be unstable due to the stability condition. Thus, we propose a second algorithm based on the *implicit* Euler scheme:

$$c^*_k = c^n_k + \Delta t [Q]_B c^*_k.$$  

(3.25)

Following the same methodology, we deduce the following algorithm:

$$c^{n+1} = c^n + \Delta t \Lambda_N(C) [Q]_B (\text{Id} - \Delta t [Q]_B)^{-1} c^n,$$  

(3.26)

with $\Lambda_N(C)$ the square matrix given by [3.24].

3.4. *Transport term*

Finally, we propose a finite volume method [31] to solve the transport equation (3.2).

Suppose $f$ is defined on a Cartesian grid in $(x, y)$, $f(x_i, y_j, \theta)$. The finite volume method consists of identifying $f(x_i, y_j, \theta)$ as the mass of particles in the cell $C_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ moving with speed $\omega(\theta) = (\cos \theta, \sin \theta)$ (see figure 4). Integrating the transport equation on the cell $C_{i,j}$ yields

$$\frac{d}{dt} f(x_i, y_j) + \cos \theta \frac{f(x_{i+\frac{1}{2}}, y_j) - f(x_{i-\frac{1}{2}}, y_j)}{\Delta x}$$  

$$+ \sin \theta \frac{f(x_i, y_{j+\frac{1}{2}}) - f(x_i, y_{j-\frac{1}{2}})}{\Delta y} = 0.$$  

(3.27)

It remains to determine the values of the interface of the cell $C_{i,j}$ (e.g. $f(x_{i+\frac{1}{2}}, y_j, \theta)$, $f(x_i, y_{j+\frac{1}{2}}, \theta)$). We use an upwind approach: for each value of $\theta$, the value of $f$
at the boundary is given by:

\[
f(x_{i+\frac{1}{2}}, y_j, \theta) = \begin{cases} 
  f(x_i, y_j, \theta) & \text{if } \cos \theta \geq 0 \\
  f(x_{i+1}, y_j, \theta) & \text{if } \cos \theta < 0.
\end{cases}
\]  

(3.28)

Figure 4: Illustration of the finite volume method. Here, the transport velocity \( \omega(\theta) \) is pointing toward the right direction, therefore the value at the right interface \( f(x_{i+\frac{1}{2}}, y_j, \theta) = f(x_i, y_j, \theta) \).

4. Numerical investigations

Using the numerical scheme developed for the kinetic equation (2.8), we now investigate the Vicsek model using either its particle description (2.1), kinetic formulation (2.8), or hydrodynamic limit (2.12)-(2.13).

We first validate our numerical scheme for the kinetic equation by solving the homogeneous equation (3.1) and analyzing its convergence toward equilibrium (2.11). In particular, we highlight the importance of preserving the invariants for the stability of large time simulation. Next, we investigate the kinetic equation within its hydrodynamic limit, i.e. letting \( \varepsilon \to 0 \) in (2.10). Our results show that at \( \varepsilon = 10^{-2} \) the solutions of the kinetic equation and the hydrodynamic limit are almost indistinguishable. For instance, we recover at the kinetic level shock and rarefaction waves. Finally, we explore the effect of boundary conditions on the dynamics. We use a domain with reflexive boundary conditions and compare the solution of the particle dynamics with the kinetic formulation once they reach a stationary state.
4.1. Convergence to equilibrium

In this subsection, we validate our numerical scheme for the kinetic equation using analytic results. Since the stationary states of the homogeneous equation are known, we can test that our scheme converges toward such analytic solutions.

With this aim, we use our numerical scheme to solve the homogeneous equation \( \partial_t f = Q(f) \) with \( Q \) defined in (2.9). We use as initial condition a sinusoid for \( f(\theta) \)

\[
f(t = 0, \theta) = 1 + \cos 2\theta.
\]

We leave the discussion about the time and velocity discretization (\( \Delta t \) and \( \Delta \theta \)) of the kinetic equation in the appendix Appendix B. In figure 5, we plot the solution \( f(t) \) at \( t = 10 \) unit times and the corresponding stationary state (i.e. Von Mises distribution (3.3)). We observe that the two curves are in perfect agreement.

The mean direction \( \bar{\theta} \) of \( f(t, \theta) \) can vary during a simulation. This is expected since the mean velocity is not conserved by the kinetic equation (2.8). However, the total mass, \( \rho(t) = \int_\theta f(t, \theta) d\theta \), has to be conserved in time. Thanks to our formulation (3.23), the mass is precisely conserved in our simulation (up to round-off error). If one would use the formulation (3.22) to update \( f \) (i.e. without the constraint), one cannot guarantee that the mass would be preserved. To analyze this property, we numerically investigate the long time behavior of the mass \( \rho(t) \) with and without taking into account the invariant of the dynamics. Thus, we use a preserving and a non-preserving scheme. In figure 6, we represent the evolution of the total mass \( \rho(t) \) in time for both cases. With the preserving scheme, the mass \( \rho(t) \) remains constant over time whereas the mass \( \rho(t) \) is increasing with the non-preserving scheme which is inaccurate.

4.2. Traveling waves

We now turn our attention to the full kinetic equation (2.4). Few analytic results are known for this equation other than hydrodynamic limit. Therefore, we will numerically investigate the kinetic equation in the hydrodynamic regime,
Figure 5: Solution of the homogeneous equation $\partial_t f = Q(f)$. The solution $f(t, \theta)$ (blue) converges to an equilibrium (red) given by a Von Mises distribution. Parameters for the simulation: $\sigma = .2$, $\Delta t = .05$, $\Delta \theta = \frac{2\pi}{12}$.

Figure 6: Evolution of the total mass $\rho = \int_{\theta} f(\theta) \, d\theta$ for the homogeneous equation with and without implementing the constraint (3.20). We observe that the mass increases if we do not enforce the constraint. Parameters for the simulation: $\sigma = .2$, $\Delta t = .1$, $\Delta \theta = \frac{2\pi}{12}$. 
i.e., we investigate the solution of (2.10) for small value of ε and compare it with the solution of the hydrodynamic model (2.12)–(2.13).

To compare the solution \( f^\varepsilon \) of the kinetic equation (2.10) and the solution \((\rho, u)\) of the hydrodynamic limit (2.12)–(2.13), we analyze the evolution of the two first moments of \( f^\varepsilon \) corresponding to the mass \( \rho^\varepsilon \) and the macroscopic velocity \( u^\varepsilon \). They are obtained through averaging of \( f^\varepsilon \) in velocity variables:

\[
\rho^\varepsilon(x) = \int_\theta f^\varepsilon(x, \theta) \, d\theta, \quad u^\varepsilon(x) = \frac{j^\varepsilon(x)}{|j^\varepsilon(x)|} \text{ with } j^\varepsilon(x) = \int_\theta \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} f^\varepsilon(x, \theta) \, d\theta.
\]

Both quantities \( \rho^\varepsilon \) and \( u^\varepsilon \) are expected to converge to the solution of the hydrodynamic system (2.12)–(2.13) as \( \varepsilon \to 0 \). Thus, we compute the solutions of the kinetic equation (2.10) with different \( \varepsilon \) (resp. \( \varepsilon \in \{1, 10^{-1}, 10^{-2}\} \)). The hydrodynamic system (2.12)–(2.13) plays the role of a benchmark in this approach.

To obtain interesting patterns, we use as an initial condition a Riemann problem in the \( x \)-direction and we suppose that the solution is homogeneous in the \( y \)-direction. Thus, for the hydrodynamic limit, the initial condition is prescribed by two values \((\rho_l, u_l)\) and \((\rho_r, u_r)\) corresponding to the values at the left and right side of the domain (resp.). For the kinetic equation, we suppose that \( f^\varepsilon \) starts from local equilibrium meaning that \( f^\varepsilon(x, \theta) \) is initially a Von Mises distribution for any \( x \). For the two remaining degrees of freedom, \( \rho^\varepsilon \) and \( \theta^\varepsilon \), we initiate them such that the moments of \( f^\varepsilon \) match with the initial condition of the hydrodynamic limit.

We investigate three different Riemann problems. For our first simulation, the solution of the macroscopic model is given by a rarefaction wave:

\[
(\rho_l, \theta_l) = (2, 1.7), \quad (\rho_r, \theta_r) = (0.218, 0.5). \tag{4.1}
\]

In figure 7, we represent the density \( \rho^\varepsilon(x) \) and angle velocity \( \theta^\varepsilon(x) \) at \( t = 4 \) unit time. We do observe that, as \( \varepsilon \to 0 \), the solution given by the kinetic equation \((\rho^\varepsilon, \theta^\varepsilon)\) gets closer and closer to the solution of the hydrodynamic model \((\rho, u)\). The effect of \( \varepsilon \) can be seen as a “smoothing parameter”. It is consistent with the derivation of the hydrodynamic limit where the second order approximation in \( \varepsilon \) gives a diffusion-type operator [32].
For our second simulation, we use a Riemann problem that generates a shock solution. We use for that the initial condition:

\[(\rho_l, \theta_l) = (1, 1.5), \quad (\rho_r, \theta_r) = (2, 1.83).\]  

(4.2)

We plot the solutions of both the kinetic equation and hydrodynamic limit at time \(t = 4\) unit time in figure 8. Once again, the moments \((\rho^\varepsilon, u^\varepsilon)\) do converge toward the shock profile given by the solution of the hydrodynamic model \((\rho, u)\). We note that there is no analytic solution for the shock wave since the hydrodynamic model is not conservative, we lack the Rankine-Hugoniot conditions to find an explicit value for the shock speed. Hence, it is quite remarkable that both the kinetic formulation and the hydrodynamic model give the same shock profile with the same speed.

An additional inspection of the solutions suggest that the solution of the kinetic model might present cusp formation. For instance, in figure 7 (right), we observe a ‘blob’ forming at \(x \approx 3.7\) unit space, that vanishes as \(\varepsilon \to 0\). There is a similar but ‘more disperse’ pattern in the figure 8 at \(x \approx 1\) unit space. The formation of cusps will have a strong importance for the understanding of the kinetic model.

For our third simulation, the Riemann problem used is a contact discontinuity (see figure 9):

\[(\rho_l, \theta_l) = (1, 1), \quad (\rho_r, \theta_r) = (1, -1).\]  

(4.3)

Here, the solution of the hydrodynamic model is more complex than a rarefaction wave or a shock wave. As for the shock profile (figure 8), there is no analytic expression for this solution. But once again, the solution given by the kinetic equation does converge as \(\varepsilon \to 0\) toward the complex profile of the hydrodynamic model. Thus, these numerical simulations at the kinetic level confirm the relevance of the profile observed at the macroscopic level.

4.3. Vortex formation

In all the previous simulations, we have used Neumann boundary conditions in order to reduce the influence of the boundary of the domain on the dynam-
Figure 7: The solutions of the Riemann problem (4.1) is given by a rarefaction wave. **Left:** the mass $\rho^\varepsilon$ for the kinetic equation (blue) and for the hydrodynamic model (red). **Right:** the velocity angle $\theta^\varepsilon$ for the kinetic equation (blue) and for the hydrodynamic model (red).

Figure 8: Shock wave solution of the Riemann problem (4.2).

Figure 9: Contact discontinuity solution of the Riemann problem (4.3).
ics. We would like now to investigate in more details the impact of boundary conditions on the dynamics. For this purpose, we compare the kinetic model \cite{2.4} and the microscopic model \cite{2.1} in a bounded domain $\Omega$ with reflexive boundary conditions. At the microscopic level, each time a particle $i$ hits the boundary (i.e. $x_i \in \partial \Omega$), its velocity $\omega_i$ is reflected:

$$\tilde{\omega}_i = S_{\partial \Omega(x)}(\omega_i) = \omega_i - 2(\omega_i, \eta)\eta$$

where $\eta$ is the unit normal vector at $\partial \Omega(x)$ (see figure 10). Similarly, at the kinetic level, reflexive boundary conditions impose that $f$ satisfies at the boundary:

$$f(x, \omega) = f(x, S_{\partial \Omega(x)}(\omega)) \quad \text{for} \quad x \in \partial \Omega.$$

In other words, we have no-flux boundary conditions. In contrast to the kinetic model, reflexive boundary conditions would be more delicate to implement for the hydrodynamic model due to the constraint $|u| = 1$. Moreover, the validity of the hydrodynamic model near the boundary is questionable due to the possible formation of boundary layers (e.g. Prandtl’s boundary layer).

![Diagram of reflexive boundary conditions](image)

Figure 10: **Left:** Reflexive boundary conditions for the particle model. Once $x_i$ reaches the boundary $\partial \Omega$, its velocity $\omega_i$ is reflected. **Right:** Reflexive boundary conditions for the kinetic model: outgoing flux (i.e. $f(x, \omega)$) equals the incoming flux (i.e. $f(x, S_{\partial \Omega}(\omega))$) at the boundary $x \in \partial \Omega$ in all direction $\omega$.

Starting from a uniform distribution in space and velocity, we run the Vicsek model for both the particle and kinetic level. Both simulations are run on a
square domain $\Omega$ of 10 space units. For the particle level, we use $10^4$ particles with a radius of interaction $R = .2$ and a time discretization of $\Delta t = .01$ unit time. For the kinetic model, we use a mesh grid $\Delta x = \Delta y = .2$ and a larger time discretization $\Delta t = .05$ to reduce the numerical viscosity. We use 32 modes to discretize the velocity distribution.

After a transient period, both simulations converge toward a stationary state consisting of a vortex-type formation. In figure 11 we represent the spatial density $\rho$ and macroscopic velocity $\mathbf{u}$ defined as:

$$
\rho(x) = \int_\theta f(x, \theta) \, d\theta, \quad \rho(x)\mathbf{u}(x) = \int_\theta \omega(\theta)f(x, \theta) \, d\theta,
$$

with $\omega(\theta) = (\cos \theta, \sin \theta)^T$. As a result of the reflexive boundary conditions and the alignment interaction, the flow $\mathbf{u}$ tends to follow the boundary. Here, the flow $\mathbf{u}$ is turning clockwise but it is equally probable that it would turn counter-clockwise since the initial distribution is uniform in space and velocity.

We notice that the particle model has more fluctuations compared to the kinetic model as we could expect. For instance, the solution of the kinetic model is invariant under rotation of angle $\pi/2$ which is not the case in the particle model due to fluctuations. To measure the agreement and the discrepancy between the two spatial distributions, we introduce the distribution $\varphi(\ell)$ measuring the average density on the squares $C_\ell$ centered at the origin with radius $\ell$:

$$
C_\ell = \{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = \ell\}.
$$

In other words, $\varphi(\ell)$ is defined for $\ell > 0$ as:

$$
\varphi(\ell) = \frac{1}{|C_\ell|} \int_{C_\ell} \rho \, ds = \frac{1}{4\ell} \int_{\{\|(x, y)\| = \ell\}} \rho \, ds. \quad (4.4)
$$

We average over squares instead of circles to take into account the geometry of the domain $\Omega$. As we observe in figure 12 the distributions $\varphi(\ell)$ for the particle and kinetic models agree very well with each other. We observe some discrepancy near $\ell \approx 0$ and $\ell \approx 5$ corresponding respectively to the center and the boundary of the domain $\Omega$ but this is expected since the number of particles at the origin fluctuates more.
Overall, the kinetic model provides a reliable description of the microscopic model without the fluctuations. Moreover, the computation time of the microscopic model would drastically increase as the number of particles $N$ becomes larger. Whereas, for the kinetic model, the computation time remains exactly the same when we increase the density.

Additional numerical investigations are required to better understand the discrepancy between the two curves. In particular, it would be of great interest to analyze the influence of the number of particles $N$. Both curves should coincide as $N \to \infty$, but we lack a good quantifier to indicate when $N$ would be “large enough”.

5. Conclusion

We presented the first numerical method for a kinetic description of the Vicsek swarming model, which poses a unique challenge as there is a distribution dependent collision invariant to satisfy when computing the interaction term. The method used a spectral representation linked with a discrete constrained optimization to compute these interactions, and shows excellent agreement with the macroscopic equations when near that regime. The numerical results emphasize the importance of enforcing the collision invariants in the system. Future work will seek to extend this formulation to the three dimensional Vicsek equation, which has two Generalized Collisional Invariants instead of one. Another interesting application is using the kinetic equation to study the dynamics of the non-conservative macroscopic equations that arise from this model, in particular, the formation of cusps. Another interesting avenue for investigation is to develop more intricate boundary conditions in order to model more advanced wall avoidance, such as birds swerving when approaching a wall.

References

Figure 11: **Left**: density $\rho$ and velocity $u$ for the particle model at $t = 200$ unit time on a square domain with reflexive boundary conditions. **Right**: $\rho$ and $u$ for the kinetic model in for the same condition. Both solutions are close to a vortex-formation.

Figure 12: The average density $\varphi(\ell)$ on the square $C_\ell$ for the particle model (blue) and kinetic model (red) at time $t = 200$ (see figure 11). The distributions agree well with each other with some discrepancy at the origin ($\ell \approx 0$) and near the boundary of the domain ($\ell \approx 5$).


Appendix A. Summary of the numerical scheme

We summarize the different steps of the numerical scheme. We use the following notations: $x_i = i \Delta x$, $y_i = j \Delta y$, $\theta_s = s \Delta \theta$.

After the initialization of the distribution $f_{i,j,s} = f(x_i, y_j, \theta_s)$, the numerical scheme consists in iterating the following splitting method.

1) **Collisional operator**: in each cell $(x_i, y_j)$

   - **Decomposition**: compute the mean direction $\bar{\theta}$ and deduce \( \{c_k\}_k \)
   \[
   c_k = \sum_s f(\theta_s + \bar{\theta}) P_{-k}(\theta_s) \frac{\Delta \theta}{M(\theta_s)}.
   \]

   - **Update**: $c^*_k = c_k + \Delta t \bar{Q}(c_k)$.

   - **Reconstruct $f$**: $f_s = \sum_k c^*_k P_k(\theta_s - \bar{\theta})$.

2) **Transport operator**: for each velocity angle $\theta_s$ and in each case $i, j$

   \[
   f_{i,j,s} = f_{i,j,s} - \frac{\Delta t \cos \theta_s}{\Delta x} (f_{i+\frac{1}{2},j,s} - f_{i-\frac{1}{2},j,s}) - \frac{\Delta t \sin \theta_s}{\Delta y} (f_{i,j+\frac{1}{2},s} - f_{i,j-\frac{1}{2},s}),
   \]

   where the indices $i \pm \frac{1}{2}$ are computed using an upwind-method. For instance,

   \[
   i + \frac{1}{2} = \begin{cases} 
   i + 1 & \text{if } \cos(\theta_s) \leq 0 \\
   i & \text{if } \cos(\theta_s) > 0 
   \end{cases}
   \]

Appendix B. Stability of the homogeneous equation

To implement the numerical scheme \([3.23]\) for the homogeneous equation $\partial_t f = Q(f)$, we have to establish a stability condition. The explicit Euler method is stable under the condition that:

\[
|\lambda_{Max}| \rho \Delta t < 1,
\]

where $\lambda_{Max}$ is the largest eigenvalue of the matrix $[Q]_B$ \([3.18]\). From the Gershgorin circle theorem, we find a rough estimate as: $|\lambda_{Max}| \sim \frac{2}{\pi} N^2$. This rough estimation is confirmed numerically (see figure \([B.13]\)).
Analytically, the solution $f$ is converging toward an equilibrium (3.3). Thus, one needs to verify that $Q$ has a zero eigenvalue. Numerically (see figure B.13), the first eigenvalue $\lambda_1$ decreases with the number of discretization intervals $N$ and stabilizes for $N > 20$ near machine precision around $10^{-15}$.

Therefore, we use in our simulation $N = 32$ discretization intervals for the angle velocity $\theta$. For the time discretization, we ensure that $\frac{\sigma}{4} N^2 \Delta t < 1$ once we use the explicit Euler scheme. There is no such constraint if we use the implicit Euler method.

Figure B.13: **Left:** the largest eigenvalue $\lambda_{Max}$ of $[Q]_B$ (3.18) depending on $N$ the number of points of discretization of the angle velocity $\theta \in (-\pi, \pi)$. **Right:** the first eigenvalue $\lambda_1$ of $[Q]_B$ depending on $N$. 