What I Learned Today

Tom Gannon

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. . (3/20/17) Today I learned that every if $R$ is a ring and if $S \subset R$ is a multiplicative subset of $R$ not containing 0, then the prime ideals of $R$ whose intersection with $S$ is empty are in one to one correspondence with the prime ideals of $RS$, the ring localizing $S$. The way I look at this is to think, well, if I want a prime ideal in $RS$ from an ideal in $R$, say, $I$, if $I \cap S = \emptyset$, then you really didn’t change the ideal. I’m not 100 percent sure how right that is—we’ll see.

(3/21/17) Today on a homework problem I learned about the Tubular Neighborhood Theorem, which says that if one is given manifolds $Z \subset Y$, then there is a neighborhood in the normal bundle of $Z$ in $Y$, written $N(Z; Y)$, containing $Z$, that is diffeomorphic to a neighborhood of $Z$. The picture I drew of this was a central circle for $Z$ and a sphere for $Y$, which makes it seem super obvious, but I’m sure it isn’t in general.

(3/22/17) So fun fact. The fact about prime ideals and localization in (3/20/17) was in the book I was reading for fun at night called Algebraic Number Fields. But today, I learned a very similar fact about prime ideals, but this time relating to quotienting, in Vakil’s notes. In particular, there’s a natural bijective correspondence between prime ideals in a ring $R$ containing an ideal $I$ and prime ideals in the quotient ring $R/I$. This in particular gives some good visualization of $\text{Spec}(R/I)$ as a subset of $\text{Spec}(R)$. This sort of explains how we can relate ideals—for example, we can view any element of $\text{Spec}(\mathbb{C}[x,y]/(x^2 - y^2))$ as the set of ideals $I \subset \mathbb{C}[x,y]$ containing $(x^2 - y^2)$. So for example, the point $(x - 1, y - 2)$ isn’t considered. This will probably have a similar story with the above localization and prime ideals.

Also today I learned a fact in DiffTop which will probably serve well in the future, well, maybe. A submanifold $Z \subset Y$ is globally cut out by the zeroes of independent coordinate functions if and only if its normal bundle in $YN(Z; Y)$ is trivial. I feel like now that we’re getting into intersection theory stuff that might come in handy? I should think about why this applies to the example I learned in class about projecting a donut down to its height—why one circle by its lonesome can’t be cut out in this case.

(3/23/17) The first thing I can think of that I learned today is the Van Kampen theorem for groupoids. In particular, if $X$ is a topological space, then $\Pi(X)$ is a groupoid whose objects are the points of $x$ and whose morphisms between objects are homotopy classes of paths between the two points. In particular, if $S = \{U\}$ is an open cover of $X$ that is closed under finite intersection such that all of those open sets are path connected*, we obtain an $S$ shaped diagram if we view $S$ as a category whose morphisms are ”inclusion”. Then the Van Kampen theorem for groupoids says that $\Pi(X) \cong \text{colim}_S(\Pi(U))$. *Although to be honest, I’m not sure if this condition was necessary. I combed through the proof a few times to see, and it didn’t seem to. Then I saw this Stack-exchange post. http://math.stackexchange.com/questions/198348/does-mays-version-of-groupoid-seifert-van-kampen-need-path-connectivity-as-a-ly Also, the ”closed under finite intersection” thing might be a similar thing to being a ”filtered category.”
(3/24/17) I don’t know if there’s a theorem I can say I learned new today, although I did learn some new insight (I imagine that will happen more if I’m doing research). There was a DiffTop homework problem about the four dimensional manifold $S^2 \times S^2$. Now, I had no idea how to visualize this manifold, but I realized that I could abstract the idea of the torus being $S^1 \times S^1$ and work with that. It turns out that, if $a \in S^2$ is chosen, then $\{a\} \times S^2$ is not homotopic or cobordant to $S^2 \times \{a\}$ . Basically, we showed on an earlier homework that if two closed manifolds $X, Z \subset Y$ are cobordant, then for any compact submanifold $C, I_2(X, C) = I_2(Z, C)$. However, those two above manifolds aren’t cobordant because for some $b \in S^2$ that isn’t $a$, $\{b\} \times S^2$ intersects one manifold one time and one manifold zero times. I also learned the formal definition of cobordism, at least in the smooth category, and learned how a manifold that is the boundary of another inherits the orientation from the large manifold.

Also also I learned something with my DRP student in graph theory. The statement is “If $G$ is a graph with $n$ vertices and chromatic number $k$ and $G^c$ has chromatic number $l$ then $n \leq kl.$” The proof is slick as fuck. Basically, color $G$ via the colors $a_1, ..., a_k$ and color its compliment with $b_1, ..., b_l$. Then $\{(a_i, b_j)\}$ colors the complete graph.

(3/25/17) Today I learned that if you have a region with $n$ holes, then you can pick a homology basis and compute any complex integral you want by just summing over the winding number of the homology basis times the integral over the specified point in the basis. That’s super cool–it basically says any weird integral can just be reduced to whatever its homology is. I also learned that you can give a prime ideal a height in a ring, and saw my first PhD defense ever! Plus, I learned the formal definition of cobordism, at least such that any critical point has invertible matrix of second derivatives. Then there are no critical points in $[a, b]$ if and only if $f^{-1}((\infty, a))$ is homeomorphic to $f^{-1}((\infty, b))$.

(3/26/17) I learned two main things today–one of them is a technical lemma involving the inherited dimension of a map $f : X \to Y \supset Z$ where $f \cap Z$ and $\partial f \cap Z$, we get a submanifold $S = f^{-1}(Z)$ where $\partial S = \partial X \cap S$. First, I learned how to put an orientation on $S$. Basically, we use the direct sum orientation, noting that if $x \in S$, then $df_x(N(S; X)) \oplus T_{f(x)}Z = T_{f(x)}Y$ so we can use the induced orientation plus the orientation property of $df_x$ to give orientation to the normal bundle, and thus the tangent bundle. Now, to get the orientation on $\partial S$, we could either do this procedure first to get orientation of $S$ and then give $\partial S$ the boundary orientation, or we can immediately note that $\partial f : \partial X \to Y$ satisfies the above procedure so we could orient $\partial S$ that way. It turns out, these differ by a factor of $(-1)^{\text{codim}(Z)}$, basically because we need to permute the outward pointing normal vector with a basis of $N(Z; Y)$ to put them in the same order in direct sum notation.

Also I learned that, at least with my stupid hour of trying, it’s nonobvious why if $p, q$ are primes and the zetas are primitive roots of unity, $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, although for some reason it seems obvious that it’s true. I’ll find out tomorrow.

(3/27/2017) I learned so much today! Although I did not learn that thing above that I said I was going to learn tomorrow. However, I did learn the definition of an elliptic function today finally–it’s just a function $f : \mathbb{C} \to \mathbb{C}$ that has two linearly independent periods. Then it relates to lattices and the like.

I also learned a lot about orientation today. In particular, you can’t orient $\mathbb{R}P^k$ when $k$ is an even number, since its pullback under the map $S^k \to \mathbb{R}P^k$ yields the two different orientations on the two different poles. On the other hand, he did mention that you can always orient $\mathbb{C}P^k$, and I don’t remember the explanation. Also, since for odd $k$ the map $S^k \to \mathbb{R}P^k$ pulls back to the orientation, then you can orient $\mathbb{R}P^k$ for odd $k$. Also topologically related, I learned the idea of a satellite knot, which has a formal definition about incompressible discs, but as explained by Duncan, it’s better thought of as a knot put in a torus, and then the torus put in a knot itself.
Weirddddddd. I also finally learned what an isotopy is for real this time—it’s just a homotopy where at each time $t$ the map is an embedding!

And I did a good chunk of algebraic geometry today too. In particular, I learned that geometric intuition of localizing things is the best way to see localizing zerodivisors, and more importantly, given a ring map $\phi : B \to A$, then we obtain an induced map $\phi^{-1} : \text{Spec}(A) \to \text{Spec}(B)$. Also, you can use this method to generalize a given map of points from a space to another and determine how to extend it to the spectrum. Although I haven’t totally done the full exercise to show why the algorithm works fully. Oh well. Also, I learned that a function $f \in R$ is not necessarily determined by its value at all points, but on the other hand, $f$ is zero at every point if and only if it is nilpotent. Also, I learned that this thing might actually work for morale boosting. I’ve learnt two pages of things. I’m really excited about how much I’m learning!

3/28/17 I also learned a lot today. I mean, I worked pretty hard all day. The first thing I did was some Algebraic Topology. It was nice, because I learned about how to prove the most general case of the Van Kampen’s theorem—where any open cover can be infinite. And essentially this works because colimits commute with colimits. This is basically because if you have two diagrams you’re varying over, you can show they’re both isomorphic to what I imagine is called the “bicategory.” May only actually did it for two specific categories, both of which were in set whose maps were inclusion, but it seems like an easy generalization. I also learned (through an awful failure of computing the fundamental group of the Klein bottle) that the coproduct in the category of Groups is the free group. Can’t believe I didn’t connect it to what I learned in Hatcher. And for more category things, I learned that for any map $f : A \to B, p : E \to B$ where $p$ is a covering space map, then the pullback map $f^*p : A \times_f E \to A$ is also a covering space! Basically given any $a$, you can choose the neighborhood around $f(a)$ to be homeomorphic, and then pull back those corresponding open sets to $A$. I’m probably not 100 percent sure on the details of this though, although I worked through it for a little while. I’ll probably work more on Thursday.

I also learned a good chunk of Algebraic Geometry today. I’m really liking it. The main thing I learned was the idea of a vanishing set, and the corresponding Zariski topology you can put on a given Spectrum. Basically, the idea is that if you have a set $S$ you can ask, ”where does $S$ vanish?” Well if you want to say $S$ vanishes at a point $p \in \text{Spec}(A)$, thinking of a point $f \in S$ as a function, we want $f \equiv 0$ mod $p$, or equivalently, $S \subset p$. So the vanishing set of a set $S$ in a ring $A$ in Algebraic Geometry is ”the set where $S$ is zero” or by above, $\{q \in \text{Spec}(A) : S \subset q\}$. I learned some nifty lemmas, including a way to think about these sorts of vanishing sets that makes it clear that the complement of these form open sets. Basically, if you want $V(S) \cup V(T)$, you want the places where $S = 0$ or $T = 0$. Well then that’s where $ST = 0$! So $V(S) \cup V(T) = (ST)$. Similarly, $\bigcap_i V(S_i) = (S_i) = \sum_i(S_i)$ since we want it to be zero everywhere.

3/29/17 One of the things I learned was a nice proposition an an application. If $R$ denotes an integral domain and $K$ denotes its quotient field, an element $a \in R$ is integral over $R$, then the coefficients of the minimal polynomial of $a$ over $K$ are also integral over $R$! In particular, in an integrally closed domain $R$, the minimal polynomial of $a$ is in $R[x]$ if and only if $a$ is integral over $R$. This can prove a pretty powerful thing—if $d$ is a squarefree integer, then the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is $\mathbb{Z} + \mathbb{Z}\sqrt{d}$ if $4 |d - 1$ and it is $\mathbb{Z} + \mathbb{Z}\sqrt{\frac{d}{2}}$ otherwise. This above proposition makes it easy to show, since you can show all of that stuff is in the corresponding integral closure, and then you can show that if $q_1 + q_2 \sqrt{d} \in \mathbb{Q}[\sqrt{d}]$ solves an integer polynomial, then $q_1 \in \mathbb{Z}$ since you simply take the ”integer part” of expanding the polynomial it solves plugging it in. Then you get that via subtracting an integer that $q_2 \sqrt{d}$ solves some integer polynomial. I also learned something today on the level of practical advice—I should ask somebody to do a reading course in the fall before the summer starts.
(3/30/17) Today I got a major insight from a number theory talk that happened at UT given by someone named Samit Dasgupta. Well, I’m not sure if it’s major, but I did learn the idea that a lot of things in number theory are proven by going from local to global principles. For example, one of the first applications of this is the Chinese Remainder Theorem. An extension of this is that 

$$Z = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}.$$ 

From this talk, I also learned you can extend the zeta function to a function on any number field simply due to the fact that if you mod out by a maximal ideal in your number field, you get a finite number of elements, so you can "multiply" over all maximal ideals (where \(\frac{1}{1-N(p)}\), where \(N(p)\) is the number of elements in \(p\).

(3/31/17) Along with a few small things and an april fools talk, there were two main things I learned today. One was a few technical details in differential topology. For example, I disambiguated a phrase in one of the homework questions about if \(i : X \to Y \supset Z\) is an inclusion map on manifolds suitable for intersection theory \(i\) "perscribes points," which I spent a pretty decent time trying to figure out what it meant. It turns out that it simply means the preimage orientation via \(i^{-1}(Z)\). Also, I learned a definition and a good picture to have in my head to prove the relation between \(I(X,Z)\) and \(I(Z,X)\)—two maps \(f : X \to Y, g : Z \to Y\) are transverse at \(y \in Y\) if for every \((x,z) \in f^{-1}(y) \times g^{-1}(y)\) that \(df_x(T_xX) + dg_z(T_zZ) = T_yY\). The informal picture was \(f\) being a north-south circle going through itself on a torus, and \(g\) also winding around the circle twice hitting that same intersection point two times, but east-west (I should learn the technical terms for these), mostly. Then you have four pairs of points to check transversality.

I also learned a pretty good categorification of covering spaces, I have to admit. Essentially, you can say that a connected groupoid \(E\) covers another \(B\) if there’s a functor \(P : E \to B\) surjective on objects and such that if \(St(x)\) denotes the morphisms starting at \(x\), the function \(f : St(x) \to St(P(x))\) is bijective. For groupoids, this means that every path with a specified start point lifts to a unique path, and formally implies statements about conjugacy relations in \(St(x) = Aut(x,x)\) for a point \(x\).

April 2017

(4/1/2017) Today I worked through the topological side of Spectrum. For example, I learned about a topological space being Noetherian, which, like the ring definition, says that there’s no infinitely descending chain of closed sets \(V_1 \supset V_2 \supset \ldots\). You can use this to show that all open sets in a Noetherian topological space are compact, since this definition is equivalent to the definition that there’s no infinitely increasing chain of open sets \(U_1 \subset U_2 \subset \ldots\). Which is a pretty nice thing. I also learned the idea of Noetherian induction along those same lines, which is basically to construct an infinite chain and then use the Noetherian condition to argue that you’ve found an argument to break "maximality." You can use this to show that any closed set is the finite union of irreducible closed sets, and if no sets contain any other sets, this ordering is unique up to rearranging.

(4/2/2017) Today I proved one. hard. thing. At least it was hard for me. And that’s that if you take \(V(I(S)) = \mathcal{S}\), where \(V\) takes subsets of \(A\) to the prime ideals they vanish on, and \(I\) takes a set of prime ideals to the functions that vanish on each of them. The reason is because \(\mathcal{S}\) is just the intersection of all closed sets containing \(S\), but then since each closed set is a vanishing set of some set, we have \(\mathcal{S} = \bigcap_{V(J_i) \supseteq S} V(J_i)\). But it turns out there’s a "minimal set" among the sets \(V(J_i)\)—it’s \(V(I(S))\)! This follows because if \(J_i\) is an ideal containing \(S\) and \(p \in S\), then \(p \in V(J_i)\), then \(J_i \subset p\) which, intersecting over all \(p\) and using the inclusion reversing nature of \(V\) establishes our claim. So the closure of any set is really just any prime ideal that is larger than the intersection of all the ideals in that set.

(4/3/2017) Today I learned a lot of cool stuff about the factorization of an ideal into a prime
ideal. You can do this in a Dedikind domain–a notherian integral domain such that if you localize any nonzero prime ideal, you get a Discrete Valuation Ring–a PID with exactly one nonzero prime ideal. Then taking a Dedikind domain \( R \) and some nonzero ideal \( Q \), you can consider the ring \( R' = R/Q \). By Noetherianness (which passes to quotients), every ideal has a product of prime ideals inside of it, so in particular zero does (which is \( Q \) in the original ring.) Then you can use the Chinese Remainder Theorem to argue since all of these ideals are maximal, you can write them as the direct sum of the quotient of each prime ideal to a power (with a technical Lemma that says the Chinese Remainder Theorem "coprime" condition still holds no matter what power you raise your maximal ideals to). Then you can basically argue that the prime ideals you got are ALL the ideals because of this direct sum notation. This gives you the factorization of an ideal \( \mathfrak{p} \) you might want to factor–you can use the fact that there’s a product of primes \( Q := p_1^{n_1} \ldots p_n^{n_n} \subset \mathfrak{p} \) and mod out by this, using the fact that any ideal is just an ideal of each slot in the quotient (CRT) ring–which are just powers of the prime ideal by our Dedikind domain assumption. You can also require this factorization be minimal on the power of all primes with a positive power–since for each prime ideal \( p_i \) if we localize all of the other prime ideals, then we again get a DVR–\( R_{p_i}/\mathfrak{p} \) is an ideal of \( R_{p_i} \), so by our DVR stuff it’s just \( R_{p_i}p_i^{n_i} \).

I also went through the proof of the Chinese Remainder Theorem for modules. And I saw something kinda cool. Let’s work in \( \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \). Then in \( \mathbb{Z}/6\mathbb{Z}, 2 \times 5 = 10 = 4 \). Now using the Chinese Remainder Theorem interpretation, \((0, 2) \times (1, 2) = (0, 4)\), which is identified with four! I don’t think I ever realized this before–it greatly simplifies a lot of the stuff I learned in my first undergraduate number theory class. Plus I learned some things about orientation theory (including that points are taken to be positive if they’re the closed subset of the image manifold!)

(4/4/2017) Today I did two main things. The first thing was I learned some insight on what’s really going into what Peter May calls the "Fundamental Theorem of Groupoids," which means that if \( p : E \to B \) is a covering of two (connected, small-assume hereout today) groupoids, and \( f : A \to B \) is a functor between two connected, small groupoids, then there’s a lift functor \( g : A \to E \) mapping a chosen basepoint \( a \in A \) to a chosen point in \( e \in p^{-1}(f(a)) \) if and only if \((f(\pi(A, a)) \subset p(\pi(E, e)))\). One direction is straightforward and just compares images. But otherwise you can construct the functor \( g \) through this recipie. For any object \( a' \in A \), where should we lift it to? Well, pick a path (really, a morphism) \( \gamma : a \to a' \). Then \( f(\gamma) \) is a path in \( B \) and we’ve specified the start point, so use the covering space to uniquely lift to a certain point. Then map \( a' \) to whatever endpoint the path got lifted to. The thing is, if we had chosen another path \( \gamma' : a \to a' \), then \( f(\gamma^{-1}\gamma') \), by assumption, is a loop in \( \pi(E, e) \) after lifting. This is why the basepoint doesn’t map. I also figured something that was obvious, but nonobvious why it was obvious. It’s easier to show morphisms lifting are unique–that’s essentially from the definition of covering–which tripped me up reading.

I also finally finished Chapter 3 of Vakil! I finished it off by noting that the vanishing set function \( V() \) and the "Innihilated by everything in my subset of Spec" function \( I() \) take prime ideals (for "vanishing sets") to irreducible closed sets, and moreover, a bijection between minimal prime ideals and irreducible components of the spectrum of a ring. The full proof is written on my printed copy–but for now I’m happy to say it’s all sorted out! It’s just a matter of remembering that a subset of \( \text{Spec}(R) \) for a ring \( R \) is basically a list of prime ideals–another obvious thing you forget from time to time.

(4/5/2017) Today I did some differential topology, specifically Lefschetz intersection theory. In particular, if \( f : X \to X \) is a map of a compact manifold to itself, then we can compute its Lefschetz number, defined as \( I(\Delta, \text{graph}(f)) \), where \( \delta \subset X \times X \) is the diagonal. This in some slightly loose way measures the fixed points of a map (although, we are implicitly requiring that \( \Delta \) is homotoped to be transverse to \( \text{graph}(f) \)–so this isn’t so obvious for a map with infinite fixed points, like the identity map.) It turns out through some technical linear algebra details that the transversality
condition at a fixed point \( x = f(x) \) to be have the transversality condition satisfied is equivalent to \( I \) not being an eigenvalue for \( df_x \). A way to interpret this is "fixed points are isolated" at the infinitesimal level. Then I worked through the case of the map being from \( \mathbb{R}^2 \to \mathbb{R}^2 \) (which extends to any two dimensional manifold since this is a local property), and got an idea of how to tell from the eigenvalues whether a Lefschetz point is a "sink" (in which liquid would go into), a "source" (in which liquid would always come from) or a "saddle," at least if the eigenvalues are positive. That interpretation also gives a semi-intuitive way to compute the Euler Characteristic of the torus of any genus, imaging the donus with many holes on its side and counting the one saddle, one sink (with Lefschetz number +1) and then the \( 2g \) (where \( g \) denotes the genus) saddles, which have Lefschetz number \(-1\).

I also did a lot of algebraic topology. I translated the result I got yesterday (through the extensive help of Peter May’s book, anyway) to show that the category of coverings of a connected, small groupoid is basically the same (read–equivalence of categories) as the orbit category of the fundamental group \( G \) at some basepoint. I also got that all \( G \) automorphisms of fibers are lifts of some loop in the fundamental group and vice versa, and the fact that a "covering of covers" exists if and only if a subconjugacy relation holds (not equality)–and this is essentially because your isomorphism need not match basepoints, so you need to conjugate to rectify that.

(4/6/2017) Today I learned the proof of the fact that the spectrum of a ring is a separated presheaf, as well glossed into the proof of the gluability axiom. Essentially, you can reduce the proof of the "identity" axiom to proving that if a function/ring element \( r \) restricts to zero on each \( D(f) \) then it is zero in the whole ring. We then use a really nice fact that says at \( D(f) \), if \( A \) is our ring, our corresponding ring at \( Spec(D(f)) \) is \( A_{\mathbb{F}} \), and then use compactness (or what Vakil calls quasicom pactness) to reduce to \( i \) taking only a finite number of values. So then we get that \( f_i^{s+r} = 0 \) after cross canceling some fractions. Then we use the fact that \( Spec(A) = \bigcup_i D(f_i) = \bigcup_i D(f_i^{s+r}) \) to note that \( (f_i^{s+r}) = A \) so we can write \( 1 = a_1 f_1^{s+1} + \ldots \) and see that \( s = 1s = 0 \).

(4/11/2017) Today I learned some facts about knot theory and the cobordism of links. First, (which is actually something I learned a while ago technically), two knots are isotopic if and only if their knot diagrams are related by a sequence of the three Reidmeister moves–the first being "unkinking/adding" a loop, the second being crossing parallel strands or going in the opposite order, and the third being "moving a straight line that is completely over two other lines over the crossing (or under)."

I also learned about the idea of "rammification," which essentially says that in an algebraic geometric sense, it’s not helpful for an ideal to factor into prime ideals to a power. For example, factoring \( Spec(\mathbb{Z}) \) over \( Spec(\mathbb{Z}[i]) \), we see that the ideal \( (2) = (1+i)^2 \). This is in a different class than all of the other prime ideals in \( \mathbb{Z} \)–something I hope to learn more about later.

(4/12/2017) Today I learned a cool consequence of the Tubular neighborhood theorem. First, the Tubular Neighborhood Theorem says that if you are given manifold \( Z \subset Y \), where \( Y \) is another manifold, then there’s a diffeomorphism from a neighborhood of \( Z \) to a neighborhood of the subset of \( Z \subset N(Z;Y) \). Now if \( Z \) also happens to be globally definable by independent functions, by a previous homework result we have that the normal bundle is actually trivial. Therefore using this tubular neighborhood theorem, if \( Z \) is a compact manifold globally definable by independent functions, we can slide \( Z \) off of itself in the neighborhood in the normal bundle, and hence in the manifold itself, which shows that \( \chi(Z) = I(\Delta, \Delta) = 0 \).

(4/13/2017) Today I learned a lot about Khovanov homology. In particular, if you have a knot diagram \( D \) and you designate one crossing, there are two ways you could have smoothed the crossing–call them \( D_0 \) and \( D_1 \) for, in the notation of Turner’s Five Lectures on Khovanov Homology, the resulting diagram obtained by 0-smoothing and 1-smoothing our diagram respectively. But "basically," we have that the complex \( C^{*,*}(D) \) we get is a direct sum \( C^{*,*}(D_0) \oplus C^{*,*}(D_1) \). Now,
the ”basically” part comes in because we haven’t taken into account grading. But by following the receipe of Khovanov homology, there’s an easy way to put grading back into the picture, depending on whether your crossing is positively or negatively oriented. Then you obtain a short exact sequence from your fake direct sum, which then immediately gives you a long exact sequence you can use to compute Khovanov homology of. One application of this is that, knowing that the Khovanov homology of the unknot is zero except in the (0, ±1) bigrading, we can use the sequence to compute the Khovanov homology of the Hopf link more easily than computing it explicitly.

(4/14/2017) Today I learned a lot about Dedikind rings. The original definition of a Dedikind ring is a Noetherian integral domain $R$ such that if $p$ is a prime ideal, then the ring $R_p$ is a Discrete Valuation Ring (DVR), meaning that it is a PID exactly one maximal ideal. This satisfies the idea of ”good” in some sense because, as I learned in the course of the two equivalent definitions of Dedikind ring, it means that we could work locally and locally, according to the definition, we have an easy ring to work with. One of the equivalent definitions of a Dedikind ring is a ring $R$ such that $p$ is a maximal ideal, then the ring $R_p$ is a Discrete Valuation Ring (DVR), with the additional requirement that every ideal contains only a finite number of prime ideals. Without totally going through the proof of why these two make sense, one way to sort of think about it is to note that another ”looser” definition of a Dedikind ring is a place where ideals can be factored into products of prime ideals.

Then I learned the proof of the other definition of a Dedikind ring, which is ”a Noetherian, integrally closed domain with every nonzero prime ideal maximal.” This sort of makes sense too if you take the ”factor into primes” being natural definition, considering if you have primes of height two, say, what would you factor them as in your natural factoring? In this proof, I also learned some Lemmas that really hammered home the point of ”work locally and see how to apply globally.” For example, one of the things I learned is that $R = \bigcap_p R_p$, which is used to show that $R$ is integrally closed (since $R_p$ is, in particular, a PID, and the intersection of integrally closed rings are integrally closed).

(4/17/2017) Today I learned a few things. To be brief on at least one of them, I learned about infinite products. In particular, you can define whether a product converges if and only if it stops being zero eventually, and then the product $\prod a_k$ converges absolutely if and only if the sum $\sigma|\omega_k - 1|$ does. That’s pretty neat, and loosely put, you can identify an entire function $\mathbb{C}$ with its genus, which roughly measures what happens after the zeroes of the function occur (more to come on that on Friday).

I also learned this amazing fact about why one forms are the way they are. In the case of a two dimensional vector space, if we want some notion of area form we want the area function $\alpha : V \times V \to \mathbb{R}$ to be linear in each variable, and we want it to send $v, v$ to 0. This forces $\alpha(v_1, v_2) = -\alpha(v_2, v_1)$. Which makes so much sense and is much more motivated than just asserting that the determinant is anticommutative. Since we don’t have normalization on an ”abstract vector space,” we have a one dimensional subspace of choices of what to make our area function. This leads to the notion of a two form on a vector space, and the wedge product and all the other fun stuff.

I also finally went through the definition of a goddamn scheme! It’s been like 6 months of me reading algebraic geometry, and I finally get to that point. Yay. And I have my first example of a scheme that is not an affine scheme, which is the infinite disjoint union of affine schemes. This isn’t an affine scheme because in an affine scheme, the whole space is compact, whereas if we have an infinite number of schemes whose topologies are disjoint, we can have an open cover with no subcover.

And finally I touched up on some Algebraic Number Fields stuff. In particular, I learned more about fractional ideals, which included taking a detour into knowing what a Noetherian
module meant. And here’s what it means—there’s a similar "ascending chain condition," but this time on submodules. Which makes a lot of sense because of the insight I had today that ideals are just submodules of the ring itself. It makes so much sense in hindsight. This is why the inverse of a fractional ideal is finitely generated by the way, since it’s contained in a finitely generated module (since if $I$ is my fractional ideal and $m$ is in the $I$, $I^{-1}m \subset R$, so $I^{-1} \subset Rm^{-1}$). The equivalence of the ACC of submodules and all submodules being finitely generated show that our inverse submodule is also a fractional ideal, since it’s finitely generated and trivially an $R$ submodule. Speaking of, I also learned some technical details which are kind of important—fractional ideals must be finitely generated as modules and nonzero.

(4/18/2017) Today I learned that when determining the long exact sequence traditionally used in Khovanov homology, we are typically just exploiting a particular $i$ (homology) grading, and the fact that if you take into account grading correctly, you get a direct sum, and hence a short exact sequence, and hence a long exact sequence induced from that. I’ve also been working hard on a problem that says if $M, R$ are fractional ideals, then $(M,R)^{-1} = M^{-1} R^{-1}$. I hope to type up a complete solution tomorrow.

(4/19/2017) Today I learned/went through problems regarding the flows on a vector space, and sorted out the difference between a flow and a vector field. So a flow is just a family of diffeomorphisms on a manifold, say $f_t$ such that $f_{t+s} = f_t f_s$ (“If you drop a leaf in a river and wait $t+s$ seconds, it’ll end up in the same spot as if you wait $s$ seconds and then drop the leaf into the $f_s$ leaf spot and wait $t$ seconds both leaves will end up at the same spot. Now, given a flow, one can partition a manifold via flow lines, which are simply fixing a point $x$ on the manifold and considering the curve $t \rightarrow f_t(x)$. So putting the tangent vector of the flow line at each point onto the manifold, we get a vector field on our manifold. And going backwards, at least in the compact case, if we’re given a vector field we can get a flow using the theory of ODE. We can also talk about the index of an isolated zero $x$ of a vector field, which is cooked up via “take a small ball not containing any zeroes of the vector field other than at $x$ and then on the boundary define a sphere to sphere map taking the point and mapping it to the direction of the sphere. I drew some pretty pictures today.

I haven’t totally gotten the number theory problem yet, but I’ve solidified some shit. For example, I think I subconsciously believed that any irreducible element $p$ has the property that $(p)$ is prime. But this isn’t true for all rings (although it is true for PIDs). For example, $(2)$ isn’t a prime ideal in $\mathbb{Z}[^{\sqrt{5}}]$; for $2|6 = (1 + \sqrt{5})(1 - \sqrt{5})$ but doesn’t divide either term in the product (which can be seen by modding our ring out by 2.) Maybe tomorrow. Maybe.

Or maybe today. First I’m going to prove a lemma that says if $U$ and $B$ are fractional ideals with $U \subset B$ and for all maximal $q U_q = B_q, U = B$. For if $b \in B$, we let $I = \{ y \in R : yb \subset U \}$ Then $I$ is an ideal of $R$ since $U$ is a fractional ideal, and since $b \in U_q, b = a/s$ for some $a \in U, s \in R \setminus q$. Therefore $s \in I$ and therefore $I$ isn’t contained in any maximal ideal, and therefore $I = R$. Therefore $1b \in U$.

Great. Now that we have that Lemma, note by exercise 1 we have $[(MN)^{-1}]_q = [(MN)_q]^{-1} = [M_q N_q]^{-1} = [M_q]^{-1} [N_q]^{-1} = (M_q^{-1})(N_q^{-1})_q = (M^{-1}N^{-1})_q$, where the double equals sign comes from the fact that the statement is true for a PID.

Now all I have to do is show that this is true for a PID. But given a fractional ideal $M$ with generators $\{ \frac{a_j}{s_j} \}$, I bet you $M = R e_r$ where $r$ is a principal generator of $M \cap R$ and $s = \gcd \{ s_j \}$. Oh and what do you know—that’s exercise true. Every fractional ideal is principal.

(4/20/2017) Today I did a LOT of number theory. In fact, I did exclusively number theory today—specifically I focused on the linear algebra you can do on a field extension $L/F$. For example, for any element $x \in L$, you can view “multiplication by $x$” as an $F$—linear map from $L$ to $L$. Picking
an $F$ basis of $L$, you can write a matrix expression for the linear map. Usually though, we don’t like to pick bases arbitrarily, and so we compute things which aren’t dependent on bases. The two we care about are the trace and norm/determinant of the map.

The first thing I learned about this today is that since you can factor any $F$ term out of the trace, there is a symmetric $F$ bilinear form on $L$ defined via $(x, y) = T_{L/F}(xy)$. Now one thing you could ask about this form is if it is nondegenerate or not–meaning whether or not $(x, L) = 0 \implies x = 0$. Today I learned the proof that this bilinear form is nondegenerate if and only if the field extension $L/F$ is separable. To see this in one direction, note that if $L/F$ isn’t separable, then $L^p \neq L$ but $L^p$ contains any separable element. Therefore you can find an element $x \in L$ that isn’t a $p^{th}$ power, where $p$ denotes the characteristic of our nonseparable field, since you can argue that $x$ is in the purely inseparable part of the field, meaning that $x^k - y^k$ is the minimal polynomial over $F$ for some $k$. So if $y \in L$ and $xy \notin L^p$, then $Tr_{L/F}(xy) = 0$ since the minimal polynomial of $xy$ is of the above form–which has second term zero. But if $xy \in L^p$, we use the fact that we can just factor it out/view it as a scalar over $L^p$ and use the transitivity of the trace–i.e. $Tr_{L/F}(xy) = Tr_{L^p/F}(Tr_{L/L^p}(xy)) = Tr_{L^p/F}(xyTr_{L/L^p}(1)) = Tr_{L/F}(xy0) = 0$ since $[L : L^p]$ must be a power of $p$.

But even more cool, we have some theorems relating the trace and the norm to Galois Theory! The specific version says that if $L/F$ is Galois, $G = Gal(L/F)$ and $x \in L$, then $Tr_{L/F}(x) = \sum_{\sigma \in G} \sigma(x)$ and $N_{L/F} = \prod_{\sigma \in G} \sigma(x)$. There’s a really cool proof of it–essentially you break it down into a chain of field extensions $F \subset K \subset L$ and you exploit the fact that over $F(x)$, the characteristic polynomial of $x$ is just $q(x)^{[L:F(x)]}$, which relates the trace/determinant to the characteristic polynomial, which can be related to roots of the characteristic polynomial (which is now approaching Galois Theory) and then if you break down $G$ into rising chain of subgroups corresponding to the above field extension, you can compute the trace/determinant and notice those also involve the roots of the characteristic polynomial. Woah!

(4/21/2017) Today I learned some things in preparation for Chandrashekar Khare’s job talk (which I hope he gets). One thing I learned was that given any field $F$, you can take the absolute Galois group $G_F$, which is defined to be the automorphisms of $F_{sep}/F$, where $F_{sep}$ is any element whose minimal polynomial over $F$ repeats no roots. Khare’s talk went into the representation theory of it, which connected it to $SL_2(\mathbb{Z})$. I also learned from Shalin’s talk that for almost every number between zero and one, if you take the geometric mean of the first $n$ numbers in the continued fraction expansion of a real number, you approach $K_0 := \prod_{r=1}^\infty (1+\frac{1}{r(r+2)})^{\log_2(r)} = 2.685452$. However, there is no specific number that we work with that we know this holds for–i.e. the only numbers we have that actually do satisfy this property were written specifically to satisfy this. I learned more things today, but went to the bar and forgot to write shit up.

(4/22/2017) Today I learned more about forms! In particular, I hammered down the definition of the alternating product of two given elements in the tensor algebra and the wedge product–in particular $S \wedge T := Alt(S \otimes T)$, at least up to a factorial sign. I also learned that given a positively oriented basis chosen, you can construct a canonical volume form, which is an element in the top exterior power which evaluates to one on the ON basis. I also learned (or discovered) today that the Galois Theory/linear transformation norms that can be applied to any field extension are the exact same norms that I learned for quadratic field extensions–which is pretty interesting.

(4/23/2017) Even more about forms today. This time, we applied forms to manifolds! In particular, you can define the notion of a $p$ form $\omega$, which, given some point $x$ on your manifold, $\omega(x)$ returns a $p$ form on the vector space $T_x X$ (where $X$, of course, is your manifold.) Then you can define the pullback of a given form via a smooth function $f : X \to Y$, which takes in forms on $Y$ and returns a form on $X$ as follows. Given a form $\omega$ on $Y$, after it’s fed a point $y \in Y$, it returns a functional on $T_y Y$. How can we move our form backwards to a point $x \in X$? Well, we
need \((f^*\omega)(x)\) to be a functional on \(T_xX\). How do we get such a functional? Well, we can push the tangent space forward through the derivative map, and then use our old form to send that into your ground field. Also, in Euclidean space, I learned that the forms \(dx_i\), where \(x_i\) are the coordinate functions, form a basis for any form, and we can define smoothness of a form \(\sum f_I x_I\) by requiring that all \(f_I\) are smooth (where \(I\) is a strictly increasing index set), and furthermore the smoothness of a form on a manifold by requiring that the pull back of a local parametrization yields a smooth form on Euclidean space.

I also created my first realistic exercise and something original, which I haven’t done much of. In particular, I proved that for all nontrivial field extensions \(K/F\), there exists a nonzero \(\alpha \in K\) such that \(Tr_{K/F}(\alpha) = 0\). I then showed that furthermore if \([K : F] > 2\) (at least, not sure about the two case), then there exists a \(\theta \in K\) such that \(\langle \theta, \theta \rangle := Tr_{K/F}(\theta) = 0\). This is different and weirder than other inner products, say over \(\mathbb{R}\). However, in \(\mathbb{Q}[i]\), \((a + bi)^2 = a^2 - b^2 + 2abi\), which in particular says that \((a + bi, a + bi) = Tr_{\mathbb{Q}(i)/\mathbb{Q}}((a + bi)^2) = a^2 - b^2 - 2ab \neq 0\), since if equality held we would have \(2a^2 = b^2 + 2ab + a^2 = (a + b)^2\). This says that either \(a = 0\) (so \(b = 0\) also) or \(2\) is rational.

(4/24/2017) Today, I learned (finally) what an analytic continuation is, and what a maximal analytic continuation is. Essentially, the idea is a dumb thing to define would be a “two analytic continuation” on a function \(f_1 : \Omega_1 \to \mathbb{C}\) and another \(f_2 : \Omega_2 \to \mathbb{C}\) as long as the functions agree on the intersection of the two domains \(\Omega_2\). But then you basically have a function split into two pieces. However, say with the logarithm example, you might have an \(f_3 : \Omega_3 \to \mathbb{C}\) which makes a “two analytic continuation” (I think the actual term is “direct”) which agrees with \(f_2\) on the common domain of intersection, but not \(f_1\). This leads to the notion of multiple valued function, and to require you have the largest possible sets of \((f_\alpha, \Omega_\alpha)\) such that any two \(f\) have a chain of analytic continuations connecting them define a maximal analytic continuation.

I also learned how to integrate today–finally. You can integrate a top form on an open set in Euclidean space \(U\), that must have the form \(\omega \wedge dx_1 \wedge ... \wedge dx_k\) by merely defining \(\int_U \omega = \int_U f dx_1 ... dx_n\). It turns out that forms respect the pullback rule (at least for orientation preserving diffeomorphisms), and that fact is built into how forms were constructed. Therefore we can define an integral of a form on a compactly supported function as just pulling back a parametrization to Euclidean space, and then (independent of choice of partition of unity) define the integral on the total space as summing over a partition of unity. I’m actually not 100 percent on these details yet. Maybe tomorrow!

I also learned how to prove that, given a Dedikind ring \(R\) and its quotient field \(K\), then if \(E/K\) is a separable field extension then \(R'\), defined to be the integral closure of \(R\) over \(E\), is also a Dedikind ring. Now, you can show it’s integrally closed since the integral closure of an integral closure is just the integral closure. Notherianness follows from the fact that you can cleverly choose a basis of \(E/K\) to ensure that all of your basis elements are actually in \(R'\) (which, by the way, it turns out that \(E\) is the field of fractions of \(R')\) and then use a dual basis argument to essentially argue that \(R' \subset \sum Rb_i\) where \(b_i\) is the dual basis of your freshly chosen basis in \(R'\). Then you get Notherianness because it’s the subset of a finitely generated \(R\) module (which, by the way, I reviewed today). Also you get the fact that every nonzero ideal is maximal from a lemma which essentially says that if you have an integrally closed ring \(A\) and \(B\) is integral over \(A\) then every nonzero ideal \(p \subset B\) has an associated nonzero prime ideal \(p \cap A \subset A\). This in particular says that if \(A\) above is also a field, then so must \(B\) be—which is essentially how you show modding out by any nonzero prime ideal of \(R\) gives you a field.

(4/25/2017) Today I learned the idea of what a moduli space. It seems to be sort of like the idea of a fiber bundle, and in fact, I can’t really tell the difference at this point. But I know that a moduli space is essentially the idea of a space where each point has a "insert your favorite mathematical
thing” living above it. The classical example is $\mathbb{C} - \to \mathbb{R}P^1$. I also learned the generalization of this, the Grassmannian, which is simply the $k$ dimensional subspace of a vector space $V$, when $k$ and $V$ are given. I also solidified what is going on with respect to the isomorphism as schemes $\text{Spec}(A) \coprod \text{Spec}(B) \cong \text{Spec}(A \times B)$. In the course of this isomorphism, I essentially showed that each closed set in $\text{Spec}(A \times B)$ is mapped to the union of two closed sets in $\text{Spec}(A) \coprod \text{Spec}(B)$, which sort of explains why the infinite disjoint union of affine schemes isn’t an affine scheme.

(4/26/2017) Today I learned (well, technically solidified, but I basically learned) what a contact form is on a manifold. A contact form on a three manifold is a form $\alpha$ which sort of explains why the infinite disjoint union of affine schemes isn’t an affine scheme. Inclusions are simple homotopy equivalences) to the “trivial” disc filling.  
biholomorphic embedding and say that a Stein filling of a manifold is essentially a Stein manifold to be talking about—essentially the question is, if we define a Stein filling to be a ”nice” filling (okay, I’m not 100 percent sure on the actual definitions, but essentially a Stein manifold admits a proper biholomorphic embedding and say that a Stein filling of a manifold is essentially a Stein manifold having that as a boundary). The statement my authors of my paper plan to explore and prove is that if $g > 2$, then any Stein filling of the unit cotangent bundle is s-cobordant (a cobordism whose inclusions are simple homotopy equivalences) to the ”trivial” disc filling.

I also learned an easy fact that if you take the talk of $\text{Spec}(A)$ at a point $p$, you obtain the ring $A_p$. This is simply because if $f \notin p$, by definition $p \in D(f)$, which in particular tells us that we have an open set containing $p$ where $f$ is invertible. Thus for any open set $U$ contained in $D(f)$, $f$ will be invertible on $U$ as well, which in particular implies $f$ is invertible in the stalk.

(4/27/2017) I learned there’s a canonical structure on the cotangent bundle of a manifold $M^k$, denoted $T^* M$, which is the manifold defined as a set as $\{(x, \tau) : x \in M, w \in T^*_x M\}$, and topologized via the product topology, with atlas defined via maps of atlases on $X$ crossed with $\mathbb{R}^k$. In particular, there’s a canonical one form on the cotangent bundle! How, you might ask? Well, the cotangent bundle comes with a canonical projection $\pi : T^* M \to M$, and let’s say we want our form to be $\omega$. Then $\omega$ should eat a point $(x, \tau)$ as above, and spit out a map from the tangent space of the cotangent space at $(x, \tau)$, which is simply the tangent space at $T^*_x M$. But we already have a projection map, which in particular tells us that we can project down to the tangent space. So why not use $d\pi$ to project onto the first coordinate, and then take $\tau$, which is already a linear map into the ground field? Which is exactly what it’s defined to be.

I also learned why it’s a little weird/deeper than you might think that there’s a bijection between ring isomorphisms $A \to A'$ and scheme isomorphisms $\text{Spec}(A') \to \text{Spec}(A)$. Basically, the problem and the ”soft deep truth” is that your whole scheme isomorphism is basically determined by how it operates on $D(1)$, since that’s the whole ring. I suspect this has to do with localization and the fact that $A = \cap_p A_p$ or the stalk version of that, but we’ll see. In algebraic geometry I also just did an exercise which essentially says that functions vanish on a closed set in locally ringed spaces. This is because in ringed spaces, if you vanish on the stalk, you can find a small enough open set to find an inverse, and then you get an inverse in that whole open set. Then since in a locally ringed space the stalks are fields, so you’re either zero or invertible. Then you can use this to argue that in a locally ringed space, if you don’t vanish anywhere then you’re invertible. This is essentially because for every point you can locally cook up an inverse, so you use gluability and then identity to show that it actually is the inverse.

(4/28/2017) Today I learned (formally) what the exterior derivative is, at least in Euclidean space. Essentially we want a linear operator that satisfies some sort of product rule on the form $\omega = \sum f_i \wedge dx_i$, but also, since $dx_i$ is basically a constant, all those terms should evaluate to zero. So we can define $d\omega = \sum df_i \wedge dx_i$. From this, you can unravel some stuff to prove that it satisfies
an "almost" product rule (which essentially means you have to account for the fact that a minus sign might appear), and more importantly, that \(d^2 = 0\), which actually later means you can define a cohomology theory with it! Wooh! Another nice thing about the exterior derivative operator is that if you have another operator that distributes over sums, has the "fake" product rule, and squares to zero, it’s determined by how it operates on functions, i.e. zero forms. So in particular there’s only one way to define a \(D\) operator on forms that agrees with the three properties above and agrees on zero forms! You can use this to show that you can do exterior derivatives on manifolds. Wooh!

I also learned an example of a nonaffine scheme today, which I would have thought before should be an affine scheme. Let \(X = \mathbb{A}^2\) and \(U = \mathbb{A}^2 - \{(0, 0)\}\), which really is \(\mathbb{A}^2 - \{(x, y)\}\). I learned how to compute the scheme at this open set \(U\), which isn’t an affine open set, since if \(f \in (x, y)\), it’s in another prime ideal—pick an irreducible element \(g\) dividing \(f\) and then \(f \in (g)\) as well. But you can show that the functions actually don’t change if you restrict to \(U\), which actually follows more formally for some result about complex geometry if the codimension is larger than one, but in this particular case, follows as \(U = D(x) \cup D(y)\) and for a function to be a function on \(U\) it must be an element of \(k[x, y, 1/x]\) which when projected agrees with the prechosen element of \(k[x, y, 1/y]\) in \(k[x, y, 1/(xy)]\). This is just \(k[x, y]\) again, so we don’t gain any functions. This actually shows that the restriction to \(U\) isn’t an affine scheme—for if it were, what its ring would have to be \(k[x, y]\) by above, and you would ask, what point is associated to \((x, y) \subset k[x, y]\)?

(4/29/2017) Today I learned (well, unfortunately, had to rebuild the foundations of), the function \(I()\), which takes a set (or a way I felt works better for me for some reason, "list") of prime ideals, and returns all of the functions (i.e. ring elements) that vanish at that point. In particular, I showed today in a better manner that \(V(I(S)) = \overline{S}\). This is because you can do some definition chasing to show that \(\subset\) holds, and to show that \(\supset\) holds, you can argue that it holds for \(S\) and then take the closure of both sides, which helps since the right hand side is closed. I also came up with a slightly better way to work with the closure of a set in \(\text{Spec} A\) for some ring \(A\)—you can think of \(\overline{S}\) as \(\bigcap_{g \in V(g) \supset S} V(g)\), since the intersection of all the closed sets are vanishing sets of ideals, and then \(V(I) = \bigcap_{g \in I} V(g)\) since you vanish on the ideal if and only if you vanish on every point of the ideal.

(4/30/2017) Today I reviewed the foundations of the Algebraic Number Fields book I’m reading. One of the first things I realized is that although this what I learned today is great, I should really not try to bullshit it. Some days you just need to go back through things and make sure you have a solid foundation on them—and that’s what I’m doing with algebraic number fields right now. One of the things I learned was a sort of "determinant trick." Essentially, if you have a set of generators \(\{m_1, ..., m_n\}\) for an \(R\) module \(M\), and you need a relation in terms of those generators (in particular, in the book, this is used if \(pM = M\) and \(R\) is a local ring, or in the case where you have an element \(b\) in an \(R[b]\) module finitely generated as an \(R\) module which isn’t annihilated by any nonzero element and you want to determine the integral relation that your element solves), you can use this method. In the case of finding an integral polynomial some \(b\) solves, simply write each \(bm_i = \sum_j a_{ij}m_j\) and then consider the matrix \(Q = bI - [a_{ij}]\). The summation above implies that \(Q(m_1, ..., m_n)^T = 0\). Note—the word "basis" doesn’t come up here because this is a module. So I don’t think this means \(Q\) is the zero transformation. Letting \(B\) be the adjoint matrix of \(Q\). Then if \(d = det(Q)\), \(BQ = dI\). Thus \(d(m_1, ..., m_n) = 0\) so \(d = 0\) so \(b\) is a root of the polynomial \(det(xI - [a_{ij}])\). Oh you know what though? I don’t think this transformation needs to be \(R[b]\) linear, which explains why it need not be the zero transformation. Okay. Cool.
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(5/1/2017) Today I learned a proof of Stokes Theorem! Which is really pretty cool. Essentially, to prove Stokes Theorem for a compactly supported form, you can argue by linearity that your form is supported on a parametrizable subset, and then if the subset doesn’t intersect the boundary at all, since your form has compact support and since you can view your integral as a bunch of iterated integrals of partial derivatives, all of which eventually vanish, then by the fundamental theorem of calculus in one variable \( \int_X d\omega = 0 \), and since \( \omega \) is not supported anywhere on the boundary, \( \int_{\partial X} \omega = 0 \). You can use a similar argument to show that if the support is on the boundary, you can reduce your integral down to all but one important integral (one where the last \( dx_n \) doesn’t appear--on the boundary, this is zero) and then show that your forms are equal there. I’m actually being a little hand wavey here, but I don’t have the book with me so I’m going to have to try and get this totally down tomorrow.

I also learned a bit about the homology of a Stein filling \( W \) of the unit cotangent bundle of a surface \( \Sigma_g \), say, \( Y_g \), where \( g \geq 2 \). It turns out that the fundamental group of this Stein filling is the same of that surface, which I haven’t proved or seen the outline for yet. I also learned (but again, without proof) that \( \pi_1(Y_g) \) is generated by all the generators of \( \pi_1(\Sigma_g) \) and an extra element \( t \) which represents the fact it’s a ”circle bundle”, with relations that this \( t \) commutes with all the other generators. I have seen the essential reason for why the inclusion map \( i : Y_g \to W \), which turns out to be surjective, actually is surjective when restricted to the ”\( \pi_1(Y_g) \)” part of \( \pi_1(W) \). Essentially, we mod out by \( \pi_1(Y_g)/\Pi \to \pi_1(W)/i_*(\Pi) \) and then use covering space theory to find a \( k = [\pi_1(W) : i_*(\Pi)] \) fold cover, which is a Stein domain since there’s an analytic property you can classify Stein domains using compact sets and bounds, and then you can restrict your coverings to the boundary to argue that since the genus is less than 2, \( k = 1 \).

Finally, I learned that you can glue sheafs together provided that you have a nice cocycle condition being met. I actually haven’t fully written out the proof of gluability, but I imagine that the proof of gluability will follow since you can use the fact your isomorphisms are maps of sheafs to get a gluing trick to work for your large set. Identity follows as if two elements are the same up when they’re hit with ”restriction then isomorphism” then they’re the same when hit with ”restriction” by taking the inverse of your isomorphism.

(5/2/2017) Today I learned that you can take a collection of schemes which have subschemes that are isomorphic to each other and you can glue them together, essentially using a sheaf gluing construction, provided that the sheaf gluing construction ”cocycle condition” holds. You can use this to construct some nonaffine schemes, such as a line with two origins. This is constructed by taking the open subset \( U := D(t) \subset X := \text{Spec}(k[t]) \) and \( V := D(u) \subset Y = \text{Spec}(k[u]) \). Now, I learned a lot more today, in particular I reviewed a good chunk about Dedikind rings, but I’m tired and going to bed.

(5/3/2017) Today I learned most of the proof of the Riemann Mapping Theorem, which says that any simply connected domain that is not the entire complex plane and has a point is conformally equivalent to the disc. Which is so freaking crazy! But here’s why—if you avoid a point, you avoid an entire ray. If you avoid an entire ray, you can shift that ray to the origin, and then you can get an analytic logarithm defined there, and hence an analytic square root. Analytic square roots in particular avoid \( -D \) if they hit \( D \), so we have a function that avoids values in a disc, so we can translate that disc over and then invert it. This shows that there’s an analytic function mapping our domain into the disc. You can also rotate it to assume that the derivative at zero is a positive real number. Friday’s class, which I’m not going to, will also show that we can get a surjective map by asking the derivative be maximized.

(5/4/2017) Today I learned the other two theorems that I have to give a talk on in a few weeks.
They say that any exact filling (i.e. any symplectic manifold whose symplectic/non degenerate two form has differential zero) of a contact manifold that admits a Calabi-Yau cap, which is a strong concave filling which has torsion as its first Chern class. Also a strong concave filling means that there is a vector field pointing into your large manifold where the Lie derivative of the sympletic form along the vector field is a positive multiple of the symplectic field.

(5/7/2017) I reviewed a bunch of things about Dedikind rings and some alternative definitions that could be used for them. In particular, I learned something of note— if \( U \) is an ideal of a ring \( R \) (Dedikind or not), given a maximal ideal \( p \), it’s not necessarily the case that \( U_p \cap R = U \). To see this, use the example of \( R = \mathbb{Z}, U = 12\mathbb{Z}, \) and \( p = 2\mathbb{Z} \). In this case, 4 is in the left side but not the right. However, \( 4 = \frac{12}{3} \) in \( R_p \) and \( 4R_p = \frac{12}{3}R_p = 12R_p \), which at least resolved the issue I had while working a specific example through the notes.

(5/8/2017) I learned what a spectral sequence was today, which is, essentially, given an \( N \in \mathbb{Z} \), modules or abelian groups or whatever of the form \( E^r_{pq} \) with a differential map where \( r \geq N \) where you can determine the \( r + 1 \)st ”page” by looking at the homology of the \( r \)th page. One application of this is computing homology—essentially, if you have an easy to compute quotient space and a class in your homology of that quotient, you can check to see if it lifts to the homology of your (possibly more difficult to work with) space. If it does, you can also check if it lives forever if your space is a filtered colimit, say \( X = \text{colim}(\ldots \to X_i \to X_{i+1} \to \ldots) \). If it lifts and then later dies at the \((n+t)^{th}\), it is taken care of by the \( n^{th} \) page. Similarly, this procedure can give us some fake stuff in homology class, but eventually that is quotiented out too.

(5/9/2017) Today I hammered down a lot of the definitions I needed to get down for the Calibi-Yau caps. In particular, a Calibi-Yau cap of a contact manifold is a strong concave filling whose first chern class is torsion. A strong filling means that if you take the differential of the contact form... well actually I learned I don’t know the definition of a strong filling certainly well. But I also learned about the AH Spectral Sequence, and a related idea which talks about the convergence of a spectral sequence. This means that essentially, after taking enough differentials, your homology doesn’t change.

(5/10/2017) Today I was destroyed by a Differential Topology exam, at least mostly. But I learned some things about it, including that you can argue that \( \mathbb{R}P^2 \times \mathbb{R}P^3 \) is not orientable, for if it were, the pullback of the inclusion map would induce some orientation on \( \mathbb{R}P^2 \). Also I learned about the Serre spectral sequence, and that you can use theorems about where things converge to not only to compute convergence, but you can work backwards—knowing that your spectral sequence converges to something tells you that elements not in that convergence have to be killed eventually by a differential from somewhere else. In quadrant one spectral sequences, you can argue that the ”killing” has to happen reasonably soon, since if you’re on the \((0,t)\) spot you only have \( t - 1 \) chances to die before you’re mapped into by zero.

(5/11/2017) I learned a technicality that I had looked over when I looked at the factoring of fractional ideals \( \mathfrak{M} \) of a Dedikind domain \( R \). In particular, choosing some nonzero \( t \in \mathfrak{M}^{-1} \cap R \), then \( t\mathfrak{M} \subset R \), so it can be factored into prime ideals, and so can \( Rt \). But then what I hadn’t noticed before was that \( \mathfrak{M}t = \mathfrak{M}Rt = \mathfrak{M}Rt \), so ”morally” \( \mathfrak{M} = \frac{\mathfrak{M}t}{t\mathfrak{M}} \), at least in the factoring sense.

(5/12/2017) Today I learned the idea of a right derived functor. Essentially, you can take an exact sequence in an abelian category \( 0 \to A \to B \to C \to 0 \) and a left exact functor (so the sequence \( 0 \to FA \to FB \to FC \) is exact) and then there are ”right derived functors” \( R^nF \) for all \( n > 0 \) such that the following sequence is exact \( 0 \to FA \to FB \to FC \to R^1FA \to R^1FB \to R^1FC \to R^2FA \to \ldots \). You then can define group cohomology for a given group \( G \) by noting the functor \((-)^G : G - Mod \to Ab \) is a left exact functor and then define the \( n^{th} \) group cohomology to be the \( n^{th} \) right derived functor with coefficients in a \( G \) module \( M \) to be \( R^n(M) \).
I also learned about the Lyndon-Hochschild-Serre spectral sequence, which relates the cohomology of a group \( G \) to the cohomology of a normal subgroup \( N < G \) and the quotient \( G/N \). In particular, the LHS spectral sequence says for any fixed \( G \) module \( A \) there is a spectral sequence \( H^p(G/N, H^q(N, A)) \Rightarrow H^{p+q}(G, A) \).

(5/13/2017) Today I learned stuff in the appendix of Algebraic Number Fields! In particular, I learned about the Normal Basis Theorem of Galois extensions (and learned how to prove the cyclic case), and Hilbert’s Theorem 90. Hilbert’s Theorem 90 says that if you have an element \( \alpha \in K \) where \( K/F \) is Galois extension of degree \( n \) and the Galois group \( G := Gal(K/F) \) is cyclic, generated by say, \( \sigma \), and \( N_{K/F}(\alpha) = 1 \), then there exists an element \( \psi \in K \) such that \( \alpha = \psi/\sigma(\psi) \).

You show this by considering elements of the form \( \lambda_i = \alpha \sigma(\alpha) \ldots \sigma^{i-1}(\alpha) \) for \( i \in \{1, 2, ..., n\} \).

Actually, writing this out, I discovered an inconsistency in my understanding about how everything connected, that I hope to rectify tomorrow. But alas, today, I’ll just say that there’s something called the Normal Basis Theorem which says in the above setup we can find a special element \( \alpha \) such that \( \alpha, \sigma \alpha, ..., \sigma^{n-1} \alpha \) is a basis of \( K/F \).

(5/14/2017) Today I learned about when two symmetric bilinear forms over \( \mathbb{Z} \) are equivalent. The definition of equivalent means that you can find an isomorphism between the two vector spaces such that the pull back of one form is the other form. On the other hand, there’s a theorem that says if you put the form into a matrix, then the matrices are isomorphic if and only if the rank, signature (i.e. the largest dimension you can make a subspace be positive definite - the largest dimension you can make a subspace be negative definite) and sign (meaning “even” if every diagonal entry is even, and odd otherwise) are equal.

(5/15/2017) Today I learned that if you have an oriented four manifold, then you have a fundamental class \([X] \in H^4(X; \mathbb{Z})\) (where hereafter we use integral homology) then you obtain a bilinear form, the intersection form, defined on \( H^2(X) \times H^2(X) \) sending two elements to their cup product (and then to \( \mathbb{Z} \) canonically). You can use this intersection form to show the thing I’m proving, which is that if you have any exact filling of the unit cotangent bundle of a surface of genus larger than one, its homology is that of the disc bundle. It’s also related to an invariant called the signature of a manifold, and in fact it relates to the two matrices \( E_8 \) and \( H \), which is the transposition matrix.

(5/16/2017) Today I learned a fact which I cannot prove yet. Let \( N \) be an exact filling of \( Y \), the unit cotangent bundle of a surface of genus \( g > 1 \). Then you can show that the homology \( H_2(N) = \langle S \rangle \cong \mathbb{Z} \) and that the map to \( H(N, Y) \) is simply multiplication by \( [S]^2 \cong k^2(2g - 2) \).

This in particular implies that all the torsion is killed off in the long exact sequence of a pair with \( H_1(Y) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/(2g - 2) \mathbb{Z} \), so \( H_1(N) \cong \mathbb{Z}^{2g} \), the same as the disc cotangent bundle. I also learned that I don’t truly understand bilinear forms that don’t have the notion of length attached to them. We’ll hopefully work through that tomorrow.

(5/17/2017) I figured it out! It turns out that the fact that if you have any function \( f : V \rightarrow F \) where \( V \) is an \( F \) vector space and \( \langle \cdot, \cdot \rangle \), a nondegenerate bilinear form, then \( f = (v, -) \) for some \( v \in V \). This is simply because the vector space of all linear maps \( V \rightarrow F \) is an \( n := dim V \) dimensional vector space. Then the map sending \( w \rightarrow \langle w, -\rangle \) is an injection (which is where nondegeneracy comes in). I also learned the proof of the fact that if you take the integral closure of a Dedikind ring in a purely inseparable finite extension, the resulting ring is also Dedikind. This is essentially because in your finite extension, there is a power \( p^q \) where you can raise all of your elements in the large field to to get in the small field. You can then argue that you get a one to one correspondence between prime ideals of your integral closure and prime ideals of your old ring by intersecting the prime ideals with the old ring. This quickly gives you the fact that each element is contained in only finitely many prime ideals, and slowly gives you the fact that if you localize any maximal ideal you obtain a DVR.
(5/18/2017) Today I learned that if you are given Dedikind rings $R \subset R'$ then there is a natural way to define the ramification index of a nonzero prime ideal $\beta \in \text{Spec}(R')$, because $\beta \cap R$ is a nonzero prime ideal of $R$. The "prime" part of this proof is trivial, and the "nonzero part" is pretty hard, especially because it’s not true in general. An obstruction is that $[L : K] < \infty$, where $K \subset L$ are the respective fraction fields (the rings $\mathbb{Z} \subset \mathbb{Z}[x]$ show this since $(x) \cap \mathbb{Z} = 0$. On the other hand, if $\alpha \in \beta$ is nonzero, then it is the root of some equation in $K[x]$, say $a_0 + a_1 x + \ldots + x^n$. Clearing out with a common denominator, we see that $\rho \alpha$ is the root of $r^n a_0 + r^{n-1} a_1 (r x) + \ldots + (r x)^n$ so $\rho \alpha$ is a nonzero element of $\beta \cap R$, where $\rho \alpha \in R$ since $R$ is integrally closed in its field of fractions. This isn’t right, but I’m tired. I’ll sort this out tomorrow. I also learned the basic idea that a map of affine schemes is determined by the map on $D(1)$. This is because a map is determined by how it operates on stalks, but on stalks there is only one prime ideal to map from and too.

(5/20/2017) Today I finally figured out my error above. Given an $\alpha \in \beta$, it’s not necessarily true that a scalar multiple of $\alpha$ is in $R$, however, if you look at the polynomial expression above, $r \alpha = -a_1 \alpha - \ldots - (\alpha^n) \in R \cap \beta$. Also I just learned what the mapping torus is–given a map $\phi : X \to X$, you can take the cylinder $X \times I$ and glue $(0, x)$ to $(1, f(x))$. I also found out a picture of what handle sliding is–which is essentially when two rainbows are next to each other, and then the leftmost rainbow (say) decides that it wants its right side to move along the other rainbow and overtake it. I also learned the other move that doesn’t change diffeomorphism, which is if you put a ball inside of a rainbow. Topology is weird.

(5/21/2017) Today I learned about projective space in algebraic geometry. Restricting discussion to the first projective space, I learned that projective space in one dimension is just taking $\text{Spec}[u]$ and $\text{Spec}[v]$ and gluing $\text{D}(t)$ to $\text{D}(u)$ via the isomorphism sending $t \to \frac{1}{u}$. This makes a picture that reminds me of the sphere where taking $\frac{1}{2}$ reflects the sphere about the center disk. I also learned what an almost complex structure was (an operator on the tangent space of a manifold which squares to negative 1–i.e. it looks like $i$.)

(5/22/2017) Today I learned about how homogeneous polynomials can determine a subscheme of projective space. Essentially the scaling can show that if you have a polynomial, like $x^2 + y^2 - z^2 = 0$, you can divide by $z$ (say) and get the equation in one of the subschemes of projective space, say $(\frac{x}{z})^2 + (\frac{y}{z})^2 - 1 = 0$ and it turns out that the gluing maps make your choice not a real choice. Also, I learned facts about graded rings, including facts about how ideals of a graded ring are closed under addition, multiplication, intersection, radicalization, and if you’re ”prime” with homogeneous elements then you’re a prime ideal. This last one comes from essentially the thing of the example if your rings are graded by $\mathbb{Z}_{\ge 0}$ and $\alpha = (a_0, \ldots, a_m, 0, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_n, 0, \ldots)$ such that $\alpha \beta \in I$ then you let $k, l$ be the maximal such that $\alpha_k, \beta_l \notin I$. Then everything smaller is in $I$ so check the $k + l$ coordinate and split it into three parts to see that $\alpha_k \beta_l \in I$.

(5/23/2017) Today I learned the more general notion of the Projective space of a graded ring. You’re supposed to think of the graded ring $k[x_1, \ldots, x_n]$ where grading is determined by the degree of each term in a polynomial expansion. Then when you want a ”point,” instead of just a prime ideal, what you want is a homogeneous prime ideal, where homogeneous means that the projection to any one particular grade is in the ideal if an element is in there (that’s talked about above). The reason you want these is the same reason you want homogeneous equations–they are the equations that actually cut out solutions in projective space.

(5/24/2017) One thing I learned today was a very good interpretation of the fact that, in the spectrum of a ring, $\text{V}(I(L)) = \overline{L}$ where $L \subset \text{Spec}(A)$ should be thought of as a list of prime ideals. Essentially what the heart of this statement is that if you have a point somewhere and it’s not lying above any of the points on your list $L$ (or on the list itself), then you can construct a function that makes you vanish on $L$ but doesn’t vanish at your point. At least if you have a finite list–although I’m pretty sure this adjusts for infinite lists. Like what if $L = \{(x - n) : n \in \mathbb{Z}\}$? I’m not sure.
Let’s see what this fact actually means in my notes.

(5/25/2017) Today I figured out what it truly means to be alone. Just kidding. I actually figured out why the topologies of regular Zariski topology on Spec$(S_f)_0$ (where $S$ is a $\mathbb{Z} \geq 0$ graded ring and $f \in S$ is a homogeneous element) is just the restriction of the topology of Spec$(S)$ to $D(f)$. My hangup was that if we have some kind of ideal $V(I) \subset$ Spec$(S_f)_0$, we need to write the associated set of prime ideals in Spec$(S)$ as $V(J) \cap D(f)$, where $J$ is a homogeneous ideal. Originally, I thought to set $J$ as a graded ideal, via $\bigoplus_i \{a \in S: \frac{1}{\deg f} a \in I\}$, but this isn’t necessarily an ideal. However, you can take $J$ to be the elements generated by these homogeneous elements and you get what you need.

(5/27/2017) Today I learned a lemma in number theory which says, among other things, that the relative degree is well defined for prime ideals a Dedekind ring. This follows for two reasons—one is that if $\beta < \beta'$ is a nonzero prime ideal, then $p = R \cap \beta$ is also a nonzero prime ideal (as I’ve shown above with the ”degree zero of a polynomial trick”) and if $S := R - p$, then $R' / \beta \cong R'_S / \beta R_S$. This is because since we’ve already moduled out by everything $p \subset \beta$, everything in $p$ already has an inverse. Therefore localizing by things that already have inverses doesn’t change anything. Then you can work with a DVR to prove that $[R' / \beta : R / p] \leq [\mathfrak{f}(R') : \mathfrak{f}(R)]$, where $\mathfrak{f}(A)$ denotes the fraction field of $A$.

(5/28/2017) Today I learned some stuff relating relative degrees and ramification indices to dimension. In particular, if we are given a prime ideal $p \subset R \subset R'$ where $R, R'$ are Dedekind rings where the quotient field of $R'$ is a finite extension of the quotient field of $R$ and $p = \beta_1^{e_1} \ldots \beta_n^{e_n}$, then it turns out that $\text{dim}(R'/pR') = \sum_{i=1}^n e_i f(\beta_i/R)$, where $f$ denotes the relative degree, i.e., $f(\beta_i/R) = [R'/\beta_i R' : R/p]$. You can take this a step further and argue that if you have that $\mathfrak{f}(R') / \mathfrak{f}(R) = \text{separable finite extension of the two Dedekind rings}$ then $\sum_{i=1}^n e_i f(\beta_i/R) = [\mathfrak{f}(R') : \mathfrak{f}(R)]$. The idea here is to use Chinese Remainder Theorem and then determine the dimension of each individual piece.

(5/29/2017) Today I learned an important theorem about ramification, which says that if $R$ is a Dedekind domain with fraction field $K$ and $L/K$ is a finite separable extension with $R'$ is the integral closure of $R$ in $L$, then the primes that ramify (i.e. factor with some prime having exponent larger than one or have a quotient field that isn’t separable over the small field) are precisely those contained in the discriminant ideal, that is, the ideal generated by the elements $\det(Tr_{L/K}(x_i x_j))$, where $\{x_i\}$ ranges over the bases of $L/K$ contained in $R'$.  

(5/30/2017) Today I learned about reduced schemes, that is, schemes $(X, \mathcal{O}_X)$ such that for all open $U \subset X, \mathcal{O}_X(U)$ has no nonzero nilpotent elements. I also learned that this can be checked on the level of stalks, since if you’re a nilpotent element on a stalk you can find an open set where this behavior occurs, and conversely obviously if you’re nilpotent on an open set, you’re nilpotent on the stalk, which is just the colimit of the restriction map diagram.

(5/31/2017) Today I that if $I, J$ are two ideals in a Dedekind ring, there is a notion of a greatest common divisor of the two ideals, which can be viewed as the product of the prime ideals which divide both. If $K$ is the common divisor, then $I + J = K$. This is because $\subset$ is obviously true, and then you can show $\frac{I}{K} + \frac{J}{K} = R$ by localizing by any prime ideal, since a prime ideal either isn’t a factor of the $I$ factor or either isn’t a factor of the $J$ ideal.

June 2017

(6/2/2017) Today I learned an important lemma which I missed in linear algebra. Well, not exactly, but basically I was thrown by the counterexample I will describe. The theorem says that if $A, B$ are finite dimensional $F$ vector spaces with a surjection $T : A \to B$, then $A \cong \ker(T) \oplus B$. This
can be proven by taking a basis of both \( \ker(T) \) and \( B \), noting that the number of elements adds up to the dimension of \( A \) and then showing they’re linearly independent by taking \( T \) of it to show that the \( B \) part coefficients are zero, and then noting that you picked a basis of the kernel, which is all that’s left after you show the \( B \) coefficients are zero. This threw me because there is a surjection \( \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \) given via multiplication by two but those aren’t vector spaces (since if \( F \) is a vector space for both \( F = \mathbb{Z}/2\mathbb{Z} \) but \( 1 + 4\mathbb{Z} + 1 + 4\mathbb{Z} \neq 0 \). You can use this to show that if \( \beta \in R^k \) is a prime ideal, then \( R/\beta \cong R/\beta \oplus \ldots \oplus \beta^{n-1}/\beta^n \).

(6/3/2017) Today I learned a good chunk of things, mostly at the Temple University Graduate Math Conference. I learned that there’s an intersection to algebraic geometry, arithmetic geometry, and dynamical systems, and that this connection is what essentially makes the Mandlebrot set. I also learned that in the same sense that covering space theory and Galois Theory are very connected, you can also translate the language of Galois theory/covering space theory to the language of algebraic geometry, which uses the word etale a lot, but with a little accent over the e which I don’t want to figure out how to do because I’m not connected to the internet right now. I also learned that a lot of arguments involving modules over principal ideal domains essentially boil down to "isolate the ideal that stuff in the last coordinate can be. Then it’s a principal ideal so pick a principal generator, and then write any element as the module element with last coordinate that principal generator direct sum something with last coordinate zero (or whatever you need).

(6/4/2017) Today I learned about the theory of groups, which is the set of first order sentences they satisfy. Obviously isomorphic groups satisfy the same theories, so you can use theories to distinguish some isomorphic groups. For example, the sentence "There exists \( x,y \) such that \( xy \neq yx \) distinguishes abelian groups from non-abelian ones. However, it turns out that the first order theory of any free group on \( n > 1 \) generators has the same first order theory as any free group on \( m > 1 \) generators. Thus they are not distinguishable from first order theory. Moreover, it turns out that any group whose first order theory is that of a free group is hyperbolic, which is pretty neat, even though I don’t know what it means for a group to be hyperbolic yet.

(6/5/2017) Today I learned about bilinear forms on free abelian groups. One of the things I learned was that if you have a subspace that is unimodular (that is, the matrix associated to the form has determinant \( \pm 1 \)) then you can write your space and the bilinear form as a direct sum of the subspace and the space orthogonal to it and the form restricted to those two things. This shows that if you happen to have any set of the correct number of elements in the free abelian group and their bilinear form associated to the set of elements has determinant \( \pm 1 \) then that set actually forms a basis for your space.

(6/6/2017) Today I learned that any scheme that is irreducible and is also reduced is an integral scheme. This is because a scheme being reducible means that any open subset is also irreducible (and in fact, these are equivalent notions), so you pick some open set \( U \) that you want to show \( \mathcal{O}(U) \) is integral. So pick \( f, g \in \mathcal{O}(U) \) and assume \( fg = 0 \). Then you can show that \( U = (V(f) \cap U) \cup (V(g) \cap U) \) (by some abuse of notation here—you’re writing the set of points where the function vanishes at the stalk as a vanishing set). These are closed, and therefore one of them must be the entire space. But then since you can restrict that function to any point and get zero, and your scheme is reduced (i.e. functions really are determined at their points) then that function must be zero in \( \mathcal{O}(U) \).

(6/7/2017) I finally freaking learned what Poincare Duality is actually saying! It’s not just saying there’s an isomorphism, it’s saying that if you have an oriented closed \( n \) manifold \( X \), what you’ve essentially done is chosen a fundamental class \( [X] \in H_n(X) \), i.e. a generator. Now Poincare duality says if I define the map \( D : H^k(X) \to H_{n-k}(X) \) via \( [\alpha] \to [\alpha] \cap [X] \) where \( \cap \) means that for every simplex, \( \alpha \) should eat the first \( k+1 \) vertices and treat it as a \( k \) simplex and spit out a number, and then multiply that number by the \( n-k \) simplex that the rightmost \( n-k+1 \) coordinates yield,
then the map $D$ gives an isomorphism. You can use this to show that in a closed four manifold the interaction form is unimodular.

(6/8/2017) Today I learned about Kummer’s Theorem, which is a theorem which helps factor a prime ideal $p$ in a Dedikind ring $R$ with fraction field $K$ with $L/K, S = R - p, R'$ the integral closure of $R$ in $L$ with $L/K$ a finite extension and in the special case where $R'_S = R_p[\theta]$ for some $\theta \in L$. Then if $f(x) \in K[x]$ is the minimal polynomial of $\theta$ over $K$, then the coefficients are in $R_p$ and so you can reduce them modulo $p$ and factor it. Then those factors of $f$ correspond with the same power and the relative degree of each prime ideal is the degree of the polynomial.

(6/9/2017) Today I learned that there’s certain properties of schemes, called affine local properties, which are properties where if they are true for an affine open set, they are true for any restriction to any distinguished open set, and if they are true for restrictions of some affine scheme $A_{f_i}$ such that the ideal generated by the $f_i$ is the entire ring, then the property holds for $Spec(A)$. This is true due to a lemma which says that you can realize any intersection of two affine schemes (in a larger scheme) as a union of open sets which are distinguished open sets in each scheme.

(6/11/2017) Today I learned that you can write any Notherian scheme as a finite union of irreducible components, and each connected component is merely just a union of some of the irreducible components. The first part comes from the fact that any Notherian topological space can be written as the finite union of closed irreducible sets, none of which contained in any other, and the irreducible components, then, are those sets. And the connected components part comes from the fact that if you have a connected component and you write that connected component as a union of irreducible components then those sets are actually irreducible components of the whole space. At least I’m pretty sure.

(6/12/2017) Today I learned about Kahler manifolds and Hyperkahler manifolds and what they’re like. Essentially Kahler manifolds are symplectic, Riemannian manifolds with a complex structure on them with these complex structures that go together and play together nicely. I also learned that a nondegenerate two form $\omega$ just means that $\omega$ as a map from $V \rightarrow V^*$ (where if you take in one vector and have one vector left to take in) is an isomorphism. And a hyperkahler manifold is just a manifold with three almost complex structures which make it a Kahler manifold and those complex structures satisfy some quaternion like relations!

(6/13/2017) Today I learned a bit more about forms (in particular, simply connected manifolds are isomorphic if and only if their intersection forms are the same). Moreover, I translated a hard problem (at least hard for me!) into a problem I made good headway on, where I’m trying to prove that if you can write the spectrum of a ring... well. I put a lot of work into it dammit! It’s hard to say what you learned in a problem.

(6/14/2017) Today I learned that my problem yesterday was just reduced to the fact that any integral domain has exactly one closed point. I also learned about connected sum, which is taking two disks and taking an orientation reversing diffeomorphism between them and identifying them. Also the intersection form of the connect sum of two manifolds is the direct sum of the two forms.

(6/15/2017) Today I learned that if you’re a locally finite type $k$-scheme, where $k$ is a field, then a point being a closed point is the same thing as the quotient field being a finite residue field. One direction of this proof uses Hilbert’s Nullstellensatz, and the other half of the proof uses the fact that if you have a function in a larger prime ideal and it has $k$ linear combination of the powers of the function are zero in the quotient ring, then whatever coefficient is on the $f^0$ power must be in that larger prime ideal, since $f$ and the smaller prime ideal are in the larger prime ideal. But then you can do an induction type argument to argue that no linear combination of the $f^i$ is zero. Which is pretty cool.

(6/16/2017) Today I learned that, given a UFD $A$ where 2 is invertible, all of the obstructions to adjoining a square root in your ring being the integral closure of the ring in its field of fractions.
Let’s say we adjoin a root of \( D \) with \( D^2 \in A \) but not \( D \). Then if \( D \) doesn’t have any repeated prime factors, you can take a monic polynomial with coefficients in \( B := A[D] \) and multiply it by the ”conjugate” (i.e. conjugate all the coefficients) polynomial and you get a polynomial with coefficients in \( A \)! Then you can argue that there has to be a degree exactly 2 polynomial which your element solves, and then use the fact that you don’t have any prime divisors to conclude that the denominator of the \( A \) coefficient on \( D \) actually has no prime divisors–i.e. it is a unit.

(6/18/2017) Today I learned something about the interacting composite extensions of rings and how they behave with their corresponding extensions of fields. Like in the field extension case, we have a chain of fields \( K \subset E, L \subset F \) with \( E \cap L = F \) (for simplicity). Then if we let \( R_K \) be a Dedekind ring whose quotient field is \( K \), and let \( R_W = \text{int}(R_K, W) \) for each field, then it turns out that if either of \( E \) or \( L \) is Galois, then if you take the discriminant \( \Delta(R_E/R_K) \) and multiply it by the ring \( R_F \) (the largest ring), that’s a subset of the composite ring \( R_E R_L \). This basically says that you only can divide by so much and get so far off from \( R_E R_L \) before you can’t be in the integral closure of \( R_K \) over \( F \) anymore. Moreover, if it just so happens that your two discriminants are relatively prime (i.e. the two ideals sum to the entire ring \( R_K \)) then it turns out you can show equality here!

I also learned about associated points and associated prime ideals, and in particular, that even though I don’t know the definition yet, I know that they are the generic points of the irreducible components of the closed subsets that can be the support of some function (i.e. element) of the ring, and that there are only finitely many of these for Noetherian rings. Interesting.

(6/19/2017) Today I learned the proof that being Noetherian is an affine local property and wrote it up. It essentially comes from the fact that if you have a finite collection of \( f_i \in A \) such that \( (f_i) = A \) and your \( A \) isn’t a Noetherian ring, then there exists a strictly increasing infinite chain of ideals. Now you can show that at each point in that chain, one of the ”ideal of the numerators” in the \( A_{f_i} \) is also not equal. Therefore, an infinite amount of these must occur somewhere! I also learned about Chern classes and characteristic classes–characteristic classes are particular classes in cohomology which are natural with respect to pullback.

(6/20/2017) Today I learned what a Stiefel Whitney class is (although I can’t necessarily spell it)–it’s a theorem that says the colimit over all the orthogonal groups as vector bundles over a given manifold has cohomology some polynomial ring over the field with two elements with one coefficient in each degree. You can take the \( i \)th Stiefel Whitney class to be the term \( x_i \). You can also use axioms to compute it (like the Whitney sum or the fact that it’s a characteristic class) and it turns out that a manifold is orientable (spin) if and only if its first (first and second) SW class vanishes!

(6/21/2017) Today I learned with a large chunk of work that if \( A \) is a ring and \( p \in \text{Spec} A \) such that \( A_p \) has nonzero nilpotent, then \( \mathfrak{p} \in \mathfrak{p} \) implies that \( A_\mathfrak{p} \) also has a nonzero nilpotent element. To see this, let \( f \) be the nonzero element, and let \( m \in \mathbb{N} \) and \( g \not\in \mathfrak{p} \) such that \( gf^m = 0 \) in \( A \). Then you can show that in \( A_\mathfrak{p}, gf \neq 0 \) but clearly \( (gf)^m = 0 \). This proof seems trivial, but I took a damn good 3 hours to overwork it. But I understand how annihilators relate to these.

(6/22/2017) Today I learned that you can define associated points for modules over a Noetherian ring too. And those points are those prime ideals in the ring whose closure is an irreducible component of the support of some element. Moreover, for rings, localizing by a set \( S \) simply deletes the associated points that intersect with \( S \)–otherwise the associated points are identical! In particular, associated points do not change from a ring to its stalk, which in turn implies that you can define the associated point on any scheme by simply defining an associated point to be a point that is an associated point for some affine open set! And this equivalently, then, means for all open sets containing it.

(6/23/2017) Today I learned that the algebraic integers in any cyclotomic extension \( \mathbb{Q}[\theta]/\mathbb{Q} \) are
simply the elements of $\mathbb{Z}[\theta]$. You can first show this for cyclotomic extensions that are merely just powers of prime ideals (which is sort of the building blocks of a lot of things in number theory) by essentially computing things explicitly. You can show that $\Delta(1, \theta, \theta^2, \ldots)$ is a (possibly very large) power of $\pm p$ and then use that to show that you can’t have any “strange” algebraic integers in prime power cyclotomic extensions. And then you can induct on the number of prime factors, using linear disjointness of fields of two cyclotomic extensions that don’t divide one another.

(6/24/2017) Today I learned that for every irreducible closed subset $K$ in some scheme, there exists a unique point in that subset such that the closure of that point is your entire subset. This follows because it is true for affine schemes, and then for an arbitrary point in your closed subset you can pick an affine closed subset containing that point. The resulting (necessarily) closed subset remaining after intersection is still irreducible, so in this affine scheme land you have a closed point! It turns out that if you could have gotten a different point this way (on ANY scheme), then they can’t intersect at all, which you can use to show that your set is not really irreducible (since you can write it as the union of the closure of one point union the of the closure of each of the other points). Then you can show that suspicious possibly infinite union of closures of some points is still closed because none of the closures of two distinct points intersect so restricted to any of the affine schemes you picked earlier and $K^C$ that suspicious set is still closed. Then I proved a small lemma which says it suffices to check closedness on an affine open cover.

(6/25/2017) Today I learned that if you have a composite of field extensions that are linearly disjoint and take the integral closures in all of them, it might not be that if you take the integral closure of the top field, then it’s just the product of the two rings. There might be some more stuff in there, but what you do know is that if you multiply anything in that stuff with anything in the discriminant ideal of one of the intermediate rings, then you get in the product of the two rings. (*assuming one of the extensions is Galois) This helps if two ideals are coprime, for example, because then equality does hold. It essentially holds by a dual basis argument and the fact that $T\!r(y_{i \bar{j}})^{-1} = T\!r(y'_{i \bar{j}})$, letting $T\!r(y_{i \bar{j}}) = \delta_{ij}$

(6/28/2017) Today I learned that you can obtain associated prime ideals of a ring/module through the composition series of that module. Essentially what’s happening here is that in a composition series (which you can argue exists in any finitely generated module over any Noetherian ring) if you’re an associated prime $p$, then there exists an $m$ in the module where $p$ is precisely the annihilator of $m$. Then you ask, ”where is the first time this $m$ appears in the series?” Either that prime you mod out by the last one by to get is $p$, or you can show that $p$ is also the annihilator of some $fm$ that appeared earlier in the series. Then you can use some kind of induction argument to show that you eventually make that prime appear!

(6/29/2017) Today I closed the book on associated points (well, at least mostly, I’m writing this a little early today). Essentially, here’s the story of associated points. If $M$ is a finitely generated module of a Noetherian ring $A$ and $m \in M$, then $Supp(m)$ is the collection of prime ideals (that I like to think of as security guards or prisons) that hold back everything that can kill $m$, i.e. $Supp(m) = \{ q : ann(m) \subset q \}$. This is because in $M_q$, you invert everything else, so if you inverted something that could have killed $m$ you did. Associated primes, then, are the prime ideals that can do this with the least amount of work. That is, they are the prime ideals that are exactly the annihilators of some elements. It turns out that if you have any element annihilated, then you only need one good associated point to guard you–if the annihilator ideal of your element isn’t prime, then there’s a ”concoction” that can kill you while the individual items remain harmless (like ammonia and bleach). All you need is to protect from one of them–i.e. you only need one of the associated primes. This is an explanation for why $Supp(m)$ is the union of the closures of the associated points $p$ where $m \neq 0$ in $M_p$.

I also finally sort of get handlebodies. Essentially they’re the same thing as doing cells as in
algebraic topology, but to make them manifolds, we have to add extra stuff as we attach to keep dimensions consistent.

(6/30/2017) Today I went through a lot of results involving modules over principal ideal domains. Essentially, all of these results boil down to the fact that given any submodule of a free module, it’s free, and moreover, you can find a basis of the large module such that the first \( k \) terms, when multiplied by some ring elements \( a_1, \ldots, a_k \), give a basis for your submodule, such that \( a_1 | a_2 | \cdots | a_k \).

Immediately this tells you that you can write your module isomorphic to a free module plus torsion \( \oplus_i R/(a_i) \) and then you can argue via the Chinese Remainder Theorem that you can also write it as the direct sum of prime power modules. You can use the prime power modules to show that the invariant factor and the prime decompositions are unique, and then you can use that for the specific principal ideal domain \( F[x] \) (where \( F \) is a field) to argue that there is a canonical form for any linear transformation, and, with more work, you can argue that a matrix satisfies its own characteristic polynomial.

July 2017

(7/1/2017) Today I learned that you can separate any finite field extension \( E/F \) into a separable extension \( K/F \) where every element in \( K \) is separable over \( F \), and a purely inseparable extension \( E/K \) where for each element \( x \in E \), there exists a power of \( p := char(F) \) such that \( x \) raised to that power of \( p \) is in \( K \). This result is mostly founded on the fact that the set of elements in a field extension separable over a base field is actually a field, and that relates to how many embeddings of a field you could, potentially, have. This means that an extension of degree \( n \) is separable if and only if for every embedding of the ground field into some field \( L \), there is an extension field \( L' \) such that this embedding of the ground field can be extended \( n \) different ways. And you notice in this proof that all you needed was the separability of some generating set, so you see that “field is separable” iff ”\( n \) extensions” iff ”set of generators is separable”. This says that you can get a real thing called the separable closure, and then the inseparable part afterward is just writing the polynomial as a power of \( x^p \) (i.e. so its derivative is no longer zero) and then realizing whatever is left has to be separable, i.e. in the ground field.

(7/2/2017) Today I learned two cool number theory things. One of the things was how to factor a prime ideal \( p \mathbb{Z} \) in the larger ring \( \mathbb{Z}[\theta] \), where \( \theta \) is a primitive \( m \)-th root of unity for some \( m \). Essentially, you can first argue using Kummer’s Theorem that if you have any prime \( q \) that doesn’t divide \( m \), then if \( \phi_m \) is the minimal polynomial of \( \theta \), then after reducing the coefficients of \( \phi_m \) modulo \( q \), you see that each irreducible factor corresponds to having a primitive root of unity in some field extension of \( \mathbb{F}_q \). So, letting \( r \) be the smallest positive integer such that the field of order \( q^r \) has a primitive \( m \)-th root of unity, that’s the splitting field of the polynomial \( \phi_m \) reduced modulo \( q \). Then you can use Kummer’s Theorem to write out the structure of the prime ideals (and moreover, if you traced back, you could compute them explicitly). Moreover, given a prime that does divide \( m \), you can use the fact that (say \( p^\ell | m \)) \( p \mathbb{Z} \) totally rammifies in \( \mathbb{Z}[\alpha] \) where \( \alpha \) is a primitive \( p^\ell \)-th root of unity. Then you can use the above to sort of “composite extension” your prime ideal, and then use the nifty \( efg = ” \)degree of field extensions” to show that this technique works (I’m being hand-waivey, but my full proof is in my notes and if anyone reads this and is interested in discussing, feel free to email me.

(7/3/2017) Today I learned the proof of quadratic reciprocity! Essentially if \( p, q \) are distinct odd primes then \( p \) being a square modulo \( q \) is essentially translatable to the factorization of \( p \mathbb{Z}[\psi] \) where \( \psi \) is a primitive \( q \)-th root of unity. Then you can use Kummer’s Theorem to turn this statement back on itself, involving reducing the polynomial \( x^2 \pm q \) and whether or not it has roots. It’s a
pretty interesting proof!

(7/5/2017) Today I learned the proof of quadratic reciprocity if one of the primes is 2. Essentially the proof runs pretty similarly, except instead of splitting the polynomial \( x^2 - q \) for some odd prime \( q \) you’re splitting the polynomial \( x^2 - x + \frac{1 + p}{2} \). I also learned that you can construct some kind of natural map on affine schemes given a ring map \( B \rightarrow A \). The interpretation of this is exciting to me. The main insight I gained today was to say this: Given a ring map \( \psi : B \rightarrow A \), thinking of elements of the ring as functions instead of just elements, and the functions are on the points \( \text{Spec}(\cdot) \). Then given a point \( p \in \text{Spec}(A) \), you can ask, what \( B \) functions vanish at \( p \) by defining \( f \in B \) ”vanishing” to mean that \( \psi(f) \) vanishes. Then as two functions don’t vanish if and only if their product doesn’t vanish, you get a prime ideal where \( p \) naturally goes! You can extend this idea to argue that you can make this a map of ringed spaces where the pullback on a base is exactly what you might think it is—with the insight above being the key to showing that it’s easy to define and well defined and commutes with restriction, etc.

(7/6/2017) Today I learned about Minkowski’s Theorem, which says that given any bounded convex (meaning closed under midpoint) set \( X \subset \mathbb{R}^n \) such that \( -X = X \) (called centrally symmetric) and a full lattice \( \mathcal{L} \) such that \( \text{vol}(X) \geq 2^n \text{vol}(\mathcal{L}) \), where \( \text{vol}(\mathcal{L}) \) denotes the volume of a fundamental region, then \( X \) contains a nonzero point of \( \mathcal{L} \).

Essentially, the reason this works is that if you’re given a set \( T \) whose \( \mathcal{L} \) translates are invariant, you can integrate over all the translates of \( \mathcal{L} \) (which you can show is actually a finite sum) to show that the volume of \( T \) must be less than the fundamental region (again, by just translating each point into the fundamental region). Then you can apply this to the set \( \frac{1}{2}X \) showing that there’s a point \( \frac{x}{2} + \lambda_1 = \frac{x}{2} + \lambda_2 \) with \( \lambda_1 \neq \lambda_2 \) and you can show that those conditions imply \( \lambda_2 - \lambda_1 = \frac{x}{2} \in X \).

(7/7/2017) Today I finished going through the proof of why a morphism of affine schemes is determined by a ring map! Essentially, this observation comes from the fact that any map of locally ringed spaces on the level of structure sheaf sends all the functions (and only the functions) that vanish at some prechosen point in the preimage of a prespecified point to the functions that vanish at the prechosen point. But we can identify points with all the functions that vanish on them, so you can use this to argue that the map of global sections determines what the map of topological spaces must be. Moreover, you can extend this to show that the category of Rings is equivalent to the category of affine schemes with arrows reversed!

(7/8/2017) Today I worked hard in two problems in algebraic geometry. One was explicitly showing that the projection map from affine \( k \)–space of dimension \( n + 1 \) projecting onto projective \( k \)–space of dimension \( n \) actually is a morphism of schemes. And it’s pretty hard, but I’m happy about the fact that I’m getting a lot out of this. Most of what I’ve learned is in the other problem, which I made more headway on. Essentially, there is a ”natural bijection” (which, to be honest, I didn’t much work through the natural part yet) between maps from a scheme \( (X, \mathcal{O}_X) \) to an affine scheme \( \text{Spec}(A) \) and maps from \( A \rightarrow \mathcal{O}_X(X) \). This is because it can be shown on the level of affine schemes and then glued!

(7/9/2017) Today I solved the above two problems! The first one, saying that the projection map from affine \( k \)–space of \( n + 1 \) dimension onto \( n \) dimensional projective \( k \)–space is actually a map of schemes, can be done in affine coordinates and then the reasons that the gluing maps agree come from a nice sexy commutative diagram I drew and took a picture of. The other problem (see yesterday) essentially did just follow as given proper gluing instructions, there really only is one map satisfying those gluing conditions. But what’s nice about this fact is that we have a ring morphism if \( A = \mathcal{O}_X(X) \), the identity! So you get a canonical map. Here are some more canonical maps I learned about today–I learned that \( \text{Spec}(\mathbb{Z}) \) is the final object in Sch (it’s a corollary of the last theorem) and that you can create a canonical map from the spec of a stalk at a point to the
whole scheme.

(7/10/2017) Today I learned that no compact complex manifold can be embedded into \( \mathbb{C}^n \). This is because if such an embedding occurred, then you could take coordinate functions \( \mathbb{C}^n \to \mathbb{C} \) and restrict them to the manifold. But these are holomorphic functions that attain maxima, since the forward image of compact sets is compact. I also learned what a linking number is, which is defined as a way to the intersection of two knots in an “oriented intersection number” kind of form.

(7/12/2017) Today I learned that the tautological bundle is the \(-1\)st indexed bundle of \( \mathbb{C}P^n \), taking its dual you get the first, and then tensoring you can get \( \mathbb{Z} \) distinct bundles of \( \mathbb{C}P^n \).

(7/14/2017) Today I learned that the class group of the integral closure of any ring over \( \mathbb{Z} \) into some finite extension is a finite group! The reason for this is that you can use lattice theory to argue that given any such ring \( R \), there exists a certain constant \( M \) such that for any ideal \( \mathfrak{U} \subset R \), there exists a nonzero \( a \in \mathfrak{U} \) such that \( N(a) < M\mathcal{N}(\mathfrak{U}) \). This helps a ton, because you can use this to show that for every fractional ideal in a ring, you can use this fact to find a multiple of that ideal that is in the ring \( R \) and has norm less than \( M \), which essentially comes from the fact that \( \frac{N(a)}{\mathcal{N}(\mathfrak{U})} < M \) and norms play nice with inverses. Then since the norms of ideals are determined by their product of primes, there’s only a finite number of primes there can be in your product for the representative with relatively small norm. Thus your class group is finite.

(7/16/2017) One thing I learned today was the specific constant that proved the above thing and one application of looking at the specific constant. It turns out that you use the specific bound on the norms to show that the positive integer generating the discriminant ideal must be larger than one, which in particular implies that some prime ideal ramifies in any nontrivial integral closure of any finite field extension over \( \mathbb{Q} \).

(7/17/2017) Today I learned some things about morphisms of projective schemes, including the fact that morphisms of graded rings induce a map on most, but not necessarily all, of the projective space of the two rings. The fact that it’s most and not all can be seen with the inclusion map \( \mathbb{C}[x, y] \to \mathbb{C}[x, y, z] \). The question is, where do we send the point \([0,0,1] \in \mathbb{P}^2 \)? Well, the problem here is that every function in \((x, y)\) vanishes at the point \([0,0,1] \) so the only thing \([0,0,1] \) could be mapped to as a point map is something where everything vanishes. But we specifically design projective space so that every point has a function that doesn’t vanish at that point!

(7/18/2017) Today I learned at least two things. One of them was the proof that you can basically think of any graded ring \( S_* \) as generated in degree one, provided that you only care about the projective space formed by that graded ring. This is essentially because if you have \( n \) generators \( x_i \) of degree \( d_i \) and define \( N := nd_1\ldots d_n \) then you can show any monomial of degree \( dN \) is in the \( S_0 \) algebra generated by \( S_N \) by an inductive argument, using a weighted average point to show you can factor out some monomial of degree \( d_1\ldots d_n n \) times. I also learned about the geometric intersection number, which is the minimum number of times two curves have to intersect, and that if you apply a Dehn twist among one of the curves \( k \) times, then your geometric intersection number is \( k \) times the old geometric intersection number squared. \( k > 0 \).

(7/19/2017) Today I learned that you can use ideas of algebraic geometry to extend something else I learned today—a formula to obtain all pythagorean triples. Since pythagorean triples are essentially rational solutions to \( x^2 + y^2 = 1 \), we can pick a ”start point”, say \((1,0)\) and then any other point has a slope associated with it (as in the stereographic projection, almost). This map is invertible so you can show that all pythagorean triples that aren’t \((1,0)\) are basically on the affine line \( \mathbb{A}^1_{\mathbb{Q}} \). But then you sort of can extend this rational function \( \text{Spec} \mathbb{Q}[x, y]/(x^2 + y^2 - 1) \to \mathbb{A}^1_{\mathbb{Q}} \) by including it into \( \mathbb{P}^1_{\mathbb{Q}} \). This includes the point we left out earlier, and hints at a theorem I will learn probably next year.

(7/20/2017) Today I learned the proof of Dirichlet’s Unit Theorem, which says that if the alge-
braic closure of some field $K/Q$ has $r$ distinct real embeddings and $2s$ distinct complex embeddings, then the group of units of the ring of algebraic integers in $K$ is a cyclic group times a free group of order $r + s - 1$, which corresponds to the fact that the group of units form a full lattice after taking an appropriate logarithm transformation, which you can show by arguing that you can pick units as generators with large "diagonal" entries when the log transformation is applied and very small, but negative, other entries.

(7/21/2017) Today I used the above Dirichlet’s Unit theorem to show that there’s a very easy method to determine the fundamental unit—the unique positive generator of the group of positive units larger than one—of the ring of integers in a quadratic extension $Q[\sqrt{d}]$ for some squarefree integer $d$. Essentially, any fundamental unit must solve the equation $x^2 - ax \pm 1$ for some positive integer $a$, and the first one that works is your fundamental unit!

(7/22/2017) Today I learned that the ring of integers in $Q[\sqrt{10}], R$, is not a principal ideal domain, and moreover, it is a ring with class group the unique group of order two. This is because the Minkowski bound gives that every element in the class group has some representative contained entirely in the ring and norm less than 4 in this particular case. This then implies that the class group is generated by the prime divisors of $2R$ and $3R$. You can use norms and ramification to show that $2R = \beta^2$ for some beta, and then show that $\beta$ cannot be principal by arguing that if it were, we could write $2 = a^2 - 10b^2$, which one can see cannot happen by taking both sides modulo 5. Then you can use the element $\theta := 4 + \sqrt{10}$, which just so happens to have norm 6, to argue $R\theta$ is $\beta$ times some divisor of $3R$. Thus since $3R$ must factor into two distinct prime ideals (by Kummer's Theorem) you can show that one is the inverse of the other, and thus the group is the group $\langle \beta : \beta^2 = 1 \rangle$.

(7/23/2017) Today I learned a way to extend the concept of an element/field being integral over a field! The idea is to note that if you’re talking about these things, you have an injection of fields $K \hookrightarrow L$. So at the level of rings, you can talk about whether a morphism $\psi : B \rightarrow A$ is integral, or whether an element $a \in A$ is integral over it, by requiring that the element $a$ solves a monic polynomial with coefficients in $\psi(B)$. I also learned/proved a million and one thing about restriction maps and inclusion maps, including that the property of being an open embedding is closed under restriction, "gluing," composition, and base change of fiber products. I also proved that the induced map of schemes induced by inclusion is a monomorphism!

(7/25/2017) Today I learned two theorems relating to algebraic geometry and integral homomorphisms/extensions. One of them is called the Lying Over Theorem, which says that given any integral extension $B \hookrightarrow A$ and a prime ideal $q \in Spec(B)$, there is an ideal $p \in Spec(A)$ that maps to $q$ under the associated map of schemes, i.e. the associated map of schemes is surjective. Also I learned about the Going Up Theorem, which says that given an integral homomorphism $B \rightarrow A$, if you just so happen to have a list of prime ideals $q_1 \subset q_2 \subset \cdots \subset q_n \subset B$ and some possibly smaller length list of prime ideals in $Spec(A)$ hitting the first few elements, then you can complete that list to primes hitting the larger ones! The map $eval_{x,0} : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x]$ can show what’s going on.

(7/26/2017) Today I learned a bunch of "finiteness" properties of morphisms, which are three definitions of the form a morphism of schemes $\tau : X \rightarrow Y$ is a property $P$ of sets if for every affine open set $U \subset Y$ the preimage is that property. (This holds for quasicompactness, quasiseparatedness, and affine-ness at least). Oh by the way—I learned what quasiseparatedness is. It’s when the intersection of two compact open sets is still compact. This can actually not happen— for example, glue two copies of $Spec(k[x_1, x_2, \ldots])$ everywhere but at the origins. Then the two affine open sets are compact but the intersection is $D(x_1) \cup D(x_2) \cup \ldots$ which isn’t compact. I also learned about the quasicompact quasiseparatedness condition, which basically says that the scheme can be covered with a finite number of affine open subsets and the intersection of any two is the finite union of
affine open sets. Then there’s the quasicompact quasiseparatedness LEMMA, which says that given a scheme, then the sheaf on the set of points where a function $s$ vanishes is just the localization of the structure sheaf.

(7/30/2017) Today I learned about the concept of a finite morphism—this is a morphism that is not only affine, but moreover the inverse image of $\text{Spec}(B)$ is a finite $B$ algebra, where finite means finitely generated as a $B$ module. This is something that you can semi-easily show is affine local on the target (at least if you know that the property of being an affine morphism is local on the target). I also learned of a way that you can view any $A$ algebra $A \to R$ as projective $R$ space, you essentially define the zero grading to be $A$ and every other level grading to be a copy of $R$. It seemed kind of dumb, but I suppose it argues that if something is true of a projective scheme, then it’s true for any finite $R$ module.

(7/31/2017) Today I struggled through a topological lemma which said that if $X = \bigcup U_\alpha$, then a set $K$ is closed in $X$ if and only if $X \cap U_\alpha$ is closed in the subspace $U_\alpha$ topology. It took me a little too long, but honestly it’s something you struggle through once and then never forget.

(8/2/2017) Today I learned a neat trick in the world of completing fields with absolute values on them. Essentially you can realize the Cauchy sequence $\{a_i\}$ in the completion as a limit of the constant sequences, i.e. \[ \lim \{a_1, a_1, \ldots, a_2, a_2, \ldots, a_3, a_3, \ldots\} \] and use this to show that every extension (and in particular, the identity) can be extended to a unique map. This is used to show that the completion of a field is unique up to isomorphism.

(8/3/2017) Today I learned that you can specify a map from affine $A$–space to affine $A$–space by saying where functions go or by saying where points go, it turns out it’s the same thing! The reason is, if I give you the map $(x, y) \to (p(x, y), g(x, y))$, say, and you at first interpret that as a pullback map of functions $\pi^# : A[x, y] \to A[x, y]$, then $(\pi^#)^{-1}(x - a, y - b) = \{ r(x, y) : \pi^#(r) \in (x - a, y - b) \} = \{ r(x, y) : r(p(x, y), g(x, y)) \in (x - a, y - b) \} "$The set of points where if you plug in $x = a$ and $y = b$ zero comes out" = $\{ r : r(p(a, b), q(a, b)) = 0 \} = (x - p(a, b), y - q(a, b))$. This makes me really happy!

Also I learned about the concept of a locally closed set, and how if you have a constructable set then it’s actually just the union of disjoint locally closed sets. I am also barking at the door of Hilbert’s Nullstellensatz!

(8/4/2017) Today I learned the proof of Nullstellensatz, assuming that we know Chevalley’s Theorem, which states that given any finite type morphism between constructable schemes, the image of any constructable set (and in particular, the space itself) is a constructable set. Combine this with the fact that the point that’s just the generic point of $\text{Spec}(k[x])$ isn’t a constructable point, and you can argue that any field generated finitely as an algebra is also generated finitely as a module, since otherwise you can embed $k[x]$ into the field and show that the image is the generic point!

(8/5/2017) Today I learned how the Nullstellensatz implies a "weak" form of the Nullstellensatz, which basically says that if $\mathfrak{m}$ is a maximal ideal of $k[x_1, \ldots, x_n]$ for some algebraically closed field $k$, then $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$. This is because you can reduce to the one variable case by intersecting with $\mathfrak{m} \cap k[x_i]$ and then arguing that if you have finite generation as a module, then $\{ 1, x_i, x_i^2, \ldots \}$ solves some polynomial equation. In particular that polynomial equation lifted up is in $\mathfrak{m}$. But you can factor it as a product of $(x - \beta_j)$ for some roots, and since $\mathfrak{m}$ is in particular prime one of those roots is in the prime idea.

(8/6/2017) Today I learned the proof of the Grothendieck Freeness Lemma (and how to spell Grothendieck.) The Grothendieck Freeness Lemma says that if $B$ is a Noetherian integral domain and $A$ is a finitely generated $B$ algebra, then for any finitely generated $A$ module $M$ there exists some nonzero $f \in B$ such that $M_f$ is a free $B_f$ module. The proof comes from the fact that this is true for $A = B$ (essentially because if $A \cong B/I$ for some nonzero ideal $I$, then you can localize
by a nonzero element to make the localization of $A$ zero, which makes it free by definition). Then you can argue that it suffices to show that $A$ is satisfies the theorem implies $A[T]$ does shows our theorem is true and then use some sexy category theory to write $M$ as a direct sum of finite $A$ modules.

(8/7/2017) Today I learned that if you have a field $K$ that is complete with some respect to a non-archimedian valuation and $R$ is the ring associated to this valuation, and the ring just so happens to be a DVR, then if you take the integral closure of that ring $R$ in a finite dimensional separable extension, that ring is a DVR also, and there’s a unique way to extend that valuation to the field extension (which essentially follows because you can show that the unique prime ideal of $R$ ramifies totally, which you can prove using the Chinese remainder theorem and the fact that idempotent elements lift.) I also learned that you can reduce the problem of showing that $Spec(A) \rightarrow Spec(B)$, a finite type morphism of Noetherian schemes with $Spec(B)$ irreducible, has a nonempty open set $V \subset B$ that is either entirely contained in the image or not at all touching the image can be reduced to the case where $B$ is an integral domain. This is essentially because as topological spaces $Spec(B) \cong Spec(B/I)$, where $I$ is an ideal contained in all prime ideals of $B$.

(8/8/2017) Today I proved that it suffices to reduce the problem of yesterday can actually be reduced to the affine case. Essentially, you can use the compactness of both of your schemes to argue that you can cover your target scheme in a finite number of affine schemes such that the preimage of each one is the finite union of some affine schemes where the induced map gives a finitely generated algebra. Then assuming it works in the affine case, you get a finite number of open $V_{ij} \subset Y$, if $\pi : X \rightarrow Y$ is your morphism. If any of them are in the image of the smaller subset then they’re in the image of the larger subset (obviously) and if the image of the smaller subset (i.e. the nice affine scheme we had discussed earlier) avoids the associated $V_{ij}$ for all of them, just take the intersection of all of them, which is still open and nonempty since the finite intersection of dense open subsets is still a dense open subset.

(8/9/2017) Today I spent the entire day showing that Chevalley’s Theorem implies that the surjectivity of any map between affine schemes on closed points implies that the map is surjective on all points. One thing I did was reduce this to the case where the thing you’re quotienting out by was a prime ideal, i.e. your variety is actually an integral domain. Another thing I did was spend a lot of the day using maximal ideals/closed points to try and prove that a prime ideal is the intersection of all maximal ideals containing it in $k[x_1,\ldots,x_n]$. This makes sense–if you imagine the functions on the curve $(x - y)$, it’s exactly the stuff that’s valid functions on all of the closed points $(x - a, y - a)$. Hopefully tomorrow.

(8/10/2017) Today I learned about a stronger form of the Nullstellensatz dealing with Jacobson rings–i.e. rings where each prime ideal is the intersection of all maximal ideals containing it. This, combined with Chevalley’s Theorem, implies that the surjectivity of a map on a $k$– variety on closed points gives the surjectivity of the entire map, for if the image contained all the closed points, the compliment, also a constructible set, contains none. But if it contains any point, you can use the fact that your ring is a Jacobson ring to show that the set of maximal ideals are dense, and thus the open set must contain a closed point.

(8/11/2017) Today I learned the proof of Chevalley’s Theorem! Essentially you can argue that a corollary of Grothendieck’s Freeness Lemma says that given any map of schemes to an irreducible Noetherian scheme $Y$ there exists a nonempty open set $U$ such that either the image contains $U$ or is entirely disjoint from it. You can break these into two cases and make an algorithm iterating finding this open set, which can’t go on forever, lest you have a decreasing chain of closed ideals. Therefore it eventually must stop, which you can show must give $U$ being the entire image.

(8/12/2017) Today I learned the proof of the Fundamental Theorem of Elimination Theory, which says that the natural morphism $\mathbb{P}_A^n \rightarrow Spec(A)$ (given by the fact that $\mathbb{P}_A^n$ is an $A$–scheme)}
is a closed map. This doesn’t sound so sexy, until you unpack what it says algebraically. This essentially says that if you’re given some polynomials in $A[x_0, ..., x_n]$, you can write polynomials in $A$ itself to tell you when homogeneous polynomials in $A[x_0, ..., x_n]$ have a nontrivial solution for substituting the $x_i$’s for elements of $A$. The essential idea is to associate whether a point $p \in \text{Spec}(A)$ is in the vanishing set by looking back at the quotient field $\kappa(p)[x_0, ..., x_n]$ and then using some linear algebra to show that $\text{Proj}(\kappa(p)[x_0, ..., x_n])$ has a point in the vanishing set if and only if a certain set of linear maps aren’t surjective–i.e. a certain larger set of determinants are zero.

(8/13/2017) Today I learned about closed embeddings. I learned a lot of things about them! First of all, a closed embedding is an affine morphism such that the induced morphism of rings on each affine open subset is a surjection. This isn’t immediately obvious, but having this on all affine open sets says that the map actually identifies the domain with a closed subset of the codomain. Moreover, I learned a bunch of easier properties, like the fact that it’s a finite (and thus finite type) morphism and the property of being a closed embedding is affine local on the target. Also I learned that there’s a necessary and sufficient condition for an ideal sheaf to give you an actual closed embedding. What needs to happen is that you have to have an ideal sheaf on each affine open subset and for any $B \to B_f$, the ideal chosen for $B_f$ must be the $f$ localization for the ideal chosen or $B$. Then you can build a closed subscheme out of it!

(8/14/2017) Today I learned a pretty fact about prime ideals and their relation to the finite intersection of ideals of a ring. Specifically, if $A$ is any ring, $p$ is a prime ideal of $A$, and $J_i \subset A$ are a finite collection of ideals in $A$, then $p \supset \cap_i J_i$, then in fact $p \supset J_j$ for some $j$. That’s pretty cool. The proof works by induction, with the base case trivial and can be reduced to the case of $i \in \{1, 2\}$ through clever use of parenthesis. Assume swapping if necessary that $J_1 \not\subset p$. Then there exists an $f \in J_1$ but $f \notin p$. But for all $g \in J_2, fg \in \cap_i J_i \subset p$. Thus $g \in p$ for all $g \in J_2$. Really pretty.

(8/15/2017) Today I learned about the power of love. Ha ha ha. Just kidding. I learned something more valuable than that–what it means to take the closed subsheaf of projective space! It’s not too hard to define because you just give a set of homogeneous polynomial equations of some degree and it determines a closed subscheme and on the different gluing patches using the standard open cover you can show that the vanishing sets agree since the different homogeneous polynomials only differ by some unit in the corresponding rings.

(8/16/2017) Today I learned about the proof that $I = \langle wz - xy, x^2 - wy, y^2 - xz \rangle$ is a prime ideal of $k[w, x, y, z]$. Essentially you can use all three of those equations to take a polynomial and write it as the sum of polynomials that just have a few variables in them, including a polynomial with just $x$ terms and a polynomial with just $y$ terms. You can then argue that $k[w, x, y, z]/I \cong k[a^3, a^2b, ab^2, b^3]$ by showing that if it just so happens that the polynomial $p(w, x, y, z) \in k[w, x, y, z]$ and $p(a^3, a^2b, ab^2, b^3) = 0$ then you can argue that those polynomials can’t have terms that ”interact” with each other–i.e. each term has unique monomials associated to it that don’t appear in any others. This shows that each polynomial is individually zero!

(8/17/2017) Today I learned about the relation of projective space and linear spaces. Essentially if you take an $n + 1$ dimensional $k$ vector space, say $W = \langle kx_0 + ... + kx_n \rangle^\perp$ for some chosen elements of a basis, then the symmetric construction $\text{Sym}(W^\perp) \cong k[x_0, x_1, ..., x_n]$ so then can define projective $W$ space $\mathbb{P}W = \text{Proj}(\text{Sym}(W^\perp))$. You can use this to prove that if you have an injection $V \hookrightarrow W$ (then you have a surjection $W^\perp \to V^\perp$ as graded rings) then you get a closed embedding $\mathbb{P}V \hookrightarrow \mathbb{P}W$.

(8/18/2017) Today I learned another proof that finite morphisms are closed. Essentially you can reduce the problem to a map of affine schemes $\text{Spec}(A) \to \text{Spec}(B)$ and then realize that you can write $\text{Spec}(A)$ as a projective space! Namely, if you have $B$ in the zero grading and a copy of
A in every other grading, you can realize $\text{Spec}(A)$ as embeddable into $\mathbb{P}^n_B$ for some $N$. And then you write the map as a composition of $\text{Spec}(A) \hookrightarrow \mathbb{P}^n_B \rightarrow \text{Spec}(B)$, which we proved the second maps was closed via the fundamental theorem of elimination theory.

(8/21/2017) Today I learned the product formula for all the possible norms over any algebraic number field. Essentially, viewing primes as the equivalence classes of valuations on the field, the product of all of the valuations is one. Essentially this follows by reducing to the rational case, almost. What’s going on is that we take any arbitrary field extension and then we write all the possible completions and associate them to all of the different prime ideals or Galois embeddings depending on whether they are archimedian or not. Then we can manipulate each particular extension of a prime ideal so that they multiply to the norm (and this uses the fact that the characteristic polynomials/norms after a certain decomposition decomposes as products) and then multiplying over all possible norms is just the same thing as multiplying the norm of an element over all possible rational norms, which gets you 1!

(8/30/2017) Today I learned some facts relating to the Bruhat-Tits tree on $SL_2$, which today mostly was about linear algebra-esque things involving fields that have a discrete valuation on them. I learned that you can normalize any invertible entries in the field using a technique called integral column/row operations, where the main shift is at first to identify where the smallest power of $\pi$, an element whose unit multiples and powers generate the field. I also learned that you can diagonalize any invertible matrix in a discretely valued field, where ”diagonalize” loosely means that you can write it as the product $k_1 dk_2$ where the $k$’s are invertible matrices whose entry and whose inverse’s entries are also in the associated discrete valuation ring. You can use this to show that given any two lattices, you can find a basis of one that you can multiply by $\pi^{m_i}$ for some $m_i, i \in \{1, 2\}$.

(8/31/2017) Today I learned what the actual tree is for a complete field with a discrete valuation $K$. In particular, the nodes of the tree are just the lattices of $K^2$ subject to equivalence that the lattice $L$ is also the lattice $\pi L$ where $\pi$ is a generator of the unique maximal ideal of the associated DVR. The nodes are then determined by the principal divisor theorem. I also learned the theorem which essentially says that what we’ve been calling a tree actually is a tree. Essentially this stems from the fact that if you have any path, where edges are if the principal divisor theorem gives that you can find a basis of one of them $(e, f)$ where a basis of the other is just $(\pi e, f)$. Then you can basically argue that any path can be translated by some $GL_2(K)$ element, using completeness, to the path Id, $\nu_1, \nu_2, ...$ where $\nu_j$ has first column $(\pi^j, 0)^t$ and second that of the identity.

(9/3/2017) Today I learned about the two rulings on the quadric surface $\text{Spec}(k[w, x, y, z]/(wz - yx))$. Another way to imagine this is that the quadric surface $X$ is parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$. I’ve showed that each point on the quadric surface has a surface from each line passing through it, which you can essentially do by doing the algebra you might think to do originally and then make it carry over scheme theoretically using the $(x - a, y - b)$ means $x = a, y = b$ idea. Today I also learned that you can make Simpsons characters in LaTeX after downloading a style file. Here’s Homer. ℹ️