# Notes on Category Theory

## Contents

1. Introduction and Motivation 2

2. Definition of a Category 2
   2.1 Examples 3

3. Functors 3
   3.1 Natural Transformations 4
   3.2 Adjoint Functors 5
   3.3 Units and Counits 5
   3.4 Initial and Terminal Objects 6
       3.4.1 Comma Categories 6

4. Representability and Yoneda’s Lemma 8
   4.1 Representables 8
   4.2 The Yoneda Embedding 9
   4.3 The Yoneda Lemma 9
   4.4 Consequences of Yoneda 11

5. Limits and Colimits 12
   5.1 (Co)Products, (Co)Equalizers, Pullbacks and Pushouts 13
   5.2 Existence of limits and colimits 15
   5.3 Limits as Representable Objects 16
   5.4 Limits as Adjoints 17
   5.5 Preserving Limits and GAFT 18

6. Abelian Categories 19

---

Note to the reader: This is a short introduction to categories and their basic notions for the curious undergraduate. These notes are based on the Fall 2015 Category Theory tutorial led by Danny Shi at Harvard. There is no single textbook that these notes follow, but *Categories for the Working Mathematician* by Mac Lane and Lang’s *Algebra* are good standard resources. For a more modern take, see Emily Riehl’s *Category Theory in Context*.
1 Introduction and Motivation

Categories are a way of abstracting the mechanics of various subjects in mathematics into a coherent theory. It is the modern standard for communicating and formulating many “algebraic” fields of mathematics, including Algebraic Topology and Algebraic Geometry.

As a warm-up, consider two abelian groups $A, B$. The reader knows that the tensor product $A \otimes B$ is defined to be an abelian group together with a map $f : A \times B \to A \otimes B$ satisfying certain properties. It is unique in the sense that, if $X$ is any other abelian group with a map $g : A \times B \to X$ satisfying the same properties as $f$, then there is a unique map $\phi$ such that the following diagram commutes:

$$
\begin{array}{c}
A \times B \xrightarrow{f} A \otimes B \\
\downarrow{g} \downarrow{\phi} \downarrow X \\
X
\end{array}
$$

In this sense $A \otimes B$ is the “smallest” such group, because any other such groups factor through $A \otimes B$ in the above diagram. This is an example of a universal property construction, and is used in various areas of mathematics. Category theory can characterize how and why this works, and most importantly, it can define it without reference to any axioms of group theory.

2 Definition of a Category

Definition 2.1. A category $\mathcal{C}$ consists of the following:

1. A collection of objects $\text{Obj}(\mathcal{C})$
2. For each $A, B \in \text{Obj}(\mathcal{C})$, there is a collection of morphisms $\text{Hom}_\mathcal{C}(A, B)$ (“maps” between $A$ and $B$)
3. For every $A \in \text{Obj}(\mathcal{C})$, there exists an identity morphism $1_A \in \text{Hom}_\mathcal{C}(A, A)$.
4. A composition rule for morphisms:
   $$
   \circ : \text{Hom}_\mathcal{C}(A, B) \times \text{Hom}_\mathcal{C}(B, C) \to \text{Hom}_\mathcal{C}(A, C)
   $$

Such that the following two axioms are satisfied:

- **Associativity**: for morphisms $g, f, h$:
  $$(h \circ g) \circ f = h \circ (g \circ f)$$

- **Identity**: for objects $A, B$ and morphism $f \in \text{Hom}_\mathcal{C}(A, B)$:
  $$\text{id}_A \subset A \xrightarrow{f} B \supset \text{id}_B \Rightarrow f \circ \text{id}_A = \text{id}_B \circ f = f$$

Categories are usually best thought of as dots and arrows between dots.

Remark 2.2. Notation for objects and morphisms varies. We often drop the $\mathcal{C}$ subscript in $\text{Hom}$ and sometimes we write $\mathcal{C}(A, B)$ instead of $\text{Hom}(A, B)$. We also often drop the “obj” notation and think of $\mathcal{C}$ itself as the set of objects (i.e. we write $A \in \mathcal{C}$ instead of $A \in \text{Obj}(\mathcal{C})$). Additionally, we denote $\text{Hom}(\mathcal{C})$ to be the collection of all morphisms in $\mathcal{C}$.

Definition 2.3. Let $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}(A, B)$. We say $A$ and $B$ are *isomorphic* $(A \cong B)$ if there exists $g \in \text{Hom}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$
2.1 Examples

Below are some familiar examples of categories.

- Sets (\(\text{Set}\)), where \(\text{Obj}(\text{Set})\) are sets and morphisms are functions between sets.
- Groups (\(\text{Grp}\)), where \(\text{Obj}(\text{Grp})\) are groups and morphisms are group homomorphisms.
- Modules (\(\text{Rmod}\)), where \(\text{Obj}(\text{Rmod})\) are \(R\) modules and morphisms are linear homomorphisms.
- Topological spaces (\(\text{Top}\)), where \(\text{Obj}(\text{Top})\) are topological spaces and morphisms are continuous functions.
- Pointed topological spaces (\(\text{Top}_*\)), where \(\text{Obj}(\text{Top}_*)\) are spaces with a specified basepoint and morphisms are continuous maps that send basepoints to basepoints.
- The trivial category, 1. This is the category with only one object (often denoted “\(\cdot\)”) and its identity morphism.

Definition 2.4. Given a category \(\mathcal{C}\), we define the dual (or opposite) category \(\mathcal{C}^{\text{op}}\) as the category whose objects are those of \(\mathcal{C}\) and \(f \in \mathcal{C}(A, B) \iff f \in \mathcal{C}^{\text{op}}(B, A)\). It is the same category but with arrows (morphisms) pointed in the reversed direction.

3 Functors

Definition 3.1. A (covariant) functor \(F : \mathcal{A} \to \mathcal{B}\) between categories is a function taking objects in \(\mathcal{A}\) to objects in \(\mathcal{B}\). Further it maps morphisms to morphisms in the following way:

\[f \in \mathcal{A}(A, A') \Rightarrow F(f) \in \mathcal{B}(F(A), F(A'))\]

Further, \(F\) must satisfy two conditions:

1. \(F(g \circ f) = F(g) \circ F(f)\)
2. \(F(\text{id}_A) = \text{id}_{F(A)}\)

Remark 3.2. Consider the category of categories (called \(\text{Cat}\)) : the objects are categories and the morphisms are functors. This allows us to compose functors and declare when two categories are equivalent. Two categories are equivalent if they are isomorphic as elements of \(\text{Cat}\).

Examples:

- Forgetful functors – functors that “forget” certain information about objects in a category:
  a) \(F : \text{GRP} \to \text{Set}\) sending groups to their underlying groups and homomorphisms to maps of sets.
  b) \(F : \text{RING} \to \text{GRP}\) sending rings to their abelian group and homomorphisms to homomorphisms.
- \(F : \text{AbGRP} \to \text{GRP}\) sending groups to themselves, forgetting that they are abelian.
- The fundamental group – This sends a topological space to its fundamental group:
  \(\pi_1 : \text{Top}_* \to \text{GRP}\)
- Singular homology – This sends a topological space to its \(n\)th singular homology group:
  \(H^n(-) : \text{TOP} \to \text{AbGRP}\)
Definition 3.3. A contravariant functor is a covariant functor $F : \mathcal{A}^{\text{op}} \to \mathcal{B}$.

Definition 3.4. A functor $F : \mathcal{A} \to \mathcal{B}$ is faithful if, given $A, A' \in \text{Obj}(\mathcal{A})$ then $F$ as a map of $\mathcal{A}(A', A)$ is injective.

Definition 3.5. A subcategory $\mathcal{C}$ of a category $\mathcal{A}$ is a category such that:
- $\text{Obj}(\mathcal{C}) \subset \text{Obj}(\mathcal{A})$
- $\mathcal{C}(C, C') \subset \mathcal{A}(C, C')$

A subcategory is called full if $\forall A, A' \in \text{Obj}(\mathcal{A})$ we have $\mathcal{C}(A, A') = \mathcal{A}(A, A')$.

3.1 Natural Transformations

Consider two functors $F, G : \mathcal{A} \to \mathcal{B}$. We would like a way to make these compatible:

Definition 3.6. A natural transformation $N$ between two functors is a family of morphisms, each of which associates an object of $\mathcal{A}$ to a morphism $N_A$ (sometimes denoted $N_A$) of $\mathcal{B}$. Any natural transformation must make the following square commute:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
\downarrow N(A) & & \downarrow N(A') \\
G(A) & \xrightarrow{G(f)} & G(A')
\end{array}
$$

Natural transformations are often denoted as double arrows:

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{N} & \mathcal{B} \\
\uparrow F & & \uparrow G
\end{array}
$$

If $N_A$ is an isomorphism for each $A$, then we call $N$ an isomorphism of functors.

Examples:
- Let $G$ be a group thought of as a one-object category. A functor $F : G \to \text{SET}$ gives a group action on a set $S$. A natural transformation of group actions is a map of sets that respects the group action.
- Let $M_n(-) : \text{CRING} \to \text{MONOID}$ be the functor sending a commutative ring to the monoid of matrices over that ring. Let $U : \text{CRING} \to \text{MONOID}$ be the forgetful functor that forgets ring addition. A natural transformation between $M_n(-)$ and $U$ is the determinant.

Remark 3.7. Natural transformations can compose to form another natural transformation. This allows us to define the functor category:

Definition 3.8. For fixed categories $\mathcal{A}, \mathcal{B}$ the functor category $[\mathcal{A}, \mathcal{B}]$ is the category whose objects are functors $\mathcal{A} \to \mathcal{B}$ and whose morphisms are natural transformations.

Definition 3.9. We say two functors are naturally isomorphic if they are isomorphic as objects in the functor category. Equivalently, $F, G : \mathcal{A} \to \mathcal{B}$ are naturally isomorphic if there is a natural transformation $\alpha : F \to G$ such that $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathcal{A}$ (such an $\alpha$ is called a natural isomorphism).
### 3.2 Adjoint Functors

**Definition 3.10.** Let \( F : \mathcal{A} \rightleftharpoons \mathcal{B} : G \) be functors. We say \( F \) is left adjoint to \( G \) (and \( G \) is right adjoint to \( F \)) if for every \( A \in \mathcal{A}, B \in \mathcal{B} \) we have a natural isomorphism:

\[
B(F(A), B) \cong A(A, G(B))
\]

In other words, this is a natural bijection. For \( g \in B(F(A), B) \), we denote the corresponding element of \( A(A, G(B)) \) as \( \eta \).

**Remark 3.11.** The use of “natural” above means something specific: it means the isomorphism is compatible with changing \( A \) and \( B \). More concretely, if \( g \in B(F(A), B) \) and we have some morphism \( q : B \to B' \), we consider the composition:

\[
F(A) \xrightarrow{g} B \xrightarrow{q} B'
\]

Applying adjointness to the composition, we obtain a map \( \eta \circ q \) : \( A \to G(B') \). However, we can also apply adjointness to \( q \) and apply \( G \) to \( q \), giving us \( G(q) : G(B) \to G(B') \). Composing these gives \( G(q) \circ \eta : A \to G(B') \). Naturality means \( \eta \circ q = G(q) \circ \eta \) (and similarly in the other direction for \( f \in A(A, G(B)) \) and a morphism \( p : A \to A' \)).

**Example:**
- The free and forgetful functors \( F \) and \( U \) are adjoint for “algebra” type categories (Groups, Rings, Vector Spaces).
- An important adjunction in homological algebra is the Hom-Tensor adjunction. The Hom functor on modules sends \( M \mapsto \text{Hom}(N, M) \) for fixed \( N \), and the Tensor functor sends \( M \mapsto M \otimes N \).

**Proposition 3.12.** Adjoints can be composed. That is, if \( F : \mathcal{A} \rightleftharpoons \mathcal{B} : G \) and \( F' : \mathcal{B} \rightleftharpoons \mathcal{C} : G' \) are adjoints, then \( F' \circ F : \mathcal{A} \rightleftharpoons \mathcal{C} : G \circ G' \) are adjoint as well.

### 3.3 Units and Counits

**Definition 3.13.** If \( F : \mathcal{A} \rightleftharpoons \mathcal{B} : G \) are adjoint, we automatically obtain a morphism \( \eta_A : A \to GF(A) \) by pulling back the identity \( i_A : F(A) \to F(A) \) for any \( A \) (i.e. \( \eta_A = i_A \)). Similarly, we obtain a morphism \( \varepsilon_B : B \to FG(B) \) by pulling back \( i_B : G(B) \to G(B) \). We see that these then define natural transformations:

\[
\eta : \text{id}_A \to GF \quad \text{(unit)}
\]

\[
\varepsilon : \text{id}_B \to FG \quad \text{(counit)}
\]

These are called the unit and counit maps. The unit and counit maps satisfy important identities, the triangle identities:

**Proposition 3.14.** (Triangle Identities) Let \( F : \mathcal{A} \rightleftharpoons \mathcal{B} : G \) be adjoint and \( \eta, \varepsilon \) be the induced unit and counit. Then the following diagrams commute for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \):

\[
\begin{align*}
F(A) &\xrightarrow{F(\eta_A)} GF(A) & &\xleftarrow{G(\varepsilon_B)} GFG(B) \\
\eta_A &\xleftarrow{\text{id}_{F(A)}} F(A) & &\varepsilon_B \xrightarrow{\text{id}_{G(B)}} G(B)
\end{align*}
\]
Corollary 3.15. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be an adjunction with unit and counit $\eta, \varepsilon$. Then for any $g : F(A) \to B$:

$$f = G(g) \circ \eta_A$$

and similarly for any $f : A \to G(B)$:

$$\overline{f} = \varepsilon_B \circ F(f)$$

In fact, if we have any two functors $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ with natural transformations $\eta$ and $\varepsilon$, and we can show they satisfy the above property, then they are actually adjoint. This gives an alternative definition of adjointness:

Theorem 3.16. (Adjointness) Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be functors. There is a one-to-one correspondence between:

- Adjunctions between $F$ and $G$.
- Pairs $(\text{id}_A \xrightarrow{\eta} GF, FG \xrightarrow{\varepsilon} \text{id}_B)$ of natural transformations satisfying the triangle identities.

Remark 3.17. This says that the unit and counit determine the whole adjunction.

3.4 Initial and Terminal Objects

Definition 3.18. In any category $\mathcal{A}$, there are two special objects:

- An initial object $I \in \mathcal{A}$ is one so that there exists exactly one morphism $I \to A$ for all $A \in \mathcal{A}$.
- A terminal object $T \in \mathcal{A}$ is one so that there exists exactly one morphism $A \to T$ for all $A \in \mathcal{A}$.

Examples:

- In Set, the initial object is $\emptyset$, and the terminal object is the singleton set $\{e\}$ for any $e$.
- In Grp, the initial object is the trivial group $\{1\}$ and the terminal object is also the trivial group $\{1\}$.
- In CRing, the initial object $\mathbb{Z}$ and the terminal object is $\{0\}$, the zero ring.

Proposition 3.19. Initial and terminal objects are unique up to isomorphism.

We will define adjoint functors using initial and terminal objects. To do this, we must define the comma category and explain some of its variants.

3.4.1 Comma Categories

Definition 3.20. Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ with functors $P : \mathcal{A} \to \mathcal{C}, Q : \mathcal{B} \to \mathcal{C}$, the comma category $(P \Rightarrow Q)$ is the following:

- Objects: $(A, B, h)$, where $A \in \mathcal{A}, B \in \mathcal{B}$ and $h \in \mathcal{C}(P(A), Q(B))$.
- Morphisms: if $(A, B, h)$ and $(A', B', h')$ are two objects, a morphism is a pair $(f, g)$, where $f : A \to A'$, $g : B \to B'$ so that the following square commutes:

$$
\begin{array}{ccc}
P(A) & \xrightarrow{P(f)} & P(A') \\
\downarrow{h} & & \downarrow{h'} \\
Q(B) & \xrightarrow{Q(g)} & Q(B')
\end{array}
$$
3.4 Initial and Terminal Objects

The best way to motivate the comma category is to picture the pullback on the diagram determined by \( P \) and \( Q \), shown below:

\[
\begin{array}{c}
\mathcal{B} \\
\downarrow Q \\
\mathcal{A} \xrightarrow{P} \mathcal{C}
\end{array}
\]

The comma category is the category that naturally fills in the top left part of the square, so it should have objects of \( A \) and of \( B \) as well as a map \( h \) that makes the diagram compatible:

\[
\begin{array}{c}
(P \Rightarrow Q) \\
\downarrow \\
\mathcal{A} \xrightarrow{P} \mathcal{C} \\
\upsilon_X \downarrow \\
Q(Y)
\end{array}
\]

That is, following the blue arrows will give us \( P(X) \) and following the red arrows gives us \( Q(Y) \) and we want a morphism \( h \) that makes these compatible. Another way to understand this is that for every \( X \in \mathcal{A} \) and \( Y \in \mathcal{B} \), the map \( h : P(X) \to Q(Y) \) determines a natural transformation from the blue functor to the red functor.

We examine two special cases of comma categories that are often used, called slice categories.

**Definition 3.21.** Given a category \( \mathcal{A} \) and an object \( A \in \mathcal{A} \), the slice category, denote \( \mathcal{A}/A \), is the comma category on the functors \( \text{id}_A : \mathcal{A} \to \mathcal{A} \) and \( (\cdot) : 1 \to \mathcal{A} \), the first of which is the identity, and the second sends the singleton object \( \cdot \) identically to \( A \). Explicitly, the slice category is:

- **Objects:** \( (X, h) \) for \( X \in \mathcal{A} \) and \( h : X \to A \).
- **Morphisms:** \( f : X \to X' \) so that the diagram commutes:

\[
\begin{array}{c}
X \\
\downarrow h \\
A \xrightarrow{\eta_A} \mathcal{A}
\end{array} \quad \begin{array}{c}
X' \\
\downarrow h' \\
A
\end{array}
\]

The second is a more general version of the slice category, where instead of the identity \( \text{id}_A : \mathcal{A} \to \mathcal{A} \), we have an arbitrary functor \( G : \mathcal{B} \to \mathcal{A} \). This is also sometimes known as the slice category and has an equivalent formulation as above. This slice category is denoted \( (A \Rightarrow G) \).

Now we return to initial and terminal objects and their relationship to adjoint functors.

**Proposition 3.22.** Let \( F : \mathcal{A} \rightleftarrows \mathcal{B} : G \) be adjoint functors, and let \( A \in \mathcal{A} \). Then \( (\cdot, F(A), \eta_A : A \to GF(A)) \) is an initial object of the slice category \( (A \Rightarrow G) \).

**Proof:** This follows from the definition of adjointness. Let \( (\cdot, B, h) \) be any object of \( (A \Rightarrow G) \). We must find a map from \( (\cdot, F(A), \eta_A) \) to \( (\cdot, B, h) \), which is a map \( \overline{h} : F(A) \to B \) that commutes properly with \( h \). Since we have \( h : A \to G(B) \), we can apply adjointness to obtain \( \overline{h} : F(A) \to B \). Applying \( G \) to \( \overline{h} \) gives the desired commuting diagram.

7
4 REPRESENTABILITY AND YONEDA’S LEMA

condition:

\[
\begin{array}{ccc}
 & A & \\
\eta_A & \downarrow & \downarrow g(\eta) \\
GF(A) & \rightarrow & G(B)
\end{array}
\]

Commutativity follows from the corollary of the Triangle Identities in §3.3.

Now we have enough information to state the second theorem of adjointness:

**Theorem (Adjointness 2).** Let \( F : \mathcal{A} \rightleftarrows \mathcal{B} : G \) be functors. Then there is a one-to-one correspondence between:

- Adjunctions of \( F \) and \( G \), and
- Natural transformations \( \eta : id_A \rightarrow GF \) such that \( \forall A \in \mathcal{A} \), the object \((\cdot, F(A), \eta_A)\) is initial in \((A \Rightarrow G)\).

We have already proven one direction in the proposition above. The other direction is left as an exercise.

4 Representability and Yoneda’s Lemma

We now turn to the topic of representable functors, which are the object of study in the famous Yoneda Lemma. To discuss these, we must introduce the idea of a small category.

**Definition 4.1.** A category \( \mathcal{C} \) is small if the objects and morphisms of \( \mathcal{C} \) are actual sets, and not a proper class (i.e. not a set of sets).

**Definition 4.2.** A category \( \mathcal{C} \) is locally small if, for all \( A, B \in \mathcal{C} \), the collection \( \mathcal{C}(A, B) \) is small.

**Examples:**

- The trivial category 1 is small.
- \( \mathsf{Set} \) is not small, but it is locally small.

Since most of the categories we work with are built from sets (groups, rings, modules, etc.), most of our common examples are not small, but are locally small.

4.1 Representables

Let \( \mathcal{C} \) be a locally small category, and let \( A \in \mathcal{C} \). Then we define the functor \( H^A : \mathcal{C} \rightarrow \mathsf{Set} \) given by \( B \mapsto \mathcal{C}(A, B) \). This acts on morphisms via precomposition, i.e. \( H^A(g : B \rightarrow B') = g \circ (\cdot) \).

**Definition 4.3.** Let \( X : \mathcal{A} \rightarrow \mathsf{Set} \) be a covariant functor. It is called representable if \( X \cong H^A \) naturally for some \( A \in \mathcal{A} \). If \( \alpha : H_A \rightarrow X \) is an isomorphism, we sometimes call \((A, \alpha)\) a representation of \( X \).

**Examples:**

- (Diagonal) Consider the functor \( \Delta : \mathsf{Set} \rightarrow \mathsf{Set} \) given by \( S \mapsto S \times S \), and \( f \mapsto (f, f) \). This can be represented by the set \( \{0, 1\} = 2 \). To see this, note that an element of \( \text{Hom}(\{0, 1\}, S) \) is determined exactly by choosing where to send 0 and 1, so it is indeed isomorphic to \( S \times S = \Delta S \).

\(^1\text{This is an isomorphism in the functor category } [\mathcal{A}, \mathsf{Set}]\)
• (Powersets) The powerset functor \( P : \text{Set}^{\text{op}} \rightarrow \text{Set} \) sends a set to its powerset, and sends maps \( f : S \rightarrow T \) to the map sending \( X \subseteq T \) to \( f^{-1}(X) \). It turns out that \( P \) can also be represented using the object \( \{0, 1\} \). First we note that \( \text{Hom}_{\text{Set}^{\text{op}}} \left( \{0, 1\}, S \right) = \text{Hom}_{\text{Set}}(S, \{0, 1\}) \), so we must demonstrate an isomorphism \( P(S) \cong \text{Hom}_{\text{Set}}(S, \{0, 1\}) \). This is given by sending a subset \( X \subseteq S \) to its indicator function \( \chi_S : S \rightarrow \{0, 1\} \).

• (Tensor products) Let \( M \) and \( N \) be \( R \)-modules. Then the bilinear functor \( \text{Bilin}(M, N; -) : \text{Rmod} \rightarrow \text{Set} \) sends a module \( P \) to the set of bilinear maps \( g : M \times N \rightarrow P \) and sends a module map \( f \) to post-composition with \( f \). Then, by the universal property of the tensor product, we have \( \text{Bilin}(M, N; P) \cong \text{Hom}(M \otimes_R N, P) \) and so this functor can be represented using the tensor product \( M \otimes_R N \).

• (Forgetful functors) Many forgetful functors are representable. In general for “algebra” type categories, the representing object is often the most canonical object in that category (or the initial object). Below is a table of the representing objects for various forgetful functors.

<table>
<thead>
<tr>
<th>Functor</th>
<th>Representing Object</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{GRP} \rightarrow \text{SET}</td>
<td>\mathbb{Z}</td>
</tr>
<tr>
<td>\text{RING} \rightarrow \text{SET}</td>
<td>\mathbb{Z}[x]</td>
</tr>
<tr>
<td>\text{RMOD} \rightarrow \text{SET}</td>
<td>\mathbb{R}</td>
</tr>
<tr>
<td>\text{VECT}_k \rightarrow \text{SET}</td>
<td>k</td>
</tr>
</tbody>
</table>

• (Adjunctions) For any adjunction \( F : A \Rightarrow B : G \), the functor \( A(A, G(-)) : B \rightarrow \text{SET} \) is isomorphic to \( H^{F(A)} \), and hence it is representable.

Though we provided many examples, it should be noted that an arbitrary functor is rarely representable.

### 4.2 The Yoneda Embedding

Suppose we have a morphism \( f : A \rightarrow A' \); then we obtain a natural transformation \( \eta_f : H^{A'} \rightarrow H^A \) given by precomposing with \( f \). This allows us to define a new functor:

\[
H^* : C^{\text{op}} \rightarrow [C, \text{SET}]
\]

\[
A \mapsto H^A
\]

\[
f \mapsto \eta_f
\]

This is an “embedding” of \( C^{\text{op}} \) as a collection of functors from \( C \) to \( \text{SET} \) (also called copresheaves).

We can dualize this whole discussion and define the functor \( H_A : C^{\text{op}} \rightarrow \text{SET} \) for \( A \in C \). This is given by \( B \mapsto C(B, A) \) and postcomposition on morphisms. We say that a functor \( X : C^{\text{op}} \rightarrow \text{SET} \) (also called a presheaf) is \text{corepresentable} if \( X \cong H_A \) for some \( A \). Similarly to before, we have a functor:

\[
H_* : C \rightarrow [C^{\text{op}}, \text{SET}]
\]

\[
A \mapsto H_A
\]

\[
f \mapsto \epsilon_f
\]

Where \( \epsilon \) is the equivalent natural transformation to \( \eta \) above. This is an embedding of \( C \) as a collection of presheaves, also called the Yoneda embedding.

### 4.3 The Yoneda Lemma

Let \( C \) be locally small and let \( X : C^{\text{op}} \rightarrow \text{SET} \) be a presheaf. We can ask how \( X \) is related to a representing functor \( H_A \) for any \( A \in C \). More specifically, we can ask how is \( X \) “seen” by \( H_A \). In other words, what does
the set of natural transformations $H_A \Rightarrow X$ look like? In the language of the functor category $[\mathcal{C}^{op}, \mathbf{Set}]$, this means we want to understand $[\mathcal{C}^{op}, \mathbf{Set}](H_A, X)$ as a set.

**Exercise:** Convince yourself that in the case of a representable presheaf $X \cong H_B$, we have $[\mathcal{C}^{op}, \mathbf{Set}](H_A, X) \cong \mathcal{C}(A, B)$.

We note that, in the special case above of $X \cong H_B$, it happens that $[\mathcal{C}^{op}, \mathbf{Set}](H_A, X) = X(A)$. This turns out to be true even if $X$ is not representable. This is the Yoneda Lemma:

**Lemma 4.4. (Yoneda)** If $X : \mathcal{C}^{op} \to \mathbf{Set}$ is a presheaf on a locally small category, then for any $A \in \mathcal{C}$:

$$[\mathcal{C}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$$

(Naturally in $X$ and $A$)

In other words, the set of natural transformations $H_A \Rightarrow X$ is in one-to-one correspondence with elements of $X(A)$.

**Proof:**

This will proceed by explicitly constructing maps in both directions that compose to the identity. In one direction, we wish to associate to a natural transformation $\alpha : H_A \Rightarrow X$ an element of $X(A)$. The “only” choice we have is to use the data that the natural transformation provides, namely a morphism $\alpha_A : H_A(A) \to X(A)$. Since $H_A(A) = \mathcal{C}(A, A)$, we can pick $1_A \in \mathcal{C}(A, A)$ and look at its image under $\alpha_A$. Thus, we have a map:

$$(\sim) : [\mathcal{C}^{op}, \mathbf{Set}](H_A, X) \to X(A)
\alpha \mapsto \alpha_A(1_A) := \hat{\alpha}$$

In the other direction, we wish to associate to any element $x \in X(A)$ a natural transformation $\hat{x}$. For any $B \in \mathcal{C}$, we must define a morphism $\hat{x}_B : H_A(B) \to X(B)$. The natural way to define this morphism is $f \mapsto X(f)(x)$. Thus we have another map:

$$(\sim) : X(A) \to [\mathcal{C}^{op}, \mathbf{Set}](H_A, X)
x \mapsto (f \mapsto X(f)(x)) := \hat{x}$$

Before we proceed, we must show that $\hat{x}$ as defined above is actually a natural transformation. That is, we must show that, given a morphism $g : B' \to B$ in $\mathcal{C}$, the following square commutes:

$$
\begin{array}{ccc}
H_A(B) & \xrightarrow{H_A(g)} & H_A(B') \\
\downarrow{\hat{x}_B} & & \downarrow{\hat{x}_{B'}} \\
X(B) & \xrightarrow{X(g)} & X(B')
\end{array}
$$

If $f \in H_A(B)$, its image going in the upper path gives $X(g \circ f)(x)$ and its image in the lower path is $X(g)(X(f)(x))$. Since $X$ is a functor, these are equal by the composition properties of functors.

Now we wish to show that $\hat{x} \circ x = x$ and $\alpha = \alpha$. We will show the latter and the former is left as an exercise. We must verify that, for any $B \in \mathcal{C}$ and $f \in H_A(B)$, we have $\alpha_B(f) = X(f)(\alpha_A(1_A))$. To see this, consider the commuting square we get from $\alpha$ being a natural transformation:

$$
\begin{array}{ccc}
H_A(A) & \xrightarrow{H_A(f)} & H_A(B) \\
\downarrow{\alpha_A} & & \downarrow{\alpha_B} \\
X(A) & \xrightarrow{X(f)} & X(B)
\end{array}
$$
If we look at the image of $1_A \in H_A(A)$ through both paths, we obtain the desired result.

The final part of this proof is to verify that this isomorphism is natural in both $A$ and $X$. We will show naturality in $X$ and we leave verifying naturality in $A$ as an exercise. Let $X, X' : C^{\text{op}} \to \text{Set}$ be presheaves with a natural transformation $\Theta : X \Rightarrow X'$. As usual, we must verify that the usual square commutes:

\[
\begin{array}{ccc}
[C^{\text{op}}, \text{Set}](H_A, X) & \xrightarrow{\text{push}(\Theta)} & [C^{\text{op}}, \text{Set}](H_A, X') \\
\downarrow \scriptstyle{\sim} & & \downarrow \scriptstyle{\sim} \\
X(A) & \xrightarrow{\Theta_A} & X'(A)
\end{array}
\]

Where $\text{push}(\Theta)$ is the postcomposition map of $\Theta$. Starting with an element $\alpha : H_A \Rightarrow X$ in the top left, the bottom path yields $\Theta_A \circ \alpha_A(1_A)$ and the top path yields $(\Theta \circ \alpha)_A(1_A)$. These are equal by the composition properties of functors.

\[\square\]

**Remark 4.5.** We have just proved what is known as the *contravariant* version of Yoneda’s lemma. The dual notion is the covariant version, which asserts that if $X : C \to \text{Set}$ is any (covariant) functor, there is a natural isomorphism $[C, \text{Set}](H^A, X) \cong X(A)$. The proof is nearly identical.

### 4.4 Consequences of Yoneda

#### Universal Elements

The first corollary of Yoneda’s Lemma is that representations of functors can be restated in terms of a universal property. If $X : C^{\text{op}} \to \text{Set}$ is a presheaf, an “element” of $X$ can be thought of as a pair $(B, x)$, where $B \in C$ and $x \in X(B)$. We now define a universal element:

**Definition 4.6.** A universal element $(A, u)$ of a functor $X : C \to \text{Set}$ is an element of $X$ such that for each $B \in C$ and each $x \in X(B)$, there exists a unique morphism $f : B \to A$ such that $X(f)(u) = x$. Thus $(A, u)$ is a universal element of $X : C^{\text{op}} \to \text{Set}$.

The Yoneda lemma tells us how universal elements are related to representations of functors:

**Corollary 4.7.** If $X : C^{\text{op}} \to \text{Set}$ is a functor and $C$ is locally small, then there is a one-to-one correspondence:

\[
\begin{array}{c}
\{\text{Representations of } X\} \\
(A, \alpha) \text{ of } X
\end{array} \iff \begin{array}{c}
\{\text{Universal elements of } X\} \\
(A, u) \text{ of } X
\end{array}
\]

**Proof:**

The key idea to this corollary is that $(A, u)$ is a universal element if and only if the induced natural transformation $\bar{u} : H_A \Rightarrow X$ (à la Yoneda) is an isomorphism, hence producing a representation of $X$. Thus, we need only demonstrate this property of universal elements. Suppose that $\bar{u}$ is an isomorphism; then for each $B$ we have an isomorphism:

\[
\bar{u}_B : H_A(B) \xrightarrow{\sim} X(B) \\
f \mapsto X(f)(u)
\]

This is a bijection of sets, which means that for any $x \in X(B)$, there is a unique preimage under the above map. This is a map $f : B \to A$ such that $X(f)(u) = x$. Thus $(A, u)$ is a universal element of $X : C^{\text{op}} \to \text{Set}$. In fact, it is easy to see that this is true if and only if $\bar{u}$ is an isomorphism.
5 LIMITS AND COLIMITS

Isomorphisms of Representables

Recall the Yoneda embedding:

\[ H_\bullet : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}] \]
\[ A \mapsto H_A \]

As a consequence of the Yoneda lemma, we have \([\mathcal{C}^{\text{op}}, \text{Set}](H_A, H_{A'}) \cong \mathcal{C}(A, A')\). This implies that the Yoneda embedding is \textit{full and faithful} (i.e. it is injective and surjective on morphisms). In other words, this shows that \(\mathcal{C}\) is equivalent (or isomorphic) to the full subcategory of (representable) presheafs in \([\mathcal{C}^{\text{op}}, \text{Set}]\).

A property of full and faithful functors that we will not prove is that they preserve isomorphisms. This means that \(H_A \cong H_{A'} \iff A \cong A'\). This shows that, for example, if \(H_A \cong X \cong H_{A'}\), then \(A \cong A'\) and so representations are unique up to isomorphism.

**Example:** Recall the property of the tensor product:

\[ \text{Bilin}(M, N; W) \cong \text{Vect}_k(U \otimes V, W) \]

The tensor product of a pair of vector spaces can be used to represent the bilinear maps between them. By above, the tensor product must therefore be unique up to isomorphism (this can also be demonstrated using the universal property definition).

**Example:** Let \(G : \mathcal{B} \to \mathcal{A}\) be a functor, and let \(F, F'\) be left adjoints of \(G\). This means:

\[ \mathcal{B}(F(A), B) \cong \mathcal{A}(A, G(B)) \cong \mathcal{B}(F'(A), B) \]

This means that \(H^F(A) \cong H^{F'}(A)\), and so \(F(A) \cong F'(A)\). This demonstrates that left adjoints are unique up to isomorphism.

5 Limits and Colimits

Limits and colimits are among the most ubiquitous objects in mathematics. Even the most basic constructions, like the product of sets, can be framed in terms of limits and colimits, providing a better understanding of their fundamental properties and uniqueness.

**Limits**

**Definition 5.1.** If \(I\) is a small category, a functor \(D : I \to \mathcal{A}\) is called a diagram in \(\mathcal{A}\) with shape \(I\).

**Definition 5.2.** Given a category \(\mathcal{A}\) and a diagram \(D : I \to \mathcal{A}\), a cone with vertex \(A \in \mathcal{A}\) of shape \(I\) is a family of morphisms \(\psi_X : A \to D(X)\) for each \(X\) such that for any morphism \(f : X \to Y\), the following diagram commutes:

\[
\begin{array}{c}
D(Y) \\
\uparrow^\psi_Y \\
A \\
\downarrow^\psi_X \\
D(X)
\end{array}
\]

\[ D(f) \]

This diagram is called the \textit{commutation diagram} for \(\psi_X\).
Remark 5.3. Sometimes we are abusive and we refer to the whole cone by its tip, but implicitly we refer to both the tip and the morphisms from the tip.

We now define the category of cones with shape \( D : I \to \mathcal{A} \) (or just shape \( I \) if we’re lazy) to be the category whose objects are cones of shape \( D : I \to \mathcal{A} \) and whose morphisms are morphisms between the vertices of the cones.

Definition 5.4. The limit of a diagram \( D : I \to \mathcal{A} \) is the terminal object in the category of cones over \( D \). It is usually denoted \( \lim_I(D) \).

Remark 5.5. We can visualize a limit in the following way:

\[
\begin{tikzpicture}
  \node (X) at (0,0) {\bullet};
  \node (Y) at (2,0) {\bullet};
  \node (Z) at (1,2) {\bullet};
  \draw[->] (X) -- (Y);
  \draw[->,dotted] (X) -- (Z);
  \draw[->] (Y) -- (Z);
\end{tikzpicture}
\]

The ellipse represents the image of \( I \) inside \( \mathcal{A} \), and the morphism of the two cones is the dotted arrow between the vertices of the cone, which is induced by the terminal object property.

\section{Colimits}

As usual, prepending “co” dualizes the discussion.

Definition 5.6. If \( I \) is a small category, and \( D : I \to \mathcal{A} \) is a diagram, a cocone with vertex \( A \in \mathcal{A} \) of shape \( D \) is a family of morphisms \( \phi_X : D(X) \to A \) such that for any morphism \( f : X \to Y \), the following diagram commutes:

\[
\begin{tikzcd}
D(Y) \ar{d}[swap]{D(f)} \ar{r}{\phi_Y} & A \\
D(X) \ar{r}{\phi_X} \ar{u}{D(Y)} & \\
\end{tikzcd}
\]

Definition 5.7. If \( I \) is a small category, and \( D : I \to \mathcal{A} \) is a diagram, the colimit \( \text{colim}_I(D) \) of \( D \) is the limit of the diagram \( D^{\text{op}} : I^{\text{op}} \to \mathcal{A}^{\text{op}} \). More explicitly, it is the initial object in the category of cocones with shape \( D \).

\section{(Co)Products, (Co)Equalizers, Pullbacks and Pushouts}

We will demonstrate a few important examples of limits and colimits that are probably familiar.

\subsection{(Co)Products}

First, let \( I \) be the category with two objects: \( \bullet \bullet \). Any diagram \( D : \bullet \bullet \to \mathcal{A} \) specifies two objects \( X, Y \in \mathcal{A} \). The limit of \( D \) is called the product of \( X \) and \( Y \), denoted \( X \times Y \). Definitionally, it is the tip of the cone given by taking the limit of \( D \). More explicitly, it is the object \( X \times Y \) together with maps \( \pi_X, \pi_Y \) such that, for any \( A \in \mathcal{A} \) with maps \( A \to X, A \to Y \), there is a unique map \( A \to X \times Y \) such that the following
commutes:

\[
\begin{array}{ccc}
A & \to & X \times Y \\
\downarrow & & \downarrow_{\pi_X} \\
X & \to & Y
\end{array}
\]

This coincides with the usual definition of the product in categories like \( \text{Set} \) and \( \text{Vect}_k \).

The colimit of \( D \) is called the coproduct, denoted \( X \sqcup Y \), and satisfies the same universal property as above (but with arrows reversed). The coproduct takes various forms (like disjoint union in \( \text{Set} \) and direct sum in \( \text{Rmod} \)), but can sometimes coincide with the product.\(^2\)

(Co)Equalizers

Now consider the category \( I = \bullet \rightrightarrows \bullet \), sometimes called the equalizer category (for reasons which will be clear shortly). A diagram \( D : I \to \mathcal{A} \) produces two objects \( X,Y \in \mathcal{A} \) and two maps \( s,t : X \to Y \). The limit of \( D \) is (the cone whose tip is) \( E \), often called the equalizer. To be more explicit about what the equalizer is, we first define the notion of a fork:

**Definition 5.8.** A fork is a diagram \( A \xrightarrow{f} X \xleftarrow{s} Y \) such that \( s \circ f = t \circ f \).

Explicitly, the equalizer is the “handle” of a fork on \( X \rightrightarrows Y \); that is, object \( E \) with a map \( f : E \to X \) so that \( s \circ f = t \circ f \). It satisfies the universal property that, for any other fork handle \( g : A \to X \), there is a unique map \( A \to E \) such that the following commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & X \\
\downarrow & & \downarrow_{s} \\
A & \xrightarrow{g} & Y
\end{array}
\]

In the category of \( \text{Set} \), the equalizer is just the subset of \( X \) on which \( s \) and \( t \) agree. In the category of \( \text{Vect}_k \), it is the subspace \( \ker(s - t) \). Notice that taking \( t = 0 \) defines the kernel of a linear map \( s \). More generally, the kernel can be defined as the equalizer of a map and an appropriate notion of a “zero map” (we expand on this in the section on Abelian Categories).

The colimit of \( D \) is called the coequalizer, and satisfies a dual universal property. Just like the equalizer can be used to define the kernel, the coequalizer can be used to define the cokernel (otherwise known as the quotient).

Pullbacks and Pushouts

Let \( I \) be the category \( \bullet \rightrightarrows \bullet \). A diagram \( D : I \to \mathcal{A} \) specifies three objects \( X,Y,Z \in \mathcal{A} \) as well as maps from \( X,Y \to Z \). A cone with shape \( D \) is an object \( A \) with maps into \( X \) and \( Y \) (and an implicit map to \( Z \) given by composition). Pictorially:

\[
\begin{array}{ccc}
A & \to & X \\
\downarrow & & \downarrow \\
Y & \to & Z
\end{array}
\]

\[ \big\{ \text{Cone with tip } A \big\} \]

The limit of \( D \) is called the pullback of \( X \) and \( Y \) by \( Z \), and (its tip) is denoted \( X \times_Z Y \). This is the object for which, given any \( A \in \mathcal{A} \) and maps \( A \to X \) and \( A \to Y \), there exists a unique map \( A \to X \times_Z Y \) that makes everything commute.
In the category of \textit{Set}, the pullback has an easy interpretation. If we have set-maps $f : X \to Z$ and $g : Y \to Z$, the pullback is:

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

In the case where both $X$ and $Y$ are inside $Z$ (and $f$ and $g$ are given by inclusion), then the pullback is (isomorphic to) the intersection of the sets, $X \cap Y$.

The colimit of $D$ is called the pushout, denoted $X \sqcup_Z Y$, and satisfies a universal property dual to the one above. In the category of sets, this can be explicitly written as:

$$X \sqcup_Z Y = X \sqcup Y / \sim$$

Where $X \sqcup Y$ is the disjoint union, and $a \sim b$ if they share a preimage inside $Z$ under $f : Z \to X$ and $g : Z \to Y$.

In the case where $Z = X \cap Y$ and $f$ and $g$ are inclusion, the pushout is the union $X \cup Y$.

**Example:** The 2-sphere $S^2$ can be constructed using an equalizer. Let $\mathcal{A}$ be the category of topological spaces, and consider the equalizer of $\mathbb{R}^3 \xrightarrow{1} \mathbb{R}$, where $t(x, y, z) = x^2 + y^2 + z^2$ and $1(x, y, z) = 1$. Then the equalizer of this diagram is the unit sphere defined implicitly via coordinates:

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

$S^2$ can also be constructed in a coordinate-free manner. Let $D^2$ be the 2-disk, and consider the diagram $D^2 \leftarrow S^1 \rightarrow D^2$, where both maps are inclusion to the boundary of $D^2$. Then the pushout of this diagram is:

$$\begin{array}{ccc}
S^1 & \longrightarrow & D^2 \\
\downarrow & & \downarrow \\
D^2 & \longrightarrow & D^2 \sqcup_{S^1} D^2
\end{array}$$

The pushout “glues” two copies of $D^2$ together along an $S^1$ boundary, producing $S^2$.

**Example:** (products in $\text{Top}_*$) The product of pointed topological spaces $(X, x_0)$ and $(Y, y_0)$ is the familiar space $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ with basepoint $(x_0, y_0)$. The coproduct is the wedge sum $X \wedge Y$, which connects the two spaces at a common basepoint.

### 5.2 Existence of limits and colimits

The existence of a limit is not always guaranteed, as an arbitrary category isn’t guaranteed to have a terminal object. For example, in the category of fields, products and coproducts don’t exist because morphisms between fields must preserve the characteristic.

**Definition 5.9.** A category $\mathcal{A}$ has (co)limits of shape $I$ if for every diagram $D : I \to \mathcal{A}$ for $I$ small the (co)limit $(\text{co}) \lim_I(D)$ exists in $\mathcal{A}$.

**Definition 5.10.** A category $\mathcal{A}$ has all (co)limits if $\mathcal{A}$ has (co)limits for all small categories $I$. Such a category is said to be (co)complete.

**Theorem 5.11. (Existence of limits)** Suppose that $\mathcal{A}$ is a category in which all products and equalizers exist. Then $\mathcal{A}$ is complete.

**Proof:** (sketch)
The idea is that the limit of any diagram \( D : I \to A \) can be constructed using products and equalizers. Specifically, the limit \( \lim_I(D) \) is the equalizer of the following diagram:

\[
\begin{array}{ccc}
\lim_I(D) & \longrightarrow & \prod_{\bullet \in I} D(\bullet) \\
& & \searrow \downarrow s \\
& & \prod_{f \in \text{Hom}(I)} F(\text{cod}(f)) \\
& \nearrow \downarrow t & \\
& \lim_I(D) & \\
\end{array}
\]

Where \( \text{cod}(f) \) denotes the codomain of \( f \) and the morphisms \( s \) and \( t \) are defined by:

\[
s = D(f) \circ \pi_{F(\text{dom}(f))} \quad (f \in \text{Hom}(I))
\]

\[
t = \pi_{F(\text{cod}(f))} \quad (f \in \text{Hom}(I))
\]

The reader is encouraged to fill in the details.

**Example:** The pullback \( X \xrightarrow{f} Z \xleftarrow{g} Y \) is the equalizer of the diagram \( X \times Y \xrightarrow{s} Z \) where \( t = f \circ \pi_X \) and \( s = g \circ \pi_Y \). For this to exist, we need only to prove that the product \( X \times Y \) exists and the equalizer of \( X \times Y \) to \( Z \) exists.

Dually, we have:

**Theorem 5.12. (Existence of colimits)** Suppose that \( A \) is a category in which all coproducts and coequalizers exist. Then \( A \) is cocomplete.

**5.3 Limits as Representable Objects**

Suppose \( I \) is a small category and let \( A \) have all limits of shape \( I \). Given a diagram \( D : I \to A \), we might ask if the functor \( \text{Cone}(\_ , D) : A \to \text{Set} \) is representable; in other words, is there an object \( R \in A \) such that \( A(\_ , R) \cong \text{Cone}(\_ , D) \) in the functor category \( [A, \text{Set}] \)? It turns out that such an object exists and is exactly the limit \( \lim_I(D) \).

**Proposition 5.13.** Let \( I, A \) and \( D \) be as above. A pair \( (R, \alpha) \) is a representation of \( \text{Cone}(\_ , D) \) if and only if \( R = \lim_I(D) \).

**Proof:**

(\( \Rightarrow \)) Recall that a representation is an isomorphism \( \alpha : A(\_ , R) \to \text{Cone}(\_ , D) \). Since \( \alpha \) is a natural transformation, by Yoneda, there is a corresponding element \( x \in \text{Cone}(R, D) \). By Corollary 4.7, \( x \) is a universal element. By definition this means that for every \( A \in A \) and each \( y \in \text{Cone}(A, D) \), there exists a unique map \( \bar{y} : A \to R \) such that \( \text{Cone}(\bar{y}, D)(x) = y \). This means that there is a unique map between the tips of \( x \) and \( y \) making the cones commute, which is the defining property of the limit. Therefore \( R = \lim_I(D) \).

(\( \Leftarrow \)) The other direction follows from reversed reasoning, once again using Corollary 4.7.

**Remark 5.14.** As usual, the case of colimits is dual: colimits can be equivalently defined as corepresentations of \( \text{Cocone}(\_ , D) \).
5.4 Limits as Adjoints

Suppose once again that $I$ is locally small and let $A$ be a category having all limits of shape $I$. Another way to define limits is through adjunctions. Consider the functor $\lim_I(-) : [I, A] \to A$, which takes a diagram $D : I \to A$ to its limit. Does this have an adjoint? If so, what is it?

**Proposition 5.15.** $\lim_I(-) : [I, A] \to A$ is indeed a functor.

**Proof:**

First we have to define how $\lim_I(-)$ acts on morphisms in $[I, A]$ (natural transformations). Suppose $D, D' : I \to A$ are diagrams with limits $R$ and $R'$, respectively, and suppose $\alpha$ is a natural transformation between $D$ and $D'$. Then we can construct a cone over $D'$ with tip $R$ by post-composing with $\alpha$ as follows. For every $Q \in I$, the limit cone of $D$ with tip $R$ gives us maps $f : R \to D(Q)$, so we then have maps $\alpha(Q) \circ f : R \to D'(Q)$, which constitute a cone over $D'$. Then, by the universal property of the limit, we have a unique map between $R$ and $R'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
R & \xrightarrow{\alpha(Q)} & D' \\
\downarrow & & \downarrow \\
R' & \xrightarrow{} & D
\end{array}
$$

We encourage the reader to verify that this functor respects compositions and identity morphisms. \qed

We can also define the diagonal functor $\Delta : A \to [I, A]$, which sends an object $A \in A$ to the diagram $\Delta_A : I \to A$. This diagram sends every object in $I$ to $A$ and every morphism to $\text{id}_A$. This is the right adjoint to the limit functor:

**Proposition 5.16.** Let $I$ be small and $A$ have all limits of shape $I$. Then $\lim_I(-) : [I, A] \cong A : \Delta$ is an adjunction.

**Remark 5.17.** Recall that left and right adjoints are unique, so this Proposition allows us to equivalently define the limit of $D \in [I, A]$ as the evaluation of the left adjoint of $\Delta$ at $D$.

**Lemma 5.18.** For every $A \in A$ and $D : I \to A$, there is an isomorphism $\text{Cone}(A, D) \cong [I, A](\Delta_A, D)$ which is natural in $A$ and $D$.

**Proof:**

Note that specifying a cone with tip $A$ is amounts to specifying a collection of morphisms $f_i : A \to D(i)$ for $i \in I$ that commute through the cone. At the same time, a natural transformation $\alpha \in [I, A](\Delta_A, D)$ is a collection of morphisms $\alpha_i : \Delta_A(i) \to D(i)$ that commute with changing $i$; but since $\Delta_A(i) = A$, this collection defines a cone with tip $A$. Therefore, these sets are in bijection.

To verify naturality in $A$, let $f : B \to A$ be a morphism. Then we must verify that the following diagram
commutes:

\[
\begin{array}{ccc}
\text{Cone}(A, D) & \xrightarrow{f} & \text{Cone}(B, D) \\
\downarrow \cong & & \downarrow \cong \\
[I, \mathcal{A}](\Delta A, D) & \xrightarrow{\sim} & [I, \mathcal{A}](\Delta B, D)
\end{array}
\]

Here \( f \) precomposes a natural transformation with \( f \) so that it becomes an element of \([I, \mathcal{A}](\Delta B, D)\). The vertical isomorphisms act on a cone by “splitting” the tip into a collection of tips indexed by \( I \). This splitting action can be applied before or after precomposing with \( f \), so the diagram commutes. We leave it as an exercise to check naturality in \( D \).

\[ \square \]

**Proof (of Proposition 5.16):**

Let \( A \in \mathcal{A} \) and \( D : I \to \mathcal{A} \) be a diagram. Recall from Proposition 5.13 that \( \mathcal{A}(\cdot, \lim_I D) \cong \text{Cone}(\cdot, D) \), which means there is a natural transformation \( \alpha \) that is an isomorphism when restricted to every object in \( \mathcal{A} \). Therefore \( \alpha(A) : \mathcal{A}(A, \lim_I D) \to \text{Cone}(A, D) \) is a natural isomorphism. By the previous Lemma, we then have:

\[
[I, \mathcal{A}](\Delta A, D) \cong \text{Cone}(A, D) \cong \mathcal{A}(A, \lim_I D)
\]

Further, these are natural isomorphisms. Therefore \( \lim_I (\cdot) \) and \( \Delta \) are adjoints.

\[ \square \]

### 5.5 Preserving Limits and GAFT

Much like with classical limits in calculus and analysis, a central question about categorical limits is if they commute with various operations. For example, for what types of functors \( F \) does \( F(\lim_I D) \) equal \( \lim_I (F \circ D) \)? In this section we’ll address prove two classes of functors for which this is true, as well as describe a partial converse to one of these.
6 Abelian Categories