Linear Systems: REDUCED ROW ECHELON FORM

From both a conceptual and computational point of view, the trouble with using the echelon form to describe properties of a matrix A is that A can be equivalent to several different echelon forms because rescaling a row preserves the echelon form - in other words, there's no unique echelon form for A. This leads us to introduce the next

> **Definition:** a matrix is said to be in **Reduced Row Echelon Form** if it is in echelon form and
>
> - the leading entry in each non-zero row is 1,
> - each leading 1 is the only non-zero entry in its column.

A typical structure for a matrix in Reduced Row Echelon Form is thus

$$\begin{bmatrix} 1 & * & 0 & 0 & * & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & 0 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that this matrix is still in echelon form but each pivot value is 1, and all the entries in a pivot column are 0 except for the pivot itself. As the pivot values cannot now be rescaled, however, the next result should come as no surprise:

> **Main Reduced Row Echelon Theorem:** each matrix is row equivalent to one and only one reduced row echelon matrix. This unique reduced row echelon matrix associated with a matrix A is usually denoted by $\mathrm{rref}(A)$.

Uniqueness of the reduced row echelon form is a property we'll make fundamental use of as the semester progresses because so many concepts and properties of a matrix A can then be described in terms of $\mathrm{rref}(A)$. But first let's investigate how the presence of the 1 and 0's in the pivot column affects the Gauss Elimination method for solving three particular systems of linear equations in 3 variables.

I. Return to the system

$$\begin{aligned} y + z &= 4\,, \\ 3x + 6y - 3z &= 3\,, \\ -2x - 3y + 7z &= 10\,, \end{aligned}$$

from Lecture 1. Its associated augmented matrix was

$$A = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{bmatrix},$$

and elementary row operations showed that

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \;(= B, \text{say},)$$

associated with the equivalent system

$$\begin{aligned} x + 2y - z &= 1\,, \\ y + z &= 4\,, \\ 4z &= 8\,. \end{aligned}$$

But then by elementary row operations acting upwards:

$$B \xrightarrow{\frac{1}{4}R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Consequently,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

which is the augmented matrix associated with the system

$$x = -1\,, \quad y = 2\,, \quad z = 2\,.$$

These solutions are exactly the same as before, of course, because elementary row operations produce equivalent systems. The point is that the 1's and 0 in the pivot columns eliminate the need for back substitution.

This refinement using the the Reduced Row Echelon Form of the Augmented matrix instead of the Echelon Form in Gaussian Elimination is usually called *Gauss-Jordan Elimination* after the German mathematician Wilhelm Jordan who used it extensively in his writings.

II. Solve for x, y and z in

$$\begin{aligned} y + z &= 4\,, \\ 3x + 6y - 3z &= 3\,, \\ -2x - 3y + 3z &= 10\,. \end{aligned}$$

The associated augmented matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 3 & 10 \end{bmatrix},$$

and

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

associated with the equivalent system

$$\begin{aligned} x \qquad - 3z &= 0\,, \\ y + z &= 0\,, \\ 0z &= 1\,. \end{aligned}$$

But there is no value of z such that $0z = 1$, so the system is *Inconsistent*. We can see this from $\text{rref}(A)$: the augmented matrix A still has three pivot columns, but the column of right hand sides is one of them!!

III. Solve for x, y and z in

$$\begin{aligned} y + z &= 4\,, \\ 3x + 6y - 3z &= 3\,, \\ -2x - 3y + 3z &= 2\,. \end{aligned}$$

The associated augmented matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 3 & 2 \end{bmatrix},$$

and

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

associated with the equivalent system

$$\begin{aligned} x \qquad - 3z &= 7\,, \\ y + z &= 4\,. \end{aligned}$$

There is no equation to specify z, so z is a *free variable*:

$$x = 7 + 3t\,, \quad y = 4 - t\,, \quad z = t\,, \quad (t \text{ arbitrary}).$$

This can also be seen from $\text{rref}(A)$: only the x- and y-columns are pivot columns.

The procedure just gone through provides an algorithm for solving a general system of m linear equations in n variables:

- form the associated augmented matrix A and compute $\text{rref}(A)$.
- If the column of right hand sides is a pivot column of A, then the system is inconsistent, otherwise

the system is consistent.

- If the system is consistent, then any variable corresponding to a pivot column is called a **basic variable**, otherwise the variable is called a **free variable**.

Your Turn Now: consider the coefficient matrix for systems I, II, and III.

- Compute the reduced row echelon form of each coefficient matrix.
- How do these differ from the reduced row echelon matrix of the associated augmented matrix?
- Does the number of pivots change? When can you use the reduced row echelon form of each coefficient matrix to solve a system of linear equations?