Homework 2: Due February 6

The statement we want to prove is the following (the same as part (a) of Problem 1.5.13 in the text).

Problem 3. Let \( \pi \) be an arbitrary element of Sym\(_X\) with \( \pi \neq I \), and let \( x_0 \) be some element of \( X \) such that \( \pi(x_0) \neq x_0 \). Set

\[
x_1 = \pi(x_0), \quad x_2 = \pi(x_1), \quad x_3 = \pi(x_2), \quad \ldots .
\]

Show that there is a number \( k \) such that \( x_0, x_1, x_2, \ldots, x_k \) are all distinct and \( \pi(x_k) = x_0 \).

The text gives a good hint about how to do it. I’m going to talk about the proof - your job is to try to make sense of what I write and to put it into your own words. First define \( x_{j+1} \) in \( X \) by \( x_{j+1} = \pi(x_j) \). Since we have assumed that \( \pi(x_0) \neq x_0 \), this means that the set \( \{x_0, x_1\} \) consists of distinct elements. But what can we say about the set \( \{x_0, x_1, x_2, \ldots, x_N\} \) for some large value of \( N \)? Now \( X \) is a finite set, so the elements \( x_0, x_1, x_2, \ldots, x_N \) can’t be distinct for all values of \( N \); for instance, they can’t all be distinct whenever \( N \) is greater than the number of elements in \( X \), though it is true when \( N = 1 \). Thus there must be a smallest value of \( N \) for which the elements \( x_0, x_1, x_2, \ldots, x_N \) are all distinct; call this smallest value \( k \). Then \( \pi(x_k) \) must be one of \( x_0, x_1, x_2, \ldots, x_k \). Suppose \( \pi(x_k) = x_j \). We have to show that \( j = 0 \). Well, what if it isn’t! Then

\[
\pi(x_k) = x_j = \pi(x_{j-1})
\]

for some \( j \geq 1 \). In this case however,

\[
x_k = \pi^{-1}(\pi(x_k)) = \pi^{-1}(\pi(x_{j-1})) = x_{j-1},
\]

which can’t be true because the elements \( x_0, x_1, x_2, \ldots, x_k \) are all distinct.

This homework is based in part on section 1.5 in the text, so it’s a good idea to begin by reading that section. Now let’s use the ideas and notation discussed in class. Let \( X \) be a finite set and Sym\(_X\) denote the group of all permutations of \( X \), i.e. the set of all 1-1 functions \( \pi : X \to X \) from \( X \) onto \( X \) with binary operation the usual composition of functions. When \( X = \{1, 2, \ldots, n\} \), we often write \( S_n \) for Sym\(_X\).

The fundamental structure theorem for Sym\(_X\) is the following
Theorem 1. Every element of $\text{Sym}_X$ can be written as a product of disjoint cycles.

One of the crucial parts of this homework is to supply a proof of this result. The full theorem states that the decomposition is unique up to the order of the disjoint cycles.

Problem 1. (Problem 1.5.3 in text) Work out the decomposition into disjoint cycles for

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 3 & 7 & 4 & 1 \end{pmatrix}$

(b) (12)(1234)

c) (12)(234)

d) (12)(23)(34)

e) (13)(1234)(13)

Problem 2. (Problem 1.5.6 in text) Compute the inverse of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 6 & 3 & 7 & 4 & 1 \end{pmatrix},$$

expressing your answer in two-line notation.

The next problem, Problem 1.5.13 in the text, is a basic step in the proof of the basic Theorem 1.

Problem 3. Let $\pi$ be an arbitrary element of $\text{Sym}_X$ with $\pi \neq I$, and let $x_0$ be some element of $X$ such that $\pi(x_0) \neq x_0$. Set

$$x_1 = \pi(x_0), \ x_2 = \pi(x_1), \ x_3 = \pi(x_2), \ \ldots.$$ 

Show that there is a number $k$ such that $x_0, x_1, x_2, \ldots, x_k$ are all distinct and $\pi(x_k) = x_0$.

Problem 4. Use the result in Problem 3 to prove Theorem 1.

Now use Theorem 1. to give a proof of the next basic structure result for $\text{Sym}_X$. 
Problem 5. Every element of $\text{Sym}_X$ can be written as a product of 2-cycles.

Finally, let’s use the result in Problem 5. to describe the group $\text{Sym}_{\text{cube}}$ of all rotational symmetries of a cube. To facilitate this we have partitioned the cube into 8 congruent sub-cubes, exactly as in a $2 \times 2$ Rubik cube except that the sub-cubes have been colored differently. Let’s call this variant of the Rubik cube a SymCUBE as shown below.

![SymCUBE, front from above](image1)

![SymCUBE, front from below](image2)

Let $X = \{R, G, Y, B\}$ where we shall identify $R$ with red, $G$ with green, $Y$ with yellow, and $B$ with blue. In the SymCUBE itself, we can identify the red blocks with one diagonal of the cube, and so on. Every $g$ in $\text{Sym}_{\text{cube}}$ then determines a permutation of the four colors; for instance, the rotation $g$ clockwise through $90^\circ$ about the center of the top face determines the permutation

$$g \sim \begin{pmatrix} R & G & Y & B \\ G & Y & B & R \end{pmatrix}.$$

Problem 6. Determine which elements of $\text{Sym}_X$, $X = \{R, G, Y, B\}$, correspond to

(i) rotation through $120^\circ$ about the diagonal determined by the red sub-cubes;

(ii) rotation through $180^\circ$ about the line joining the midpoints of the edges of the cube determined by adjacent red and blue cubes;

(iii) use your result from part (ii) to list all 2-cycles in $\text{Sym}_X$, $X = \{R, G, Y, B\}$, determined by rotations of the SymCube.
Problem 7. Use part (iii) of Problem 6 together with the basic decomposition result in Problem 5 for $\text{Sym}_X$, $X = \{R, G, Y, B\}$, to show that the group $\text{Sym}_{\text{cube}}$ of all rotational symmetries of a cube can be identified with $\text{Sym}_X$, $X = \{R, G, Y, B\}$. 