In class we defined the important notion of Normal Subgroup. If \( H \) is a subgroup of a group \( G \), then \( H \) is said to be Normal in \( G \) when \( xH = Hx \) for all \( x \) in \( G \). To show that \( H \) is normal in \( G \), therefore, we have to show that \( xH = Hx \) for all \( x \) in \( G \), whereas to show that \( H \) is NOT normal in \( G \) all we have to do is show that \( xH \neq Hx \) for some \( x \) in \( G \). So let’s look at a few examples. Clearly, every subgroup of an abelian group is normal, but if \( G \) is not abelian the situation is more complicated.

**Problem 1:** If \( G = \{I, R, R^2, R^3, S, SR, SR^2, SR^3\} \) is the symmetry group of a square, show that the group \( H = \{I, S\} \) is not normal in \( G \), but \( H = \{I, R, R^2, R^3\} \) is normal in \( G \).

**Problem 2:** If

\[
G = \{I, R, R^2, R^3, R^4, R^5, R^6, R^7, S, SR, SR^2, SR^3, SR^4, SR^5, SR^6, SR^7\}
\]

is the symmetry group of a regular octagon centered at the origin where \( R \) is rotation through \( \pi/4 \) about the origin and \( S \) is reflection about the \( x \) axis, show that the group \( H = \{I, R^2, R^4, R^6\} \) is normal in \( G \).
As we also saw in class, if $H$ is normal in $G$, then the set of all equivalence classes with respect to the equivalence relation
\[ y \equiv_H x \quad \text{if and only if} \quad y^{-1}x \in H \]
becomes a group, often called a Factor Group of $G$, and denoted by $G/H$, when the group operation of these equivalence classes is defined by
\[ Hx \circ Hy = Hxy. \]
It’s an interesting exercise to identify $G/H$ for given $G$ and $H$. For instance, in class we saw that $G/H$ can be identified with the group $\mathbb{Z}_n$ of integers with addition mod $n$ as group operation when
\[ G = \mathbb{Z}, \quad H = n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}. \]
Recall now the group
\[ K_4 = \{I, A, B, AB\}, \quad A^2 = B^2 = I, \]
only called the Klein-4 group.

**Problem 3:** Show that $K_4$ can be identified with the symmetry group of a rectangle by identifying $A$ and $B$ as transformations of the rectangle.

**Problem 4:** Show that $H = \{I, R^2, R^4, R^6\}$ can be identified with the group of symmetries of the colored octagon

![Colored Octagon](image)

**Problem 5:** Show that when $G$ is the group of symmetries of a mono-chromatic regular octagon and $H = \{I, R^2, R^4, R^6\}$ is its subgroup preserving the four colors in the corresponding colored regular octagon given immediately above, then the factor group $G/H$ can be identified with the Klein-4 group.

**Hint:** Try to express $A, B$ in terms of $S, R$ and $H$ using the four colors as a guide, by coloring a rectangle with the four colors).