HOMEWORK 3: DUE OCTOBER 10

In class we saw how the field of complex number could be realized as a subset of the ring $\mathbb{R}^{2 \times 2}$ of $2 \times 2$ matrices having real entries. Similar, more general, constructions using the ring $\mathbb{C}^{2 \times 2}$ of $2 \times 2$ matrices having complex entries are important in both mathematics and physics. The following problems give some indication of how the algebraic structure of matrices can be used.

The matrices
\[
\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
are called Pauli matrices after the famous physicist who introduced them in his work on quantum mechanics (here and from now on, $i$ will denote $\sqrt{-1}$); the associated Pauli matrices are defined by
\[
\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

**Question 1.** Show that
\[
\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I, \quad e_0^2 = I, \quad e_1^2 = e_2^2 = e_3^2 = -I
\]
where $I$ is the $2 \times 2$ Identity matrix. Show also that
\[
\sigma_j \sigma_k = -i\sigma_\ell, \quad e_j e_k = e_\ell
\]
when \{j, k, \ell\} is a cyclic permutation of \{1, 2, 3\}.

In class we saw that the family of all matrices
\[
z = a\sigma_0 + b\sigma_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a, b \in \mathbb{R}
\]
could be identified with the field $\mathbb{C}$ of all complex numbers \{a+bi : a, b \in \mathbb{R}\}; in particular, the determinant
\[
\det \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a^2 + b^2
\]
of the matrix could be identified with $z\overline{z}$ and hence with $|z|^2$. In the nineteenth century the Irish mathematician Hamilton along with a number of other mathematicians tried to extend the complex field $\mathbb{C}$, a vector space over the real field $\mathbb{R}$ of dimension 2 to a three-dimensional vector space over $\mathbb{R}$. He was unsuccessful for many years until he realized that he actually needed to look for a FOUR-dimensional vector space. To see this, first observe that each of $e_1, e_2, e_3$ can be thought of a ‘square root’ of $-1$. Now denote by $\mathbb{H}$ the family

$$x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad x_0, x_1, x_2, x_3 \in \mathbb{R}.$$  

Because of the four variables, the elements of $\mathbb{H}$ are known as quaternions.

**Question 2.** Show that $\mathbb{H}$ can realized as the family of all $2 \times 2$ matrices

$$\begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{bmatrix}, \quad z_1, z_2 \in \mathbb{C}$$

of $2 \times 2$ matrices having complex entries and that

$$\det \begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{bmatrix} = |z_1|^2 + |z_2|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$  

**Question 3.** (i) Show that $\mathbb{H}$ is a ring under the usual addition and multiplication of matrices having $e_0$ as multiplicative identity.

(ii) Show that each non-zero element of $\mathbb{H}$ has a multiplicative inverse.

(iii) Why is $\mathbb{H}$ not a field despite properties (i) and (ii)?

One reason why quaternions are important is because of connections to the usual dot and vector products of vectors in 3-space. Note first that in complete analogy with the case of complex numbers we call

$$\mathcal{R}(x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) = x_0e_0,$$

$$\mathcal{I}(x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) = x_1e_1 + x_2e_2 + x_3$$

the respective *Real* and *Imaginary* parts of a quaternion. Now recall from calculus the definition of the *dot product*

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_2v_2$$

and *vector product*

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

of vectors

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

in 3-space, where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the usual unit vectors in 3-space.
Question 4. Instead of identifying the triples \((u_1, u_2, u_3), (v_1, v_2, v_3)\) with vectors we can identify them with matrices

\[
\mathbf{u} = (u_1, u_2, u_3) \sim u_1 e_1 + u_2 e_2 + u_3 e_3 = \begin{bmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{bmatrix}
\]

and

\[
\mathbf{v} = (v_1, v_2, v_3) \sim v_1 e_1 + v_2 e_2 + v_3 e_3 = \begin{bmatrix} iv_1 & v_2 + iv_3 \\ -v_2 + iv_3 & -iv_1 \end{bmatrix},
\]

i.e., as purely imaginary elements in \(\mathbb{H}\). Express the product of \(\mathbf{u}, \mathbf{v}\) in \(\mathbb{H}\) in terms of the usual dot and vector products of vectors.

The first four questions have illustrated how we can use the algebraic structure of matrices to provide different realizations of familiar ideas from 3-space, from Euclidean 3-space that is. It should not surprising, therefore, to find that we can realize the sort of spaces encountered in physics - Minkowski space - in terms of matrices. This is what the Pauli matrices are for. Recall that the Euclidean length on 3-space is defined by

\[
\| (x_1, x_2, x_3) \| = \sqrt{x_1^2 + x_2^2 + x_3^2}
\]

with obvious modifications for arbitrary dimension. Since the sum of squares is always non-negative, the length of \((x_1, x_2, x_3)\) is always well-defined; in particular, the two-dimensional sphere

\[
\Sigma_2 = \left\{ \mathbf{x} = (x_1, x_2, x_3) : \|\mathbf{x}\|^2 = 1 \right\}
\]

makes good sense. The geometry associated with \(\Sigma_2\) is standard Spherical Geometry. By contrast, space-time in physics is the four-dimensional space in which the length of a four vector \((x_1, x_2, x_3, t)\) is defined by

\[
\| (x_1, x_2, x_3, t) \|^2 = x_1^2 + x_2^2 + x_3^2 - t^2;
\]

notice that with this notion of length \(\| (x_1, x_2, x_3, t) \|^2\) may be negative. For obvious reasons we’ll denote the vector space of all such vectors by \(\mathbb{R}^{3,1}\).

Question 5. After identifying the four vector \((x_1, x_2, x_3, t)\) with the matrix

\[
\mathbf{x} = t\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,
\]

express the determinant of \(\mathbf{x}\) in terms of the Minkowski length \((\ddagger)\).

Now we can look at several different surfaces:

(i) three-dimensional hyperbolic space

\[
\left\{ \mathbf{x} : \|\mathbf{x}\|^2 = 1 \right\},
\]
(ii) three-dimensional light cone

\[ \{ x : \|x\|^2 = 0 \}, \]

(iii) three-dimensional imaginary hyperbolic space

\[ \{ x : \|x\|^2 = -1 \}, \]

each having a geometry. But the real significance of having Pauli matrices comes from the ability to factor differential operators, and this is where the famous physicist P. A. M. Dirac gets involved. Define a differential operator \( \mathcal{D} \), the famous Dirac Operator, by

\[ \mathcal{D} = \sigma_0 \frac{\partial}{\partial t} + \sigma_1 \frac{\partial}{\partial t} + \sigma_2 \frac{\partial}{\partial t} + \sigma_3 \frac{\partial}{\partial t} \]

Then \( \mathcal{D}f \) makes good sense when \( f \) takes values in \( \mathbb{R}^{3,1} \). What’s really important about \( \mathcal{D} \) is expressed in the next problem.

**Question 5.** Express the second order operator \( \mathcal{D}^2 f = \mathcal{D}(\mathcal{D}f) \) in terms of second order derivatives of a function \( f \) taking values in \( \mathbb{R}^{3,1} \). (You should end up with what’s known as the wave operator. What’s important is the idea that the Dirac Operator factors this wave operator. It’s all connected with the spin of an electron.)

Clearly this homework touches, but only touches, on a large number of ideas from geometry, calculus and physics. The point is to suggest that algebraic structures are at the heart of mathematics and of its applications. In this regard, don’t forget that the previous homework related number theory and group theory to encryption, a totally different application.