Notes and Comments

Part I, section 2. A. The introduction of the affine group and the operators $\{\pi(z) : z \in \mathbb{C}_+\}$ in theorem (2.17) are the first indication that representation-theoretic ideas lie at the heart of the mathematical theory of wavelets, as Grossman realized very early on. Of course this is very natural since such ideas lie at the heart of Fourier analysis also, but the theory of wavelets as developed in these notes avoided emphasizing these representation-theoretic ideas to keep the presentation reasonably simple. Nonetheless, any serious study of the *theory* of wavelets should take account of the relevant group theory. So we shall begin to present some of the details here, and many more will be presented in the notes and comments on subsequent sections.

Let G be a locally compact group and $(\mathcal{H}, \pi)$ a continuous representation of G on some Hilbert space $\mathcal{H}$. Such a representation will be said to be *unitary* if each operator $\pi(g)$ is a unitary operator on $\mathcal{H}$, i.e., $\pi^*(g) = \pi(g)^{-1}$ where $\pi^*(g)$ is the adjoint of $\pi(g)$ and $\pi(g)^{-1}$ is its inverse. But for the moment, we shall assume only that $(\mathcal{H}, \pi)$ is continuous, not necessarily unitary. Then for each fixed ξ, η in $\mathcal{H}$

$$F_{\xi,\eta}(g) = \big(\xi, \pi^*(g)\eta\big) \qquad (g \in G) \tag{A.1}$$

is a continuous function on G since $(\mathcal{H}, \pi)$ is a continuous representation. In particular, for each fixed η in $\mathcal{H}$,

$$\xi \to F_{\xi,\eta} \qquad , \qquad \mathcal{H} \to \mathcal{C}(G) \ , \tag{A.2}$$

defines a mapping from $\mathcal{H}$ into the space, $\mathcal{C}(G)$, of all continuous complex-valued functions on G. On the other hand, the *right regular* representation $(\mathcal{C}(G), \rho)$ of G on $\mathcal{C}(G)$ is defined by

$$\big(\rho(g)f\big)(x) = f(xg) \qquad (x, g \in G) \ . \tag{A.3}$$

The defining condition (A.1) is chosen precisely so that

$$\big(\rho(g)F_{\xi,\eta}\big)(x) = F(xg) = \big(\xi, \pi^*(xg)\eta\big) = \big(\xi, \pi^*(g)\pi^*(x)\eta\big) = \big(\pi(g)\xi, \pi^*(x)\eta\big) \ ,$$

i.e., so that

$$\rho(g)F_{\xi,\eta} = F_{\pi(g)\xi,\eta} \ . \tag{A.4}$$

In technical terms, the mapping $\xi \to F_{\xi,\eta}$ is thus said to be *intertwining* or G-*equivariant* because it intertwines the representation π of G on $\mathcal{H}$ and the representation ρ of G on $\mathcal{C}(G)$. Later we shall see that this mapping is itself only a special case of a very important family of mappings known as *Cauchy-Szegö transforms* or τ-*quotient* mappings.

B. Affine Group. As a first illustration of the representation-theoretic approach to wavelet-type decompositions we consider the example of the (connected) group G of affine transformations of $\mathbb{R}^n$. Algebraically, this group is given by the semi-direct product

$$G = \mathbb{R}^n \ \circledS \ (0,\infty) = \big\{(x,s) : x \in \mathbb{R}^n \ , \ 0 < s < \infty\big\} \tag{B.1}$$

with group multiplication

$$(x,s)(y,t) = (tx + y, st) \ . \tag{B.2}$$

Such a choice of group operation ensures that

$$g = (x,s) : v \to v.g = sv + x \qquad (v \in \mathbb{R}^n) \tag{B.3}$$

defines a continuous transformation group on $\mathbb{R}^n$, the group generated by translations and dilations; indeed, if $g_1 = (x,s)$ and $g_2 = (y,t)$, then

$$\begin{aligned}(v.g_1).g_2 &= (sv + x).g_2 = t(sv + x) + y \\ &= stv + tx + y = v.(g_1g_2) \ .\end{aligned}$$

The induced action

$$\big(U(g)f\big)(v) = f(v.g) \qquad (v \in \mathbb{R}^n) \tag{B.4}$$

on the space $\mathcal{C}(\mathbb{R}^n)$ thus defines a continuous representation of G since

$$\begin{aligned}\Big(U(g_1)\big(U(g_2)f\big)\Big)(v) &= \big(U(g_2)f\big)(v.g_1) \\ &= f\big((v.g_1).g_2\big) = f(v.g_1g_2) = \big(U(g_1g_2)f\big)(v) \ .\end{aligned}$$

More generally, for each fixed α in $\mathbb{C}$,

$$(\text{B.5}) \qquad \big(\pi_\alpha(g)f\big)(v) = s^{\frac{1}{2}n} f(v.g) \qquad (v \in \mathbb{R}^n)$$

also gives a continuous representation of G on $\mathcal{L}^2(\mathbb{R}^n)$; in fact, this is a continuous unitary representation of G which will be basic to the construction of pre-historic wavelets. As preparation for this we establish some important properties of the affine group and its representation on $L^2(\mathbb{R}^n)$.

First, it is often convenient when tying analysis on the affine group G to Euclidean Fourier analysis to identity G with the upper half-space

$$(\text{B.6})(\text{i}) \qquad \mathbb{R}^{n+1}_+ = \big\{z : z = (x,s)\ ,\ x \in \mathbb{R}^n\ ,\ s > 0\big\}$$

and then identity $\mathbb{R}^n$ with the 'boundary' of $\mathbb{R}^{n+1}_+$. In this way, (B.2) imposes a group structure

$$(\text{B.6})(\text{ii}) \qquad z.w = (x,s).(y,t) = (tx+s, st) \qquad (z, w \in \mathbb{R}^{n+1}_+)$$

on $\mathbb{R}^{n+1}_+$; equivalently, we can (and shall) use (B.2) to define an action of G both as a left transformation group

$$(\text{B.7})(\text{i}) \qquad g : z \longrightarrow g.z = (sa + x, sb) \qquad \big(g = (a,b)\big)$$

and a right transformation group

$$(\text{B.7})(\text{ii}) \qquad g : z \longrightarrow z.g = (bx + a, sb) \qquad \big(g = (a,b)\big)$$

on $\mathbb{R}^{n+1}_+$. The left and right G-invariant integrals on $\mathbb{R}^{n+1}_+$ can be readily identified also.

(B.8) THEOREM. *Under the action of the affine group as both a left and right transformation group on* $\mathbb{R}^{n+1}_+$,

$$(\text{i}) \qquad \int_{\mathbb{R}^n} \int_0^\infty f\big((x,s).g\big)\, \frac{ds\,dx}{s^{n+1}} = \int_{\mathbb{R}^n} \int_0^\infty f(x,s)\, \frac{ds\,dx}{s^{n+1}}$$

and

(ii) $$\int_{\mathbb{R}^n}\int_0^\infty f\big(g.(x,s)\big)\,\frac{ds\,dx}{s} = \int_{\mathbb{R}^n}\int_0^\infty f(x,s)\,\frac{ds\,dx}{s}$$

for all compactly supported continuous functions for $\mathbb{R}^{n+1}_+$ and all g in G.

Proof. In view of (B.7)(ii),

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_0^\infty f\big((x,s).g\big)\,\frac{ds\,dx}{s^{n+1}} &= \int_{\mathbb{R}^n}\int_0^\infty f(bx+a,sb)\,\frac{ds\,dx}{s^{n+1}}\\
&= b^n\int_0^\infty\int_{\mathbb{R}^n} f(bx+a,s)\,\frac{dx\,ds}{s^{n+1}} = \int_0^\infty\int_{\mathbb{R}^n} f(x+a,s)\,\frac{dx\,ds}{s^{n+1}}\\
&= \int_0^\infty\int_{\mathbb{R}^n} f(x,s)\,\frac{dx\,ds}{s^{n+1}} = \int_{\mathbb{R}^n}\int_0^\infty f(x,s)\,\frac{ds\,dx}{s^{n+1}}\ ,
\end{aligned}$$

using the dilation and invariance properties of Lebesgue measure on Euclidean space. This establishes (i).

(ii) On the other hand, by (B.7)(i),

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_0^\infty f\big(g.(x,s)\big)\,\frac{ds\,dx}{s} &= \int_0^\infty\int_{\mathbb{R}^n} f(sa+x,sb)\,\frac{dx\,ds}{s}\\
&= \int_0^\infty\int_{\mathbb{R}^n} f(x,sb)\,\frac{dx\,ds}{s} = \int_{\mathbb{R}^n}\int_0^\infty f(x,sb)\,\frac{dx\,ds}{s}\ ,
\end{aligned}$$

from which property (ii) follows easily. ■

Theorem (B.8) shows that the affine is a non-unimodular group in the sense that its left and right group invariant integrals are different.

C. Choose a family $\{\phi_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$. In (1.19) we said that this family was stable if

(C.1) $$A\Big(\sum_i|\alpha_i|^2\Big)^{1/2} \le \Big\|\sum_i\alpha_i\phi_i\Big\|_{\mathcal{H}} \le B\Big(\sum_i|\alpha_i|^2\Big)^{1/2}\ ;$$

in particular, $T:\{\alpha_i\}\to\sum_i\alpha_i\phi_i$ is a bounded operator from $\ell^2(I)$ into $\mathcal{H}$. On the other hand, in (2.21) we established the inequalities

(C.2) $$A\|f\|_{\mathcal{H}} \le \Big(\sum_i|(f,\phi_i)|^2\Big)^{1/2} \le B\|f\|_{\mathcal{H}}$$

assuming only that T was a bounded operator from $\ell^2(I)$ onto $\mathcal{H}$. In the literature the family $\{\phi_i\}$ is said to be a frame when (C.2) holds. To reconcile this definition with (2.20) we prove the following result.

(C.3) THEOREM. *Let* $\{\phi_i\}_{i\in I}$ *be a family in* $\mathcal{H}$ *satisfying* (C.2). *Then* $T : \{\alpha_i\} \to \sum_i \alpha_i\phi_i$ *defines a bounded operator from* $\ell^2(I)$ *onto* $\mathcal{H}$.

Proof. Since

$$|(T\,\boldsymbol{\alpha}, f)| = \Big|\sum_i \alpha_i(\phi_i, f)\Big| \le \|\,\boldsymbol{\alpha}\,\|_{\ell^2}\Big(\sum_i |(f,\phi_i)|^2\Big)^{1/2} ,$$

the right hand inequality in (C.2) ensures that T is bounded with operator norm at most B; in addition, the adjoint

$$T^* : f \longrightarrow \big\{(f,\phi_i)\big\}$$

of T will be bounded from $\mathcal{H}$ into $\ell^2(I)$. But the left hand inequality in (C.2) ensures that

$$\begin{aligned} A^2\|f\|^2 \le \sum_i |(f,\phi_i)|^2 &= (T^*f, T^*f) \\ &= (TT^*f, f) \le \|TT^*f\|\;\|f\| \,. \end{aligned}$$

Consequently, (C.2) ensures that

$$A^2\|f\|_{\mathcal{H}} \le \|TT^*f\|_{\mathcal{H}} \le B^2\|f\|_{\mathcal{H}} ,$$

i.e., TT^* is a positive bounded operator on $\mathcal{H}$. By the spectral theorem therefore, the inverse $(TT^*)^{-1}$ of TT^* also will be a bounded operator on $\mathcal{H}$. That T maps $\ell^2(I)$ onto $\mathcal{H}$ now follows easily: given f in $\mathcal{H}$, set $\alpha = T^*(TT^*)^{-1}f$. Then

$$T\alpha = (TT^*)(TT^*)^{-1}f = f ,$$

completing the proof. ∎

The proof above also helps bring out the distinction between stable families and frames.

(C.4) THEOREM. *Let* $T : \{\alpha_i\} \to \sum_i \alpha_i\phi_i$ *be a bounded linear operator from* $\ell^2(I)$ *into* $\mathcal{H}$. *Then*

(i) $\{\phi_i\}_{i\in I}$ *is a stable family if and only if* T^*T *is invertible on* $\ell^2(I)$,

(ii) $\{\phi_i\}_{i\in I}$ *is a frame if and only if* TT^* *is invertible on* $\mathcal{H}$.

Proof. (i) Condition (C.1) for stability is equivalent to the inequalities

$$A^2 I_{\ell^2} \leq T^*T \leq B^2 I_{\ell^2} \tag{$*$}$$

where I_{ℓ^2} is the identity operator on $\ell^2(I)$. But T^*T is a non-negative hermitian operator, so by the spectral theorem for hermitian operators, property $(*)$ will hold if and only if the spectrum of T^*T lies in a closed interval $[A^2, B^2]$ in $(0, \infty)$, i.e., if and only if T^*T is invertible on $\ell^2(I)$.

(ii) Condition (C.2) for frames is equivalent to the inequalities

$$A^2 I_{\mathcal{H}} \leq TT^* \leq B^2 I_{\mathcal{H}}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Arguing as part (i) we see that $\{\phi_i\}_{i\in I}$ will be a frame if and only if the spectrum of TT^* lies in the closed interval $[A^2, B^2]$ in $(0, \infty)$, i.e., if and only if TT^* is invertible on $\mathcal{H}$. ∎

Part I, section 5. Properties of the space $\mathfrak{M}(\mathbb{R})$ as they (may) apply to the theory of wavelets:

A. Following (5.9) in Part I, define an integral kernel $L = L(x,y)$ by

$$L(x,y) = \sum_k \phi(x+k)\overline{\psi(y+k)} \qquad (x,y \in \mathbb{R}) \ .$$

When ϕ and ψ are functions in $\mathfrak{M}(\mathbb{R})$ the function is well-defined and continuous (why continuous?), so the associated integral operator

$$(\mathcal{L}f)(x) = \int_{-\infty}^{\infty} L(x,y)f(y)\,dy = \sum_k \Big(\int_{-\infty}^{\infty} f(y)\overline{\psi(y+k)}\,dy \Big) \phi(x+k)$$

is well-defined. In theorem (5.10) we used the Wiener condition on the function side to show that $\mathcal{L}$ is bounded on $L^2(\mathbb{R})$; in fact, estimate I(2.6) can be used to show that $\mathcal{L}$ is a bounded operator on each space $L^p(\mathbb{R})$, $1 \le p < \infty$.

PROBLEMS: 1. Show that $\mathcal{L}$ is bounded operator from $L^2(\mathbb{R})$ into $\mathcal{V}_\phi$ for every choice of ψ in $\mathfrak{M}(\mathbb{R})$.

2. Assume now that ϕ is a stable function in $\mathfrak{M}(\mathbb{R})$. Show that $\mathcal{L} : L^2(\mathbb{R}) \to \mathcal{V}_\phi$ maps $L^2(\mathbb{R})$ onto $\mathcal{V}_\phi$ when $\psi = \tilde{\phi}$; more precisely, show that $\mathcal{L}$ is a *projection* of $L^2(\mathbb{R})$ onto $\mathcal{V}_\phi$ when $\psi = \tilde{\phi}$, i.e., $\mathcal{L}^2 = \mathcal{L}$.

B. Fix stable function ϕ and for each $t > 0$ define $L_t = L_t(x,y)$ by

$$L_t(x,y) = \frac{1}{t} \sum_k \phi\Big(\frac{x}{t} + k\Big)\overline{\tilde{\phi}\Big(\frac{y}{t} + k\Big)}$$

and let

$$(\mathcal{L}_t f)(x) = \int_{-\infty}^{\infty} L_t(x,y)f(y)\,dy$$

be the associated integral operator. Show that $\mathcal{L}_t$ is a bounded operator on $L^p(\mathbb{R})$ such that

$$\sup_{t>0} \|\mathcal{L}_t\|_{L(L^p)} < \infty$$

(this result remains true replacing $\tilde{\phi}$ by any ψ in $\mathfrak{M}(\mathbb{R})$). Following Mallat's theorem (in his TAMS paper) we can ask the following questions.

Question 1: *Is it always true that*

$$\lim_{t\to\infty} \mathcal{L}_t f = 0 \qquad \big(f \in L^2(\mathbb{R})\big)$$

Question 2: *What conditions on ϕ ensure that*

$$\lim_{t\to 0} \mathcal{L}_t f = f \qquad \big(f \in L^2(\mathbb{R})\big)\ .$$

The significance of these questions shows from the fact

$$\mathcal{L}_t : L^2(\mathbb{R}) \to \mathcal{V}_m \qquad \left(t = \frac{1}{2^m}\right)\ ,$$

so the subspace chain conditions

$$\bigcap_m \mathcal{V}_m = \{0\}\ , \qquad \overline{\bigcup_m \mathcal{V}_m} = L^2(\mathbb{R})$$

would follow from knowing the answers to questions 1 and 2.

C. In connection with the 2-scale dilation property

$$\phi(x) = \sum_k c_k \phi(2x+k)\ ,$$

show that the condition $\{c_k\} \in \ell^1(\mathbb{Z})$ follows automatically from the assumption that ϕ is a stable function in $\mathfrak{M}(\mathbb{R})$. Show also that the coefficients c_k can be computed explicitly by the formula

$$c_k = \int_{-\infty}^{\infty} \phi\big(\tfrac{1}{2}x\big)\tilde{\phi}(x+k)\,dx\ .$$

D. We now that any ϕ in $\mathfrak{M}(\mathbb{R})$ belongs also to every $L^p(\mathbb{R})$-space. So, in particular, the Fourier transform $\hat{\phi}$ of ϕ is well-defined and continuous; in fact, by definition, $\hat{\phi} = \hat{\phi}(\xi)$ satisfies the Wiener condition. In connection with the hypotheses needed for the continuous wavelet transform, we can ask the following question.

Question : *Assume that ϕ is a stable function in $\mathfrak{M}(\mathbb{R})$ such that*

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x)\,dx = 0\ .$$

What extra conditions (if any) are needed to ensure that

$$\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\ \frac{d\xi}{|\xi|}$$

is finite?

Part II, section 4. (A) Recurrence formula for M_{2m}. In lemma (4.9) we established a recurrence formula

$$N_m(x) = \frac{1}{m-1}\big(xN_{m-1}(x) + (m-x)N_{m-1}(x-1)\big) \tag{A.1}$$

for the *forward* B-spline N_m that was extremely useful in establishing the basic properties of the N_m. Since the *central* B-spline M_{2m} is just an integer translate

$$M_{2m}(x) = N_{2m}(x+m) \tag{A.2}$$

of N_{2m}, there will be a corresponding recurrence formula for M_{2m}.

(A.3) THEOREM. *The central B-splines satisfy the recurrence formula*

$$\begin{aligned} M_{2m}(x) = \frac{1}{(2m-1)(2m-2)}\Big\{&(x+m)^2 M_{2m-2}(x+1) \\ &+ 2\big(m(m-1)-x^2\big)M_{2m-2}(x) + (x-m)^2 M_{2m-2}(x-1)\Big\} \end{aligned}$$

for each $m = 2, 3, \ldots$.

Proof. Applying (A.1) twice we see that

$$\begin{aligned} N_m(x) &= \frac{1}{m-1}\big(xN_{m-1}(x) + (m-x)N_{m-1}(x-1)\big) \\ &= \frac{1}{(m-1)(m-2)}\Big\{x\big(xN_{m-2}(x) + (m-1-x)N_{m-2}(x-1)\big) \\ &\qquad + (m-x)\big((x-1)N_{m-2}(x-1) + (m-x)N_{m-2}(x-2)\big)\Big\}\ . \end{aligned}$$

Consequently,

$$\begin{aligned} N_m(x) = \frac{1}{(m-1)(m-2)}\big\{&x^2 N_{m-2}(x) + (2mx - 2x^2 - m)N_{m-2}(x-1) \\ &+ (m-x)^2 N_{m-2}(x-2)\big\}\ , \end{aligned}$$

and so

$$\begin{aligned} N_{2m}(x) = \frac{1}{(2m-1)(2m-2)}\big\{&x^2 N_{2m-2}(x) + 2(2mx - x^2 - m)N_{2m-2}(x-1) \\ &+ (2m-x)^2 N_{2m-2}(x-2)\big\}\ . \end{aligned}$$

In view of (A.2) therefore,

$$M_{2m}(x) = N_{2m}(x+m) = \frac{1}{(2m-1)(2m-2)}\Big\{(x+m)^2 N_{2m-2}(x+m) + 2\big(m(m-1)-x^2\big)N_{2m-2}(x+m-1) + (x+m)^2 N_{2m-2}(x+m-2)\Big\} .$$

The recurrence formula for central B-splines now follows immediately.

(B) Euler-Frobenius polynomials. There is a remarkable connection between cardinal splines and some polynomials introduced much earlier by Euler and Frobenius for quite different reasons, of course.

(B.1) DEFINITION. *For each* $n = 0, 1, \ldots$ *define* $\Pi_n(t)$ *by*

$$\Pi_0(t) = 1 \ , \quad t\Pi_n(t) = (1-t)^{n+1}\Big(t\frac{d}{dt}\Big)^n\Big(\frac{t}{1-t}\Big) \qquad (n > 1) \ .$$

While it is not clear initially that $\Pi_n(t)$ is defined when $t = 0$ or 1, the examples

(B.2) $$\Pi_1(t) = 1 \ , \quad \Pi_2(t) = 1 + t \ , \quad \Pi_3(t) = 1 + 4t + t^2$$

suggest that Π_n is a polynomial of degree $n-1$. This is the case: indeed, the recurrence formula

(B.3) $$\Pi_{n+1}(t) = (1+nt)\Pi_n(t) + (1-t)t\frac{d\Pi_n}{dt}$$

follows easily from (B.1), and this together with an induction argument shows that

$$\Pi_n(t) = \sum_{k=0}^{n-1} a_k t^k \ .$$

In fact, since

(B.4) $$t\frac{d}{dt} : t^k \to kt^k \ ,$$

it is clear from (B.2) that

(B.5) $$\Pi_n(t) = t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_{n-1}t + 1 \ ,$$

i.e., Π_n is a monic polynomial such that $\Pi_n(0) = 1$. These polynomials have a long history dating back to Euler and they are now called *Euler-Frobenius polynomials* because of the significant contributions made by Euler and Frobenius in developing their properties. Since the values of the individual coefficients in (B.4) have an important role to play we set

$$\Pi_n(t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k \qquad (n \geq 1)\ ; \tag{B.6}$$

it is convenient to extend the definition of $\left\langle {n \atop k} \right\rangle$ to all k by setting

$$\left\langle {n \atop k} \right\rangle = \frac{1}{k!}\Big(\frac{d}{dt}\Big)^k \Pi_n\Big|_{t=0}\ , \qquad (0 \leq k \leq n-1)\ , \tag{B.7)(i}$$

and

$$\left\langle {n \atop k} \right\rangle = 0 \qquad (k < 0\ ,\ k \geq n)\ . \tag{B.7)(ii}$$

With this extension

$$\Pi_n(t) = \sum_{k=-\infty}^{\infty} \left\langle {n \atop k} \right\rangle t^k\ . \tag{B.8}$$

The numbers $\left\langle {n \atop k} \right\rangle$ are called *Eulerian numbers* and later we shall see that each $\left\langle {n \atop k} \right\rangle$ is an integer. Values of $\left\langle {n \atop k} \right\rangle$ for small n and k are collected together in the following table.

Euler's triangle.

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$	$\left\langle {n \atop 7} \right\rangle$	$\left\langle {n \atop 8} \right\rangle$	$\left\langle {n \atop 9} \right\rangle$
0	1									
1	1	0								
2	1	1	0							
3	1	4	1	0						
4	1	11	11	1	0					
5	1	26	66	26	1	0				
6	1	57	302	302	57	1	0			
7	1	120	1191	2416	1191	120	1	0		
8	1	247	4293	15619	15619	4293	247	1	0	
9	1	502	14608	88234	156190	88234	14608	502	1	0

As Schoenberg points out in [Scho], however, the Euler-Frobenius polynomials have been re-discovered more recently by nearly everyone working on spline interpolation. This connection stems from the fact that Eulerian numbers are intimately related to the values of the forward B-spline at its knots.

(B.8) THEOREM. *The Eulerian numbers have the property*

$$\text{(i)} \qquad \left\langle {n \atop k} \right\rangle = n! N_{n+1}(k+1) \qquad (k \in \mathbb{Z}) \ ;$$

in particular, each $\left\langle {n \atop k} \right\rangle$ *is an integer and*

$$\text{(ii)} \qquad \left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle \ .$$

Proof. Since N_2 is just the tent function

$$N_2(x) = \begin{cases} x & (0 \le x < 1) \ , \\ 2-x & (1 \le x < 2) \ , \\ 0 & \text{(otherwise)}, \end{cases}$$

it is clear that

$$N_2(1) = 1 \ , \quad N_2(k) = 0 \qquad (k \neq 1) \ .$$

On the other hand, by (B.2) and (B.7)(ii),

$$\left\langle {1 \atop 0} \right\rangle = 1 \ , \quad \left\langle {1 \atop k} \right\rangle = 0 \qquad (k \neq 0) \ .$$

Thus (B.8)(i) holds for all k when $n = 1$; in addition $\left\langle {1 \atop k} \right\rangle \in \mathbb{Z}$. Let us now ignore the case $n > 1$ for the moment and establish first the recurrence formula (B.8)(ii). In view of (B.3), (B.4) and (B.8),

$$\begin{aligned} \sum_k \left\langle {n+1 \atop k} \right\rangle t^k &= (1+nt)\sum_k \left\langle {n \atop k} \right\rangle t^k + (1-t)\sum_k k\left\langle {n \atop k} \right\rangle t^k \\ &= \sum_k (k+1)\left\langle {n \atop k} \right\rangle t^k + \sum_k (n-k)\left\langle {n \atop k} \right\rangle t^{k+1} \\ &= \sum_k \left((k+1)\left\langle {n \atop k} \right\rangle + (n-k+1)\left\langle {n \atop k-1} \right\rangle \right) t^k \ . \end{aligned}$$

On replacing $n+1$ by n we obtain (B.8)(ii). A simple induction argument using (B.8)(ii) now shows that $\left\langle {n \atop k} \right\rangle \in \mathbb{Z}$ since we have seen already that $\left\langle {1 \atop k} \right\rangle \in \mathbb{Z}$.

To complete the proof we return to (B.8)(i) and establish the case $n > 1$. For this it is enough to show that $n!N_{n+1}(k+1)$ satisfies the same recurrence relation (B.8)(ii) as $\left\langle {n \atop k} \right\rangle$. Now

$$N_{n+1}(x) = \frac{1}{n}\big(xN_n(x) + (n-x+1)N_n(x-1)\big)$$

and so

$$N_{n+1}(k+1) = \frac{1}{n}\big((k+1)N_n(k+1) + (n-k)N_n(k)\big) \ .$$

Thus

$$n!N_{n+1}(k+1) = (k+1)\big((n-1)!N_n(k+1)\big) + (n-k)\big((n-1)!N_n(k)\big) \ .$$

Hence the identity

$$\left\langle {n \atop k} \right\rangle = n!N_{n+1}(k+1)$$

holds for all $n \geq 1$ and all integers k. ∎