Part II. Wavelet Analysis

1. Continuous wavelet transform. The Fourier integral representation

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi \tag{1.1)(i}$$

of a function in terms of its Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\xi}\, dy \tag{1.1)(ii}$$

exploits in a fundamental way the additive and multiplicative structures on real numbers. The Schrödinger operators

$$\big(\tau(z)f\big)(x) = e^{i\pi st} f(x+s) e^{2\pi i xt} \qquad (z = s+it) \tag{1.2}$$

introduced in section 4 of part I capture these basic structures both on the function side and on the transform side, so to speak. In physics terms, $\{s+it : s,t \in \mathbb{R}\}$ is *phase space*, and for each ϕ in $L^2(\mathbb{R})$ the mapping

$$f \to \mathcal{F}_\phi(z) = \big(f, \tau^*(z)\phi\big) \qquad (z \in \mathbb{C}) \tag{1.3}$$

associates to each f a continuous function F_ϕ on phase space. In concrete terms,

$$\mathcal{F}_\phi(z) = e^{-\pi i st} \int_{-\infty}^{\infty} f(x)\overline{\phi(x-s)}\, e^{2\pi i xt}\, dx \ .$$

This is the fundamental Bargmann-Segal-Wigner transform. Remarkably enough, its properties are completely analogous to those of the Fourier transform — some might argue that they are better.

(1.4) Theorem. *When $\|\phi\|_{L^2} = 1$, the mapping $f \to \mathcal{F}_\phi(z)$, $z = s+it$, is a unitary mapping from $L^2(\mathbb{R})$ into $L^2(\mathbb{C})$ in the sense that the identities*

(i) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}_\phi(z)|^2\, ds\, dt = \int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \textit{(Plancherel)},$$

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\overline{\mathcal{G}_\phi(z)}\, ds\, dt = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \qquad \textit{(Parseval)}$$

hold for all f, g in $L^2(\mathbb{R})$; furthermore, f can be recovered from $\mathcal{F}_\phi$ via the inverse mapping

$$\text{(iii)} \qquad f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\big(\tau^*(z)\phi\big)(x)\, ds\, dt \ .$$

Thus, in Hilbert space notation

$$(1.5)(\text{i}) \qquad (\mathcal{F}_\phi, \mathcal{G}_\phi) = (f, g) \qquad \big(f, g \in L^2(\mathbb{R})\big)$$

and

$$(1.5)(\text{ii}) \qquad f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(f, \tau^*(z)\phi\big)\tau^*(z)\phi\, ds\, dt \qquad \big(f \in L^2(\mathbb{R})\big)\ .$$

Proof. Since the Parseval identity reduces to the Plancherel identity when $f = g$, it is enough to establish (ii) and (iii). So fix f, g in $L^2(\mathbb{R})$. Then

$$\mathcal{F}_\phi(z)\overline{\mathcal{G}_\phi(z)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\,\phi(y-s)\overline{\phi(x-s)}\, e^{2\pi i(x-y)t}\, dx\, dy\ ;$$

consequently, after a change in the order of integration,

$$\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\overline{\mathcal{G}_\phi(z)}\, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\Big(\int_{-\infty}^{\infty} \phi(y+s)\overline{\phi(x+s)}\, ds\Big) e^{2\pi i(x-y)t}\, dx\, dy\ .$$

But by (4.8) and Parseval's theorem for the Fourier transform,

$$\int_{-\infty}^{\infty} \phi(y+s)\overline{\phi(x+s)}\, ds = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 e^{2\pi i(y-x)\xi}\, d\xi\ ,$$

and so,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\overline{\mathcal{G}_\phi(z)}\, ds &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\Big(\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 e^{2\pi i(y-x)(\xi-t)}\, d\xi\Big)\, dx\, dy \\
&= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \Big(\int_{-\infty}^{\infty} f(x)e^{-2\pi i x(\xi-t)}\, dx\Big)\overline{\Big(\int_{-\infty}^{\infty} g(y)e^{-2\pi i y(\xi-t)}\, dy\Big)}\, d\xi \\
&= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \widehat{f}(\xi-t)\overline{\hat{g}(\xi-t)}\, d\xi\ ,
\end{aligned}$$

interchanging the order of integration once again. Thus, finally,

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\overline{\mathcal{G}(z)}\, ds\, dt &= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \Big(\int_{-\infty}^{\infty} \widehat{f}(\xi-t)\overline{\hat{g}(\xi-t)}\, dt\Big)\, d\xi \\
&= \Big(\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\, d\xi\Big) \int_{-\infty}^{\infty} \widehat{f}(t)\overline{\hat{g}(t)}\, dt\ ,
\end{aligned}$$

from which (1.4)(ii) follows because of the Plancherel and Parseval identities for the Fourier transform.

The proof of the integral representation (1.4)(iii) is much the same. Fix f in $L^2(\mathbb{R})$. Then

$$\mathcal{F}_\phi(z)\tau^*(z)\phi = \int_{-\infty}^{\infty} f(y)\overline{\phi(y-s)}\,\phi(x-s)e^{2\pi i(x-y)t}\,dy\ ,$$

and so, after a change in the order of integration,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\tau^*(z)\phi\,ds &= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} \phi(x+s)\overline{\phi(y+s)}\,ds\right)e^{2\pi i(y-x)t}\,dy \\ &= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 e^{2\pi i(x-y)\xi}\,d\xi\right)e^{2\pi i(y-x)t}\,dy\ .\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\tau^*(z)\phi\,ds &= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\left(\int_{-\infty}^{\infty} f(y)e^{-2\pi i y(\xi-t)}\,dy\right)e^{2\pi i x(\xi-t)}\,d\xi \\ &= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\widehat{f}(\xi-t)e^{2\pi i x(\xi-t)}\,d\xi\ .\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}_\phi(z)\tau^*(z)\phi\,ds\,dt &= \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\left(\int_{-\infty}^{\infty} \widehat{f}(\xi-t)e^{2\pi i x(\xi-t)}\,dt\right)d\xi \\ &= \left(\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2\,d\xi\right)\int_{-\infty}^{\infty} \widehat{f}(t)e^{2\pi i x t}\,dt\end{aligned}$$

from which the integral representation (1.4)(iii) follows using the Fourier transform version of Plancherel and the Fourier integral representation (1.1). This completes the proof. ∎

Strictly speaking, theorem (1.4) has been established only for those f in $L^2(\mathbb{R})$ for which (1.1)(i) can be used to define the Fourier transform as well as those for which (1.1)(ii) holds pointwise, say functions in $\mathfrak{M}(\mathbb{R})$. Since such functions are dense in $L^2(\mathbb{R})$, however, the sort of limit in mean arguments used to establish theorem (4.__) in Part I can then be used to establish theorem (1.4) in complete generality. To dwell on such niceties could well obscure the basic ideas underlying such results as (1.4), so we shall omit them and proceed more formally.

While (1.4) applied to all norm one ϕ in $L^2(\mathbb{R})$, the most important example probably is the special case when

$$\phi(x) = \sqrt{2}\; e^{-\pi x^2} \; .$$

For then $\|\phi\|_{L^2} = 1$ and

$$\begin{aligned}\mathcal{F}_\phi(z) &= \sqrt{2}\; e^{-\pi ist} \int_{-\infty}^{\infty} f(x) e^{-\pi(x-s)^2} e^{2\pi ixt}\, dx \\ &= \sqrt{2} \int_{-\infty}^{\infty} f(x) e^{-\pi\{(x-s)^2 + 2ist - 2ixt\}}\, dx \\ &= \sqrt{2}\; e^{-\pi(s^2+ist)} \int_{-\infty}^{\infty} f(x) e^{-\pi x^2} e^{2\pi xz}\, dx \; .\end{aligned}$$

But

$$s^2 + ist = \tfrac{1}{2}(s^2+t^2) + \tfrac{1}{2}(s+it)^2 = \tfrac{1}{2}(|z|^2 + z^2) \; .$$

Consequently,

$$\mathcal{F}_\phi(z) = \sqrt{2}\; e^{-\frac{1}{2}\pi|z|^2} e^{-\frac{1}{2}\pi z^2} \int_{-\infty}^{\infty} f(x) e^{-\pi x^2} e^{2\pi xz}\, dx \; .$$

To overcome the perceived deficiencies of the Short Time Fourier transform the geophysicist Morlet suggested replacing the Heisenberg group and the corresponding Schrödinger operators $\tau(z)$, $z \in \mathbb{C}$, on $L^2(\mathbb{R})$ by the affine group and the operators $\pi(z)$ on $L^2(\mathbb{R})$ as defined earlier in section 2 of Part I. It was the physicist Grossman, however, who pointed out the group-theoretic features of Morlet's ideas, and shortly afterwards the mathematician Yves Meyer who pointed out that many of the same ideas had been used for a considerable time in harmonic analysis and partial differential equations, particularly by Calderón and the Chicago school.

Fix ψ in $L^2(\mathbb{R})$ and, as in section 2 of Part I, define the operator $\pi(z)$ on $L^2(\mathbb{R})$ by

(1.6)(i) $$\big(\pi(z)\psi\big)(x) = t^{1/2}\psi(tx+s) \qquad (z = s+it\;,\; t>0) \; .$$

Just as with the operators $\tau(z)$, there is an associated transform

(1.6)(ii) $$f \longrightarrow F_\psi(z) = \big(f, \pi^*(z)\psi\big) \qquad (z \in \mathbb{C}_+) \; ;$$

in concrete terms, therefore,

(1.6)(iii) $$F_\psi(z) = \frac{1}{t^{1/2}} \int_{-\infty}^{\infty} f(x) \overline{\psi\big(\frac{x-s}{t}\big)}\, dx = t^{1/2}(f * \psi_t^*)(s) \qquad (z = s+it) \; .$$

The properties of the mapping $f \to F_\psi$ are collected together in the next theorem.

(1.7) THEOREM. *When ψ is a function in $L^2(\mathbb{R})$ such that*

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \qquad (\xi \in \mathbb{R}) \ ,$$

the mapping $f \to F_\psi(z)$, $z = s + it$, is a linear mapping from $L^2(\mathbb{R})$ into $L^2(\mathbb{C}_+)$ which is unitary in the sense that the identities

(i) $$\int_0^\infty \int_{-\infty}^\infty |F_\psi(z)|^2 \frac{ds\,dt}{t^2} = \int_{-\infty}^\infty |f(x)|^2\,dx \qquad \textit{(Plancherel)}$$

(ii) $$\int_0^\infty \int_{-\infty}^\infty F_\psi(z)\overline{G_\psi(z)}\,\frac{ds\,dt}{t^2} = \int_{-\infty}^\infty f(x)\overline{g(x)}\,dx \qquad \textit{(Parseval)}$$

hold for all f, g in $L^2(\mathbb{R})$; furthermore, f can be recovered from F_ψ by the inverse transform

(iii) $$f(x) = \int_0^\infty \int_{-\infty}^\infty F_\psi(z)\big(\pi^*(z)\psi\big)(x)\,\frac{ds\,dt}{t^2} \ .$$

In most applications ψ will be a function in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, ensuring that $\hat{\psi}$ is continuous. But then $|\hat{\psi}(\xi)|^2/|\xi|$ will not be integrable at $\xi = 0$ unless $\hat{\psi}(0) = 0$, i.e., unless ψ *is wavelike* in the sense used earlier of meaning that ψ has average zero. Thus we shall speak of a function ψ in $L^1(\mathbb{R})$ as being *normalized* and *wavelike* if

(1.8) $$\text{(i)} \quad \psi \in L^2(\mathbb{R}) \ , \qquad \text{(ii)} \quad \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \qquad (\xi \in \mathbb{R}) \ .$$

This prompts the following definition.

(1.9) DEFINITION. *When ψ is a normalized, wavelike function in $L^1(\mathbb{R})$, the mapping*

$$f \longrightarrow F_\psi(z) = \frac{1}{t^{1/2}} \int_{-\infty}^\infty f(x)\overline{\psi\big(\frac{x-s}{t}\big)}\,dx \qquad (z = s + it)$$

is called the ***(right) continuous wavelet transform****, while*

$$F \longrightarrow f(x) = \frac{1}{t^{1/2}} \int_0^\infty \int_{-\infty}^\infty F(z)\psi\Big(\frac{x-s}{t}\Big)\,\frac{ds\,dt}{t^2}$$

is called the ***inverse (right) continuous wavelet transform****.*

Theorem (1.7) therefore, shows that each f in $L^2(\mathbb{R})$ can be represented as the inverse (right) continuous wavelet transform of some function $F = F(z)$ for which

$$\int_0^\infty \int_{-\infty}^\infty |F(s+it)|^2\,\frac{ds\,dt}{t^2}$$

is finite; in fact, (1.9) shows that we can take for F the function F_ψ in (1.6)(iii). Examples of normalized wavelike function are easy to construct. The simplest way is to assume that $\hat{\psi}$ is even or odd and that

$$\int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|}$$

is finite; for then

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = \int_0^\infty |\hat{\psi}(\pm t)|^2 \frac{dt}{t} = \tfrac{1}{2} \int_{-\infty}^{\infty} |\hat{\psi}(t)|^2 \frac{dt}{|t|} ,$$

whether ξ is positive or negative. In this case the function

$$\psi_0(x) = \left(2 \Big/ \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|}\right) \psi(x)$$

will be normalized. For example, set

$$\text{(1.10)(i)} \qquad \psi_1(x) = \text{const.}(e^{-\pi x^2} - \sqrt{2}\, e^{-\frac{1}{2}\pi x^2})$$

and

$$\text{(1.10)(ii)} \qquad \psi_2(x) = \text{const.} \sin 2\pi x e^{-\pi x^2} ,$$

so that

$$\hat{\psi}_1(\xi) = \text{const.}(1 - e^{-\pi \xi^2}) e^{-\pi \xi^2}$$

and

$$\hat{\psi}_2(\xi) = \text{const.}(e^{-\pi(\xi-1)^2} - e^{\pi(\xi+1)^2}) .$$

Then ψ_1 and ψ_2 are wavelike for every choice of constant, and clearly will be normalized for specific choices of constant.

Proof of (1.7). The proof of all three identities is much the same as in the corresponding results for the Schrödinger operators $\tau(z)$. Again therefore we shall establish the Parseval identity first, proceeding formally. So fix f, g in $L^2(\mathbb{R})$ (a more rigorous proof would start with f, g in $\mathfrak{M}(\mathbb{R})$ and then extend by density arguments). Then

$$F_\psi(z)\overline{G_\psi(z)} = \frac{1}{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\overline{g(y)\, \psi\Big(\frac{x-s}{t}\Big)} \psi\Big(\frac{y-s}{t}\Big)\, dx\, dy ;$$

consequently, after a change in the order of integration,

$$\int_{-\infty}^{\infty} F_\psi(z)\overline{G_\psi(z)}\,ds = \frac{1}{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\Big(\int_{-\infty}^{\infty} \psi\Big(\frac{y-s}{t}\Big)\overline{\psi\Big(\frac{x-s}{t}\Big)}\,ds\Big)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f(x)\overline{g(y)}\Big(\int_{-\infty}^{\infty} \psi\Big(s+\frac{y}{t}\Big)\overline{\psi\Big(s+\frac{x}{t}\Big)}\,ds\Big)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{g(y)}\Big(\int_{-\infty}^{\infty} |\hat\psi(\xi)|^2 e^{\frac{2\pi i}{t}(y-x)\xi}\,d\xi\Big)\,dx\,dy\ .$$

Thus

$$\int_{-\infty}^{\infty} F_\psi(z)\overline{G_\psi(z)}\,ds = \int_{-\infty}^{\infty} |\hat\psi(\xi)|^2 \widehat{f}\Big(\frac{1}{t}\xi\Big)\overline{\widehat{g}\Big(\frac{1}{t}\xi\Big)}\,d\xi$$

$$= t\int_{-\infty}^{\infty} |\hat\psi(t\xi)|^2 \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi\ ,$$

after yet another change in the order of integration. Hence, finally,

$$\int_{0}^{\infty}\int_{-\infty}^{\infty} F_\psi(z)\overline{G_\psi(z)}\,\frac{ds\,dt}{t^2}$$

$$= \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\Big(\int_0^{\infty} |\hat\psi(t\xi)|^2\,\frac{dt}{t}\Big)\,d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$$

provided

$$\int_0^{\infty} |\hat\psi(t\xi)|^2\,\frac{dt}{t} = 1 \qquad (\xi\in\mathbb{R})\ .$$

Parseval's identity for the mapping $f \to F_\psi$ now follows immediately from Parseval's theorem for the Fourier transform $f \to \widehat{f}$.

To establish the integral representation (1.7)(iii), observe that

$$\int_{-\infty}^{\infty} F_\psi(z)\pi^*(z)\phi\,ds = \frac{1}{t}\int_{-\infty}^{\infty} f(y)\Big(\int_{-\infty}^{\infty} \overline{\psi\Big(\frac{y-s}{t}\Big)}\psi\Big(\frac{x-s}{t}\Big)\,ds\Big)\,dy$$

$$= t\int_{-\infty}^{\infty} |\hat\psi(t\xi)|^2 \widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

by similar arguments to the ones used before. Thus

$$\int_0^{\infty}\int_{-\infty}^{\infty} F_\psi(z)\pi^*(z)\phi\,\frac{ds\,dt}{t^2} = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}\Big(\int_0^{\infty} |\hat\psi(t\xi)|^2\,\frac{dt}{t}\Big)\,d\xi$$

$$= \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

provided ψ is normalized and wavelike. This completes the proof since the Fourier integral representation of f ensures that

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi \ . \qquad \blacksquare$$

2. Multi-resolution analysis. Having established many of the Hilbert space properties of individual $\mathcal{V}_\phi$-spaces, we turn now to the very important idea of Multi-resolution analysis associated with those ϕ satisfying a so-called *two-scale dilation* equation

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k)\ . \tag{2.1}$$

Before studying these ideas in detail however, it will be very instructive to look back on some image analysis ideas put forward by Burt and Adelson in the early 1980's. Fix a stable function ϕ. We can then think of the graph of any f in $\mathcal{V}_\phi$ as some image. More generally, since all of the previous theory for $\mathcal{V}_\phi$ as a subspace of $L^2(\mathbb{R})$ extends quite routinely to analogous subspaces of $L^2(\mathbb{R}^2)$, we might interpret any $f = f(x,y)$ in such a space $\mathcal{V}_\phi$ in $L^2(\mathbb{R}^2)$ as a two-dimensional image. For instance, f could then be a mathematical realization of 'Katy'.

From an image analysis point of view, Katy's 'tiger-stripes' are particularly important because they provide 'sharp edges' in the image, whereas the wall on which Katy sits forms a large area of constant color and texture. So in thinking of the graphs of functions $f = f(x)$ as one-dimensional images we should look for corresponding features — regions where the graph is changing rapidly, i.e., rapid changes of concavity, and regions where the graph has constant behaviour. In mathematical terms, therefore, we want functions f such that f'' is changing rapidly on some intervals, whereas f' is small on others. One such function is

$$f(x) = e^{-x^2} \sin\left(\frac{\pi\,\mathrm{sgn}(x)}{x^4 + \frac{1}{8}}\right) .$$

Its graph is given in figure 2.

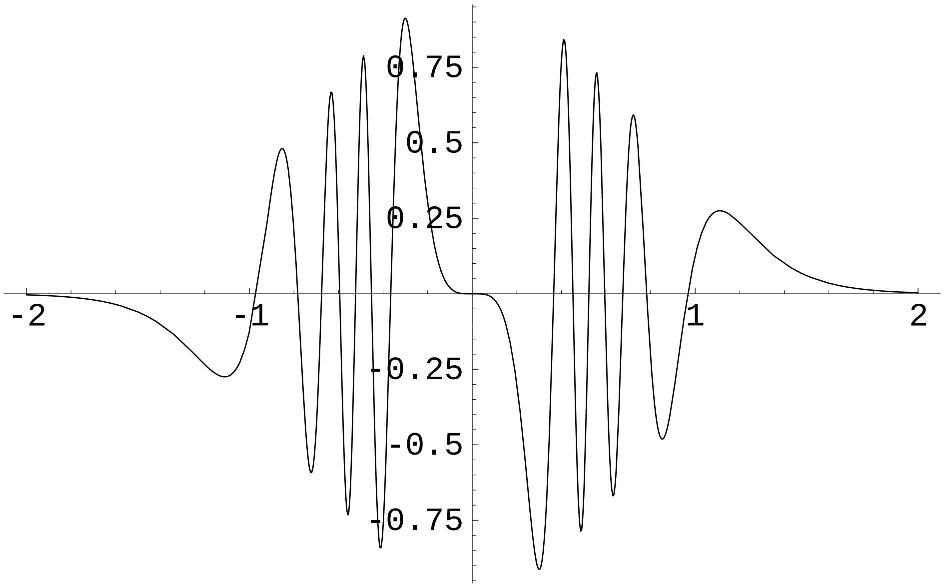

Figure 2

The work of Burt and Adelson arose in the decomposition and reconstruction of images, two-dimensional images in their case. Now one of the goals of a decomposition scheme for two dimensional images is to remove the very high correlations that may exist between neighboring pixels — in the Katy image, for instance, there will be an extremely high correlation through the whole of the wall area, but very little correlation in the area of the tiger strips. Several different schemes have been proposed for achieving this decomposition. Typically, they use a prediction method in which the value at a pixel is predicted by a weighted average from either previously encoded or from neighboring pixels, and only the difference between the actual pixel value and the predicted value is encoded. Using neighboring pixels for prediction seems more natural and intuitively should lead to greater data compression than schemes which use only previously encoded pixel values. The scheme proposed by Burt and Adelson combines both of these ideas "[in] *technique* *for image encoding in which local operators of many scales but identical shape serve as basis functions.*"

But let us see how these ideas work in the one-dimensional case. Fix a stable function ϕ in $\mathfrak{M}(\mathbb{R})$ and regard the graph of any function

$$f(x) = \sum_k a_k \phi(x-k)$$

in $\mathcal{V}_\phi$ as a (one-dimensional) image. We can think of the sequence $\mathbf{a} = \{a_k\}$ of coefficients as representing the pixel values at level 0 or unit spatial frequency since the integer translates $\phi(x+k)$ of ϕ are a basis for $\mathcal{V}_\phi$. Thus it will be natural to change the notation for $\mathcal{V}_\phi$ and write $\mathcal{V}_0$ instead to reflect this 0^{th}-level or unit spatial frequency. To achieve a different spatial frequency, higher or lower, we replace the basis functions $\{\phi(x-k : k \in \mathbb{Z}\}$ by $\{\phi(2^m x - k) : k \in \mathbb{Z}\}$, and set

$$\mathcal{V}_m = \left\{ \sum_k 2^{\frac{1}{2}m} a_k \phi(2^m x - k) : \{a_k\} \in \ell^2(\mathbb{Z}) \right\} .$$

The dilation factor $2^{\frac{1}{2}m}$ ensures that

$$\int_{-\infty}^{\infty} \Big| \sum_k 2^{\frac{1}{2}m} a_k \phi(2^m x - k) \Big|^2 dx = \int_{-\infty}^{\infty} \Big| \sum_k a_k \phi(x-k) \Big|^2 dx ,$$

and in turn the stable property of ϕ ensures that

$$A\Big(\sum_k |a_k|^2\Big)^{1/2} \le \Big(\int_{-\infty}^{\infty} \Big| \sum_k 2^{\frac{1}{2}m} a_k \phi(2^m x - k) \Big|^2 dx\Big)^{1/2} \le B\Big(\sum_k |a_k|^2\Big)^{1/2} .$$

Thus each $\mathcal{V}_m$ is a closed subspace of $L^2(\mathbb{R})$, and in complete analogy with (3.16), a function f in $L^2(\mathbb{R})$ belongs to $\mathcal{V}_m$ if and only if

$$\widehat{f}(\xi) = \nu(\xi) \hat{\phi}\Big(\frac{1}{2^m}\xi\Big) \qquad (\xi \in \mathbb{R})$$

for some function $\nu = \nu(\xi)$ which has period 2^m and lies in $L^2[0, 2^m]$. To achieve the averaging and decimation, let us suppose that there exists a function ψ together with sequences $\boldsymbol{\alpha} = \{\alpha_j\}$, $\boldsymbol{\beta} = \{\beta_j\}$, $\boldsymbol{\gamma} = \{\gamma_j\}$ and $\boldsymbol{\delta} = \{\delta_j\}$ in $\ell^1(\mathbb{R})$ such that the decomposition

$$\begin{aligned} \phi(2x) &= \sum_j \alpha_j \phi(x-j) + \sum_j \beta_j \psi(x-j) \\ \phi(2x+1) &= \sum_j \gamma_j \phi(x-j) + \sum_j \delta_j \psi(x-j) \end{aligned} \tag{2.2}$$

hold for all x. Anticipating the various wavelet constructions we shall see that any such ψ is potentially a good wavelet mother function when (2.2) holds for ψ. Of course it is far from obvious that any such ψ exist! Assuming they do, we can rewrite any f_0 in $\mathcal{V}_0$ as

$$
\begin{aligned}
f_0(x) &= \sum_k a_k \phi(x-k) = \sum_k a_{2k}\phi(x-2k) + \sum_k a_{2k+1}\phi(x-2k-1) \\
&= \sum_k a_{2k}\phi\Big(2\big(\tfrac{1}{2}x-k\big)\Big) + \sum_k a_{2k+1}\phi\Big(2\big(\tfrac{1}{2}x-k\big)-1\Big) \\
&= \sum_{j,k} \alpha_j a_{2k}\phi\Big(\tfrac{1}{2}x-j-k\Big) + \sum_{j,k} \beta_j a_{2k}\psi\Big(\tfrac{1}{2}x-j-k\Big) \\
&\qquad + \sum_{j,k} \gamma_j a_{2k+1}\phi\Big(\tfrac{1}{2}x-j-k\Big) + \sum_{j,k} \delta_j a_{2k+1}\psi\Big(\tfrac{1}{2}x-j-k\Big)\ .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
f(x) &= \sum_k \Big(\sum_j \alpha_j a_{2k-2j} + \sum_j \gamma_j a_{2k-2j+1}\Big)\phi\Big(\tfrac{1}{2}x-k\Big) \\
&\qquad + \sum_k \Big(\sum_j \beta_j a_{2k-2j} + \sum_j \delta_j a_{2k-2j+1}\Big)\psi\Big(\tfrac{1}{2}x-k\Big)\ .
\end{aligned}
$$

Hence, when (2.2) hold, each function

$$
f_0(x) = \sum_k a_k^{(0)}\phi(x-k) \qquad (f_0 \in \mathcal{V}_0)
$$

in $\mathcal{V}_0$ can be written as

$$
f_0(x) = \sum_k a_k^{(1)}\phi\Big(\tfrac{1}{2}x-k\Big) + \sum_k b_k^{(1)}\psi\Big(\tfrac{1}{2}x-k\Big) = f_1(x) + g_1(x)
$$

with

$$
a_k^{(1)} = \sum_j \alpha_j a_{2k-2j}^{(0)} + \sum_j \gamma_j a_{2k-2j+1}^{(0)}
$$

and

$$
b_k^{(1)} = \sum_j \beta_j a_{2k-2j}^{(0)} + \sum_j \delta_j a_{2k-2j+1}^{(0)}\ .
$$

But by the discrete convolution inequality (2.11) in Part I,

$$\Big(\sum_k |a_k^{(1)}|^2\Big)^{1/2} \le \Big(\sum_k \Big|\sum_j \alpha_j a_{2k-2j}^{(0)}\Big|^2\Big)^{1/2} + \Big(\sum_k \Big|\sum_j \gamma_j a_{2k-2j+1}\Big|^2\Big)^{1/2}$$

$$\le \Big(\sum_j |\alpha_j| + \sum_j |\gamma_j|\Big)\Big(\sum_k |a_k^{(0)}|^2\Big)^{1/2}$$

since $\{\alpha_j\}$ and $\{\gamma_j\}$ belong to $\ell^1(\mathbb{Z})$; similarly,

$$\Big(\sum_k |b_k^{(1)}|^2\Big)^{1/2} \le \Big(\sum_j |\beta_j| + \sum_j |\delta_j|\Big)\Big(\sum_k |a_k^{(0)}|^2\Big)^{1/2} .$$

Thus the hypotheses in (2.2) ensure that the coefficients of f_1 and g_1 belong to $\ell^2(\mathbb{Z})$; in particular, f_1 is a function in $\mathcal{V}_{-1}$. Now we come to the crucial multi-resolution idea. If we knew in advance that $\mathcal{V}_{-1}$ was a subspace of $\mathcal{V}_0$, i.e., $\mathcal{V}_{-1} \subseteq \mathcal{V}_0$, then both f_1 and g_1 would belong to $\mathcal{V}_0$ so that each f_0 in $\mathcal{V}_0$ would be represented as the sum $f_1 + g_1$ where f_1 is a "low detail" function at a level of resolution one step lower than that of f_0, while g_1 would be a "high detail" function in $\mathcal{V}_0$ at the same level of resolution as f_0. But by definition,

$$f(x) \in \mathcal{V}_0 \Leftrightarrow f(2^m x) \in \mathcal{V}_m$$

so the condition $\mathcal{V}_1 \subseteq \mathcal{V}_0$ is equivalent to the condition

(2.3)(i) $$\{0\} \subseteq \cdots \subseteq \mathcal{V}_{-m} \subseteq \cdots \subseteq \mathcal{V}_{-1} \subseteq \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_m \subseteq L^2(\mathbb{R}) .$$

Hence we can continue this decomposition by applying it to f_1 showing that

$$f_0 = f_1 + g_1 = f_2 + g_2 + g_1 = \cdots$$

with f_2 in $\mathcal{V}_{-2}$, g_2 in $\mathcal{V}_{-1}$ and g_1 in $\mathcal{V}_0$. Clearly this decomposition can be continued indefinitely. This is the essence of the Laplacian pyramid idea of Burt and Adelson.

Formally, we shall show that a stable function ϕ in $\mathfrak{M}(\mathbb{R})$ generates a multi-resolution if the associated subspaces

$$\mathcal{V}_m = \Big\{ 2^{\frac{1}{2}m} \sum_k a_k \phi(2^m x - k) : \sum_k |a_k|^2 < \infty \Big\}$$

satisfy (2.3)(i) together with the condition

$$\text{(2.3)(ii)} \qquad \bigcap_m \mathcal{V}_m = \{0\} \quad , \quad \overline{\bigcup_m \mathcal{V}_m} = L^2(\mathbb{R}) \ .$$

Condition (2.3)(i) can be expressed in a much more direct and useful way. If $\mathcal{V}_0 \subseteq \mathcal{V}_1$, then ϕ itself will belong to $\mathcal{V}_1$, so

$$\text{(2.4)} \qquad \phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k)$$

for some sequence $\{h_k\}$ in $L^2(\mathbb{R})$. In fact, if $\{\tilde{\phi}(x-k) : k \in \mathbb{Z}\}$ is the bi-orthogonal family associated to $\{\phi(x-k) : k \in \mathbb{Z}\}$, then

$$\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \phi(x)\overline{\tilde{\phi}(2x-k)}\, dx = h_k \ ,$$

so

$$\begin{aligned}\sum_k |h_k| &= \frac{1}{\sqrt{2}}\sum_k \Big| \int_{-\infty}^{\infty} \phi(x)\tilde{\phi}(2x-k)\, dx \Big| \\ &\le \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} |\phi(x)| \sum_k |\tilde{\phi}(2x-k)|\, dx \le \text{const.}\, \|\phi\|_{L^1} \|\tilde{\phi}\|_{\mathfrak{M}} \ .\end{aligned}$$

Thus automatically, the coefficients $\{h_k\}$ in (2.4) belong to ℓ^1 as well as to ℓ^2. Equation (2.4) is said to be a *two-scale dilation equation.* Already we have seen that ϕ must satisfy such a two-scale equation if (2.3)(i) holds. But the converse is also true.

(2.5) Theorem. *The family of subspaces $\{\mathcal{V}_m\}$ satisfies the nested condition* (2.3)(i) *if and only if ϕ satisfies a two-scale dilation equation.*

Proof. We have only to show that $\mathcal{V}_0 \subseteq \mathcal{V}_1$ when (2.4) holds. But if f belongs to $\mathcal{V}_0$, then

$$\begin{aligned} f(x) &= \sum_\ell a_\ell \phi(x-\ell) = \sqrt{2}\sum_\ell a_\ell \Big(\sum_k h_k \phi(2x-2\ell-k)\Big) \\ &= \sqrt{2}\sum_\ell a_\ell \Big(\sum_k h_{k-2\ell}\phi(2x-k)\Big) = \sqrt{2}\sum_k \Big(\sum_\ell a_\ell h_{k-2\ell}\Big)\phi(2x-k) \ .\end{aligned}$$

Since $\{h_k\} \in \ell^1$, the convolution of the ℓ^2-sequence $\{a_\ell\}$ with $\{h_k\}$ is again in ℓ^2. Hence f belongs also to $\mathcal{V}_1$. ∎

On taking Fourier transforms, we can express the two-scale dilation equation in a very useful form:

$$\text{(2.6)(i)} \qquad \hat{\phi}(\xi) = \frac{1}{\sqrt{2}}\left(\sum_k h_k e^{-\pi i k \xi}\right)\hat{\phi}\left(\tfrac{1}{2}\xi\right) = H\left(\tfrac{1}{2}\xi\right)\hat{\phi}\left(\tfrac{1}{2}\xi\right)$$

where

$$\text{(2.6)(ii)} \qquad H(\xi) = \frac{1}{\sqrt{2}}\sum_k h_k e^{-2\pi i k \xi} \ .$$

There are many interesting examples of such equations.

(2.7) Examples. (i) (Cubic splines). *Since*

$$\sin 2\theta = 2\cos\theta\sin\theta \ ,$$

clearly

$$\hat{\beta}_4(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^4 = \left(\cos\frac{\pi}{2}\xi\right)^4\left(\frac{\sin\frac{\pi}{2}\xi}{\frac{\pi}{2}\xi}\right)^4 = H\left(\tfrac{1}{2}\xi\right)\hat{\beta}_4\left(\tfrac{1}{2}\xi\right)$$

with

$$H(\xi) = (\cos\pi\xi)^4 \ .$$

(ii) (Butterworth filter condition)

$$H_m(\xi) = \frac{1}{1 + (\tan\frac{1}{2}\pi\xi)^m} \ .$$

(iii) (Daubechies condition)

$$\phi(x) = \frac{1}{\nu^2+1}\big(\nu(\nu+1)\phi(2x) + (1-\nu)\phi(2x-1) + (1+\nu)\phi(2x-2) + \nu(\nu-1)\phi(2x-3)\big) \ .$$

Now let's see how the two-scale dilation equation is used to construct wavelets.

A very important family of wavelets was constructed by Ingrid Daubechies in 1988. In her construction the emphasis is placed on the coefficients specifying the two-scale dilation property of ϕ by contrast with the Chui construction where the emphasis is on the simplicity and natural occurrence of ϕ. But the Daubechies construction produces compactly supported wavelet mother functions $\psi = \psi(x)$ which have a specified degree of smoothness; furthermore, the set of integer translates and dilates

$$\left\{2^{\frac{1}{2}m}\psi(2^m x - k) : k, m \in \mathbb{Z}\right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. In every respect, therefore, except possibly for simplicity and natural occurrence, such ψ are ideal wavelet mothers.

To begin the Daubechies construction we choose any ϕ in $\mathfrak{M}(\mathbb{R})$ such that

(2.8)(i) $\quad \{\phi(x+k) : k \in \mathbb{Z}\}$ *is orthonormal in* $L^2(\mathbb{R})$,

(2.8)(ii) $\quad \phi(x) = \sqrt{2}\sum_k h_k \phi(2x+k)$ *for some* h_i *in* $\mathbb{R}$.

As we have seen this ensures that the associated family of subspaces

$$\mathcal{V}_m = \left\{2^{\frac{1}{2}m}\sum_k a_k \phi(2^m x - k) : \sum_k |a_k|^2 < \infty\right\}$$

is nested; later we shall see that property (2.8) of a multi-resolution is satisfied automatically and that $\hat{\phi}(0) \neq 0$. Let's first see what conditions are already imposed on the coefficients $\{h_k\}$.

(2.9) LEMMA. *The coefficients* $\{h_k\}$ *in the two-scale dilation equation for a function* ϕ *in* $\mathfrak{M}(\mathbb{R})$ *satisfy*

$$\text{(i)} \quad \sum_k |h_k| < \infty\ , \qquad \text{(ii)} \quad \sum_k |h_k|^2 = 1\ , \qquad \text{(iii)} \quad \sum_k h_k = \sqrt{2}$$

whenever (2.8)(i) *and* (2.8)(ii) *are satisfied.*

Proof. Property (i) was proved earlier. As for property (ii), it follows from the orthonormality of the $\{\phi(x-k) : k \in \mathbb{Z}\}$

$$1 = \int_{-\infty}^{\infty} |\phi(x)|^2\,dx = 2\sum_{j,k} h_j \bar{h}_k \int_{-\infty}^{\infty} \phi(2x-j)\overline{\phi(2x-k)}\,dx$$

$$= \sum_{j,k} h_j \bar{h}_k \int_{-\infty}^{\infty} \phi(x-j)\overline{\phi(x-k)}\,dx = \sum_k |h_k|^2\ .$$

On the other hand,

$$\int_{-\infty}^{\infty} \phi(x)\,dx = \sqrt{2}\sum_k h_k \int_{-\infty}^{\infty} \phi(2x-k)\,dx = \frac{1}{\sqrt{2}}\Big(\sum_k h_k\Big)\int_{-\infty}^{\infty} \phi(x)\,dx\ ,$$

which shows that

$$\hat{\phi}(0) = \frac{1}{\sqrt{2}}\Big(\sum_k h_k\Big)\hat{\phi}(0)$$

establishing property (iii) since $\hat{\phi}(0) \neq 0$. ∎

In view of Lemma (2.9), therefore, the function

$$H(\xi) = \frac{1}{\sqrt{2}}\sum_k h_k e^{2\pi i k\xi} \tag{2.10)(i}$$

defines a continuous, period 1 function having absolutely convergent Fourier series; furthermore, because of (2.9)(iii),

$$H(0) = 1\ . \tag{2.10)(ii}$$

This function $H = H(\xi)$ is often referred to as the *symbol* of ϕ. The coefficients $\{h_k\}$ can also be computed by exploiting the orthonormality of the integer translates $\phi(x-k)$, $k \in \mathbb{Z}$:

$$\sqrt{2}\int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x-j)}\,dx = 2\sum_k h_k \int_{-\infty}^{\infty} \phi(2x-k)\overline{\phi(2x-j)}\,dx = h_j\ . \tag{2.11}$$

This leads immediately to the second property of the coefficients.

(2.12) Lemma. *If the function ϕ in $\mathfrak{M}(\mathbb{R})$ satisfying* (5.1) *has compact support, then only finitely many of the h_k are non-zero; in particular, the symbol*

$$H(\xi) = \frac{1}{\sqrt{2}}\sum_k h_k e^{-2\pi i k\xi}$$

of ϕ is then a trigonometric polynomial.

Now we come to the crucial definition.

(2.13) Definition. *For each ϕ in $\mathfrak{M}(\mathbb{R})$ satisfying* (2.1) *define the associated wavelet mother function by*

$$\psi(x) = \sqrt{2}\sum_k (-1)^{k-1} h_{1-k}\phi(2x-k) \ .$$

One explanation for the precise form of this definition of the wavelet mother will emerge. To continue the sequence of basic lemmas note that ψ belongs to $\mathfrak{M}(\mathbb{R})$, and hence to $L^2(\mathbb{R})$, since $\{h_k\}$ is a sequence in $\ell^1(\mathbb{Z})$; in fact,

$$\begin{aligned}\int_{-\infty}^{\infty} |\psi(x)|^2\,dx &= 2\sum_{j,k}(-1)^{j+k} h_{1-j}h_{1-k}\int_{-\infty}^{\infty}\phi(2x-j)\overline{\phi(2x-k)}\,dx \\ &= \sum_k |h_k|^2 = 1\end{aligned}$$

because of (2.9)(ii). Consequently, ψ belongs both to $\mathfrak{M}(\mathbb{R})$ and to the closed subspace $\mathcal{V}_1$ of $L^2(\mathbb{R})$ having $\{\sqrt{2}\,\phi(2x-k) : k \in \mathbb{Z}\}$ as basis. It has many other fundamental properties.

(2.14) Lemma. *When $H = H(\xi)$ is defined by* (2.3)(i), *the Fourier transforms of ϕ and the associated mother function ψ satisfy*

$$\text{(i)}\quad \hat{\phi}(\xi) = H\big(\tfrac12\xi\big)\hat{\phi}\big(\tfrac12\xi\big)\ , \qquad \text{(ii)}\quad \hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac12\xi+\tfrac12\big)}\,\hat{\phi}\big(\tfrac12\xi\big)\ .$$

Applying these factorizations again we see that

$$\hat{\phi}(\xi) = H\big(\tfrac12\xi\big)H\big(\tfrac14\xi\big)\hat{\phi}\big(\tfrac14\xi\big)$$

while

$$\hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac12\xi+\tfrac12\big)}\,H\big(\tfrac14\xi\big)\,\hat{\phi}\big(\tfrac14\xi\big)\ ;$$

and so after successive iterations we obtain the following descriptions of ϕ and ψ.

(2.15) Corollary. *The symbol $H = H(\xi)$ completely determines ϕ and ψ in the sense that*

$$\hat{\phi}(\xi) = \Big(\prod_{m=1}^{\infty} H\big(\tfrac12\xi\big)\Big)\hat{\phi}(0)\quad ,\quad \hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac12\xi+\tfrac12\big)}\Big(\prod_{m=2}^{\infty} H\big(\tfrac12\xi\big)\Big)\,\hat{\phi}(0)\ .$$

Proof of (2.14). The proof consists of two routine calculations. By (2.9)(ii),

$$\hat{\phi}(\xi) = \sqrt{2}\int_{-\infty}^{\infty}\Big(\sum_k h_k\phi(2x-k)\Big)e^{-2\pi i x\xi} = \frac{1}{\sqrt{2}}\sum_k h_k\int_{-\infty}^{\infty}\phi(x-k)e^{-\pi i x\xi}\,dx$$

$$= \frac{1}{\sqrt{2}}\Big(\sum_k h_k e^{-\pi i k\xi}\Big)\hat{\phi}\big(\tfrac{1}{2}\xi\big) = H\big(\tfrac{1}{2}\xi\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ .$$

On the other hand, by the same arguments starting with the defining relation (2.13),

$$\begin{aligned}\hat{\psi}(\xi) &= \frac{1}{\sqrt{2}}\Big(\sum_k(-1)^{k-1}h_{1-k}e^{\pi i k\xi}\Big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\\ &= \frac{1}{\sqrt{2}}e^{\pi i\xi}\Big(\sum_k e^{\pi i k}h_{1-k}e^{\pi i(k-1)\xi}\Big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\\ &= \frac{1}{\sqrt{2}}e^{\pi i\xi}\Big(\sum_k h_{1-k}e^{\pi i(k-1)(\xi+1)}\Big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ .\end{aligned}$$

But

$$\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)} = \sum_k h_k e^{-\pi i k(\xi+1)} = \sum_k h_{1-k}e^{-\pi i(k-1)(\xi+1)}$$

since the h_k are assume to be real; consequently,

$$\hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)}\,\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ ,$$

completing the proof. ∎

Now let $\Phi = \Phi(\xi)$ and $\Psi = \Psi(\xi)$ be the overlap functions corresponding to ϕ and ψ respectively. Of course, we know in advance that $\Phi(\xi) = 1$, but let's ignore that for the moment and use solely (2.14). Then

$$\begin{aligned}\Phi^2(\xi) &= \sum_k|\hat{\phi}(\xi+k)|^2 = \sum_k\Big\{|\hat{\phi}(\xi+2k)|^2 + |\hat{\phi}(\xi+2k+1)|^2\Big\}\\ &= \sum_k\Big\{|H\big(\tfrac{1}{2}\xi+k\big)|^2|\hat{\phi}\big(\tfrac{1}{2}\xi+k\big)|^2 + |H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}+k\big)|^2|\hat{\phi}\big(\tfrac{1}{2}\xi+\tfrac{1}{2}+k\big)|^2\Big\}\ .\end{aligned}$$

But H has period 1, so

$$\Phi^2(\xi) = |H\big(\tfrac{1}{2}\xi\big)|^2\sum_k|\hat{\phi}\big(\tfrac{1}{2}\xi+k\big)|^2 + |H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)|^2\sum_k|\hat{\phi}\big(\tfrac{1}{2}\xi+\tfrac{1}{2}+k\big)|^2\ .$$

Consequently,

$$\Phi^2(\xi) = |H\left(\tfrac{1}{2}\xi\right)|^2\Phi^2\left(\tfrac{1}{2}\xi\right) + |H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)|^2\Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right) . \tag{2.16)(i}$$

Similarly,

$$\Psi^2(\xi) = |H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)|^2\sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi+k\right)|^2 + |H\left(\tfrac{1}{2}\xi\right)|^2\sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi+\tfrac{1}{2}+k\right)|^2 ,$$

and so

$$\Psi^2(\xi) = |H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)|^2\Phi^2\left(\tfrac{1}{2}\xi\right) + |H\left(\tfrac{1}{2}\xi\right)|^2\Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right) . \tag{2.16)(ii}$$

Hence, as a *consequence of the integer translates* $\{\phi(x-k) : k \in \mathbb{Z}\}$ *being assumed to be an orthonormal family*, the symbol of ϕ satisfies the fundamental equation

$$|H\left(\tfrac{1}{2}\xi\right)|^2 + |H^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)|^2 = 1 \qquad (\xi \in \mathbb{R}) \tag{2.17}$$

and in turn, the overlap function of ψ satisfies

$$\Psi(\xi) = \left(|H\left(\tfrac{1}{2}\xi\right)|^2 + |H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)|^2\right)^{1/2} = 1 \qquad (\xi \in \mathbb{R})$$

In particular, the set of integer translates $\{\psi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal family. Still more is true along these lines.

(2.18) LEMMA. *When ϕ is a function in $\mathfrak{M}(\mathbb{R})$ satisfying* (2.8) *and ψ is the associated wavelet mother function, then each integer translate $\phi(x-\ell)$ of ϕ is orthogonal to each integer translate $\psi(x-m)$ of ψ in the sense that*

$$\int_{-\infty}^{\infty} \phi(x-\ell)\overline{\psi(x-m)}\,dx = 0 \qquad (\ell, m \in \mathbb{Z}) .$$

To appreciate the significance of this last lemma, let

$$\mathcal{V}_0 = \left\{\sum_k a_k\phi(x-k) : \{a_k\} \in \ell^2(\mathbb{Z})\right\} ,$$

be the closed subspace of $L^2(\mathbb{R})$ having $\{\phi(x-k) : k \in \mathbb{Z}\}$ as orthonormal basis, and let

$$\mathcal{W}_0 = \left\{\sum_k b_k\psi(x-k) : \{b_k\} \in \ell^2(\mathbb{Z})\right\}$$

together with

$$\mathcal{V}_1 = \left\{ \sum_k \sqrt{2}\, c_k \phi(2x-k) : \{c_k\} \in \ell^2(\mathbb{Z}) \right\}$$

be the corresponding subspaces having $\{\psi(x-k) : k \in \mathbb{Z}\}$ and $\{\sqrt{2}\,\phi(2x-k) : k \in \mathbb{Z}\}$ respectively as orthonormal bases. Then by (2.8) and (2.13) both $\mathcal{V}_0$ and $\mathcal{W}_0$ are closed subspaces of $\mathcal{V}_1$, while lemma (2.18) ensures that $\mathcal{V}_0$ is orthogonal to $\mathcal{W}_0$. Thus $\mathcal{V}_0$ and $\mathcal{W}_0$ are orthogonal subspaces of $\mathcal{V}_1$. In fact, the Laplacian pyramid idea of Burt and Adelson shows that $\mathcal{V}_1$ is the sum of these two orthogonal subspaces.

(2.19) LEMMA. *When $\{h_k\}$ is the sequence of coefficients in the two-scale dilation equation for a function ϕ in $\mathfrak{M}(\mathbb{R})$ satisfying (2.8), then*

(i) $$\phi(2x) = \frac{1}{\sqrt{2}} \left\{ \sum_k h_{2k}\phi(x-k) - \sum_k h_{2k+1}\psi(x-k) \right\} ,$$

while

(ii) $$\phi(2x+1) = \frac{1}{\sqrt{2}} \left\{ \sum_k h_{2k+1}\phi(x-k) + \sum_k h_{2k}\psi(x-k) \right\} .$$

In particular, each

(iii) $$f(x) = \sqrt{2}\, \sum_k a_k^{(1)} \phi(2x-k) \qquad (f \in \mathcal{V}_1)$$

in $\mathcal{V}_1$ has an orthogonal sum decomposition

(iv) $$f(x) = \sum_k a_k^{(0)} \phi(x-k) + \sum_k b_k^{(0)} \psi(x-k) = f_0 + g_0 .$$

with f_0 in $\mathcal{V}_0$, g_0 in $\mathcal{W}_0$, and

(v) $$a_k^{(0)} = \sum_j a_{j-2k}^{(1)} h_j \quad , \quad b_k^{(0)} = \sum_j (-1)^j a_{2k+1-j}^{(1)} h_j$$

Proof of (2.18). Because of the two-scale dilation property of ϕ and the defining property of ψ,

$$\int_{-\infty}^{\infty} \phi(x-\ell)\overline{\psi(x-m)}\, dx = 2 \sum_{j,k} (-1)^{k-1} h_j h_{1-k} \int_{-\infty}^{\infty} \phi(2x-2\ell-j)\overline{\phi(2x-2m-k)}\, dx$$

$$= \sum_{j,k} (-1)^{k-1} h_j h_{1-k} \int_{-\infty}^{\infty} \phi(x-2\ell-j)\phi(x-2m-k)\, dx = \sum_k (-1)^{k-1} h_{k+2m-2\ell} h_{1-k} .$$

Now change the index of summation by setting $n = 1 - k - 2m + 2\ell$. Then

$$\int_{-\infty}^{\infty} \phi(x-\ell)\overline{\psi(x-m)}\,dx = \sum_n (-1)^n h_{1-n} h_{n+2m-2\ell} = -\int_{-\infty}^{\infty} \phi(x-\ell)\overline{\psi(x-m)}\,dx \ ,$$

and so each integer translate $\phi(x-\ell)$ is orthogonal to each integer translate $\psi(x+m)$, completing the proof. ∎

Proof of (2.19). Suppose

$$\phi(2x) = \sum_j \alpha_j \phi(x-j) + \sum_j \beta_j \psi(x-j)$$

for some choice of sequences $\{\alpha_j\}$ and $\{\beta_j\}$. Then by orthonormality,

$$\alpha_k = \int_{-\infty}^{\infty} \phi(x)\overline{\phi(x-k)}\,dx \quad , \quad \beta_k = \int_{-\infty}^{\infty} \phi(2x)\overline{\psi(x-k)}\,dx \ .$$

Thus

$$\alpha_k = \sqrt{2}\ \sum_j h_j \int_{-\infty}^{\infty} \phi(2x)\overline{\phi(2x-2j-k)}\,dx = \frac{1}{\sqrt{2}} h_{-2k} \ .$$

Similarly,

$$\beta_k = \sqrt{2}\ \sum_j (-1)^{j-1} h_{1-j} \int_{-\infty}^{\infty} \phi(2x)\overline{\phi(2x-2j-k)}\,dx = -\frac{1}{\sqrt{2}} h_{1+2k} \ .$$

Entirely analogous arguments starting from the decomposition

$$\phi(2x+1) = \sum_j \gamma_j \phi(x-j) + \sum_j \delta_j \psi(x-j)$$

show that

$$\gamma_k = \frac{1}{\sqrt{2}} h_{1-2k} \quad , \quad \delta_k = \frac{1}{\sqrt{2}} h_{2k} \ .$$

This establishes (i) and (ii). Now each f in $\mathcal{V}_1$ has a representation

$$f(x) = \sqrt{2}\ \sum_k a_k^{(1)} \phi(2x-k)$$

for some sequence $\{a_k^{(1)}\}$ in $\ell^2(\mathbb{Z})$. So by (i) and (ii),

$$\begin{aligned} f(x) &= \sqrt{2}\Big\{\sum_k a_{2k}^{(1)} \phi(2x-2k) + \sum_k a_{2k+1}^{(1)} \phi(2x-2k-1)\Big\} \\ &= \sum_{j,k} \{a_{2k}^{(1)} h_{2j} \phi(x-k-j) - a_{2k}^{(1)} h_{2j+1} \psi(x-k-j)\} \\ &\qquad + \sum_{j,k} \{a_{2k+1}^{(1)} h_{2j+1} \phi(x-k-j) + a_{2k+1}^{(1)} h_{2j} \psi(x-k-j)\} \ . \end{aligned}$$

Thus

$$f(x) = \sum_k \Bigl(\sum_j a^{(1)}_{2j-2k} h_{2j} + a^{(1)}_{2j+1-2k} h_{2j+1}\Bigr) \phi(x-k)$$

$$+ \sum_k \Bigl(\sum_j a^{(1)}_{2k-2j+1} h_{2j} - a^{(1)}_{2k-2j} h_{2j+1}\Bigr) \psi(x-k) \ .$$

Properties (iv) and (v) now follow easily, completing the proof. ■

The precise form of decomposition (2.19)(iv) can be established also using (2.17). All the pieces have now been assembled so that a general wavelet decomposition result can be stated.

(2.20) THEOREM. *Let ϕ be a function in $\mathfrak{M}(\mathbb{R})$ such that*

(i) $\{\phi(x-k) : k \in \mathbb{Z}\}$ *is orthonormal in* $L^2(\mathbb{R})$, (ii) $\phi(x) = \sqrt{2}\,\sum_k h_k \phi(2x-k)$,

and define $\psi = \psi(x)$ by

$$\psi(x) = \sqrt{2}\,\sum_k (-1)^{k-1} h_{1-k} \phi(2x-k) \ .$$

Then $\{2^{\frac{1}{2}m}\psi(2^m x - k) : k, m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Thus each f in $L^2(\mathbb{R})$ admits an orthonormal wavelet-type decomposition

$$f(x) = \sum_{k,m} 2^m \Bigl(\int_{-\infty}^{\infty} f(y)\overline{\psi(2^m y - k)}\, dy\Bigr) \psi(2^m x - k)$$

in which the orthonormal basis elements are generated from a single wavelet mother function ψ by translation and dilation. More remarkable still perhaps is the fact that everything in theorem (2.20) and the results leading up to it depend solely on the coefficients $\{h_k\}$ in the two-scale; dilation equation for ϕ. The basic idea underlying the Daubechies construction is that this sole dependence on $\{h_k\}$ means that one should start with a sequence $\{h_k\}$, *then* construct ϕ and ψ, and hence obtain orthonormal wavelet-type decompositions.

3. Pre-historic wavelets. The idea of representing a function as a possibly infinite linear combination of functions having a special structure has had a long history going back to the introduction of Fourier series and Sturm-Liouville expansions in early nineteenth century, of course. Following Charles Fefferman's seminal work in H^p-theory in the 1970's, however, there was a vigorous development of the theory of so-called atomic and molecular decompositions of various H^p-spaces. Recall that a real-valued, compactly supported function $a = a(x)$ on $\mathbb{R}$ is said to be an *atom* of

$$\text{(3.1)} \qquad \text{(a)} \quad \sup_x |a(x)| \le 1 \qquad \text{(b)} \quad \hat{a}(0) = 0 \ .$$

The proof of theorem (3.) uses, either directly or indirectly, a great deal of the sophisticated modern theory of H^p-theory. But its impact was very far-reaching because it freed H^p-spaces from the confines of analytic function theory. It simplified the proof of many results and allowed H^p-theory to be extended to many different settings where no notion of analyticity was present. Then around 1978 Coifman and Rochberg gave a quite different decomposition for functions in a Bergman space $B(U)$ on domains U in one and several complex variables. When $B(U)$ is a Hilbert space of analytic functions on U the Coifman-Rochberg decomposition takes the form:

(i) *if $\Phi = \Phi(z, \zeta)$ is the Bergman kernel for $B(U)$ then for fixed lattice $\Lambda = \{\zeta_i : i \in I\}$ in U coefficients $\mathbf{a} = \{a_i\}_{i\in I}$ in a weighted $\ell^2(I)$-space the series*

$$\sum_{i\in I} a_i \Phi(z, \zeta_i) \qquad (z \in U)$$

defines a function in $B(U)$, and conversely,

(ii) *if the lattice Λ is chosen sufficiently fine, then every F in $B(U)$ can be written as*

$$F(z) = \sum_{i\in I} a_i \Phi(z, \zeta_i)$$

for some $\{a_i\}$ in the weighted $\ell^2(I)$-space.

In the language of frames therefore, the family $\Phi(z, \zeta_i)\}_{i\in I}$ is a frame for $B(U)$ provided the lattice $\{\zeta_i\}$ is sufficiently fine. This decomposition was noteworthy for at least two important reasons:

(a) *the basic building blocks* $\Phi(z, \zeta_i)$ *used in the decomposition all had a very specific structure: they were generated from one single mother function — the Bergman kernel* $\Phi = \Phi(z, \zeta)$ *— by evaluation at a fixed lattice* $\{\lambda_i\}$ *in* U*;*

(b) *the proof of properties* (i) *and* (ii) *was 'elementary' by contrast with the proof of the atomic decomposition for Hardy spaces. It depended on carefully estimating a Riemann sum approximation of the reproducing formula*

$$F(z) = \int_U F(\zeta)\Phi(z, \zeta)\, d\zeta \qquad (z \in U)$$

for functions in $B(U)$.

The simplicity of the proof of (i) and (ii) led immediately to substantial generalizations because the success of the proof depended only on the boundedness of one special operator on $B(U)$. This meant that analogous decompositions could be developed for a wide variety of Banach function spaces independently of connections to several complex variables. In this section we shall see how these modifications to the Coifman-Rochberg proof can be used to establish frame results for $L^2(\mathbb{R})$. With the benefit of hindsight one can discern many current ideas in wavelet theory in the work of Coifman-Rochberg and its generalizations, explaining the jocular use of the term pre-historic wavelets for what are really examples of frames. Whether they have practical importance, however, remains to be seen. Nonetheless, the construction is very simple and applies to a large class of functions.

(3.5) DEFINITION. *Denote by* $\mathcal{S}^{(1)}$ *the space of functions* ψ *on* $\mathbb{R}$ *such that*

(i) ψ *and its first derivative* ψ' *are continuous,*

(ii) *the inequalities*

$$|\psi(x)| \le \frac{A}{(1+x^2)^{3/2}} \quad , \quad |\psi'(x)| \le \frac{A}{(1+x^2)^{3/2}}$$

hold for some constant A *which many vary with* ψ.

Clearly $\mathcal{S}^{(1)}$ is a subspace of every $L^p(\mathbb{R})$-space, $1 \le p \le \infty$, that contains all compactly supported functions having at least one continuous derivative; in fact theory (5.7) in Part I ensures that $\mathcal{S}^{(1)}$ is a subspace of $\mathfrak{M}(\mathbb{R})$. The Fourier transform of functions in $\mathcal{S}^{(1)}$ can be further controlled.

(3.6) LEMMA. *The integral*

$$\int_{-\infty}^{\infty} |\hat{\psi}(\xi) - \hat{\psi}(0)|^2 \, \frac{d\xi}{|\xi|}$$

is finite for every ψ in $\mathcal{S}^{(1)}$.

This lemma ensures that a suitable constant multiple of any ψ in $\mathcal{S}^{(1)}$ can be used to define the continuous wavelet transform of functions in $L^2(\mathbb{R})$ provided ψ is wavelike, i.e., $\hat{\psi}(0) = 0$, and $|\hat{\psi}(-\xi)| = |\hat{\psi}(\xi)|$. For then

$$\psi_0(x) = \left(2 \Big/ \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \, \frac{d\xi}{|\xi|}\right) \psi(x)$$

is both wavelike and normalized. Both of

$$\text{(3.7)(i)} \qquad \psi_1(x) = e^{-\pi x^2} - \sqrt{2}\, e^{-\frac{1}{2}\pi x^2} \quad , \quad \psi_2(x) = (\sin 2\pi x) e^{-\pi x^2} \ ,$$

which were introduced earlier in (1.   ), are clearly examples of wavelike functions in $\mathcal{S}^{(1)}$ such that $|\hat{\psi}(-\xi)| = |\hat{\psi}(\xi)|$. Sufficiently high-order derivatives of the Poisson and Gauss-Weierstrass semi-groups

$$P(x,t) = \frac{1}{\pi} \, \frac{t}{x^2 + t^2} \quad , \quad W(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}$$

provide still examples. For instance, set

$$\text{(3.7)(ii)} \qquad \psi_3(x) = \left(\frac{\partial}{\partial t}\right)^2 P(x,t)\Big|_{t=1} \quad , \quad \psi_4(x) = \left(\frac{\partial}{\partial t}\right) W(x,t)\Big|_{t=1} \ .$$

Then

$$\text{(3.8)(i)} \qquad \psi_3(x) = \frac{2}{\pi} \, \frac{1 - 3x^2}{(1 + x^2)^3} \quad , \quad \psi_4(x) = \frac{1}{8\pi^{1/2}} (x^2 - 2) e^{-x^2/4} \ ,$$

and

$$\text{(3.8)(ii)} \qquad \hat{\psi}_3(\xi) = (2\pi\xi)^2 e^{-2\pi|\xi|} \quad , \quad \hat{\psi}_4(\xi) = (2\pi\xi)^2 e^{-4\pi^2\xi^2} \ ,$$

so ψ_3 and ψ_4 also are wavelike like functions in $\mathcal{S}^{(1)}$ such that $|\hat{\psi}(-\xi)| = |\hat{\psi}(\xi)|$.

Proof of (3.6). By definition

$$\hat{\psi}(\xi) - \hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x) \{e^{-2\pi i x\xi} - 1\} \, dx \ .$$

Consequently,

$$\begin{aligned}|\hat\psi(\xi)-\hat\psi(0)| &\le \int_{-\infty}^{\infty} |\psi(x)|\;|e^{-2\pi i x\xi}-1|\,dx \\ &\le 2\int_{-\infty}^{\infty}(1+x^2)^{1/2}|\psi(x)|\;\frac{|\sin \pi x\xi|}{(1+x^2)^{1/2}}\,dx\ .\end{aligned}$$

Thus

$$\begin{aligned}|\hat\psi(\xi)-\hat\psi(0)|^2 &\le 2\Big(\int_{-\infty}^{\infty}(1+x^2)|\phi(x)|^2\Big)\Big(\int_{-\infty}^{\infty}\frac{\sin^2\pi x\xi}{1+x^2}\,dx\Big) \\ &\le A\Big(\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^2}\,dx\Big)\Big(\int_{-\infty}^{\infty}\frac{1-\cos 2\pi x\xi}{1+x^2}\,dx\Big) \\ &= \tfrac12\pi^2 A(1-e^{-2\pi|\xi|})\ .\end{aligned}$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty}|\hat\psi(\xi)-\hat\psi(0)|^2\,\frac{d\xi}{|\xi|} &\le \tfrac12\pi^2 A\int_{-\infty}^{\infty}(1-e^{-2\pi|\xi|})\,\frac{d\xi}{|\xi|} \\ &= \pi^2 A\int_{0}^{\infty}(1-e^{-t})\,\frac{dt}{t}<\infty\ ,\end{aligned}$$

establishing the finiteness of the left hand integral. ■

At last we can state, examine and then prove the most basic result of this section. It will be the first of five fundamental constructions wavelet-type decompositions of $L^2(\mathbb{R})$ that will be given in this and the next three sections.

(3.9) THEOREM. *Let ψ be any normalized wavelike function in $\mathcal{S}^{(1)}$. Then there exists $R>1$ so that the family*

$$\Big\{r^{\frac12 m}\psi\big(r^m x+(1-r)k\big) : k,m\in\mathbb{Z}\Big\}$$

is a frame for $L^2(\mathbb{R})$ for each r, $1<r<R$.

4. Splines as pre-wavelets. In a series of beautiful papers Charles Chui *et al.* constructed families of pre-wavelets using splines. This construction is likely to be very useful as splines arise naturally in many areas of applied mathematics and can be treated very conveniently using numerical techniques. But the development of the theory of wavelets using splines involves some fascinating excursions into mathematics developed originally by Euler, so the theory is of considerable interest in its own right.

Mathematically, a spline is a piecewise polynomial function. To be more precise, let $\{x_k : k \in \mathbb{Z}\}$ be a fixed sequence of points in $(-\infty, \infty)$. Then a function f is said to be a *polynomial spline* of degree m with *knots* $\{x_k : k \in \mathbb{Z}\}$ when

(i) *f has continuous derivatives on $(-\infty, \infty)$ up to order $m-1$,*

(ii) *between knots f is a polynomial of degree at most m.*

Thus

$$f(x) = \sum_{k=-\infty}^{\infty} P_k(x)\chi_k(x) \qquad (\chi_k = \chi_{[x_k, x_{k+1})}) \tag{4.1}$$

where $\chi_{[x_k,x_{k+1})}$ is the characteristic function of the interval $[x_k, x_{k+1})$ and P_k is a polynomial in $\mathcal{P}_m$. There is an entirely analogous definition for splines on a finite sub-interval of $(-\infty, \infty)$. The term *spline* comes from drafting as well as from ship-building. For instance, by using a French curve on a drafting table, one fits a suitably smooth curve through a given set of points. On the other hand, in ship-building a spline is a flexible strip that can be held by weights so that it passes through each of a given set of points but goes smoothly from one interval to the next according to the laws of beam flexure. In mathematical terms these correspond to finding a polynomial spline f which takes on pre-assigned values $f(x_k) = \alpha_k$ at the knots. For us the knots x_k will always be located a constant distance apart — in technical language, the knots are said to be *equi-distant* or have *constant mesh size*. The case $x_k = k$, $k \in \mathbb{Z}$, will be particularly important.

(4.2) DEFINITION. *For each integer $m = 0, 1, \ldots$ we shall denote by $\mathfrak{S}_m$ the set of all real or complex-valued degree m polynomial splines whose knots are at the integers $\{k : k \in \mathbb{Z}\}$. Those f in $\mathfrak{S}_m$ for which*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

will be denoted by $\mathcal{L}_m$.

The functions in $\mathfrak{S}_m$ are usually called *Cardinal* splines, while those in $\mathcal{L}_m$ are said to be *Square-integrable* Cardinal splines. Thus each f in $\mathfrak{S}_m$ can be written as

$$f(x) = \sum_{k=-\infty}^{\infty} P_k(x)\chi_1(x-k) \qquad (\chi_1 = \chi_{[0,1]}) \tag{4.3}$$

where $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1)$.

(4.4) EXAMPLES. (i) *Every degree* m *polynomial is a degree* m *Cardinal spline.*

(ii) *More generally than* (i), *set*

$$x_+ = \max(0,x) \quad , \quad x_+^m = \big(\max(0,x)\big)^m \ .$$

Then x_+^m *lies in* $\mathfrak{S}_m$*; in fact, every finite linear combination*

$$f(x) = a_1(x-k_1)_+^m + a_2(x-k_2)_+^m + \cdots + a_n(x-k_n)_+^m$$

is a degree m *Cardinal spline whatever the choice of scalars* $a_1,\ldots,a_n$ *and integers* $k_1,\ldots,k_n$.

The most 'basic' of all Cardinal splines are conveniently defined through (4.4)(ii), taking as coefficients the binomial coefficients

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}$$

For small values of m the binomial coefficients are conveniently described diagrammatically by Pascal's Triangle:

Pascal's triangle.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

(4.5) DEFINITION. *For each integer $m = 1, 2, \ldots$ set*

$$N_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (x-k)_+^{m-1} .$$

This degree $(m-1)$ spline is called the ***Forward Cardinal B-spline****, **B** for **Basis**.*

The simplest examples of B-splines have appeared several times already in various guises since, as simple calculations show,

(4.6)(i) $$N_1(x) = \chi_1(x) = \begin{cases} 1 & (0 \le x < 1) , \\ 0 & (\text{otherwise}), \end{cases}$$

is just the characteristic function of $[0, 1)$ while

(4.6)(ii) $$N_2(x) = \begin{cases} x & (0 \le x < 1) , \\ 2 - x & (1 \le x < 2) , \\ 0 & (\text{otherwise}), \end{cases}$$

so that N_2 is the tent function centered at $x = 1$. Graphs of all the B-splines $N_1, \ldots, N_4$ are given below.

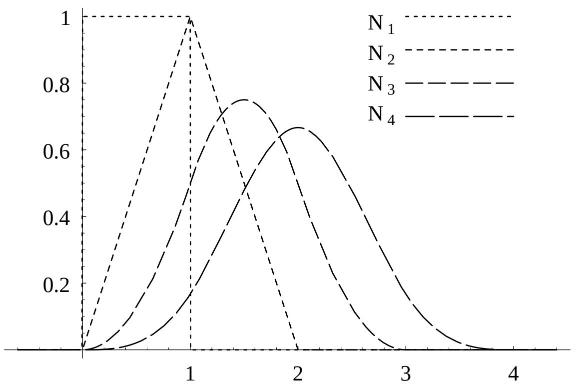

Figure 3

Notice particularly how these graphs foretell a basic property of B-splines:

$$(4.7) \qquad N_m(x) = 0 \;\; (x \le 0) \;, \;\; N_m(x) > 0 \;\; (0 < x < m) \;, \;\; N_m(x) = 0 \;\; (x \ge m) \;,$$

i.e., the support of N_m is precisely the interval $[0, m]$, which accounts for the *forward* part of the name for N_m. The compactness of the support of N_m ensures also that every infinite sum

$$f(x) = \sum_{k=-\infty}^{\infty} a_k N_m(x-k)$$

is well defined because at most m functions in the sum are non-zero at any given value of x, and so each such sum is a Cardinal spline of degree $(m-1)$. In fact, an even more precise result is true. It is one which is fundamental to any study of splines, and it clearly accounts for the name B-spline or *Basis* spline.

Consequently, every infinite sum

$$f(x) = \sum_{k=-\infty}^{\infty} a_k N_m(x-k)$$

is well defined because at most m functions in the sum are non-zero at any given value of x, and so each such sum is a Cardinal spline of degree $(m-1)$. In fact, an even more precise result is true. It is one which is fundamental to any study of splines, and it clearly accounts for the name B-spline or Basis spline. Let $s(\mathbb{Z})$ denote the space of all sequences.

(4.8) THEOREM. *The set of integer translates* $\{N_m(x-k) : k \in \mathbb{Z}\}$ *of the Cardinal* B*-spline* N_m *is linearly independent, and the function*

$$f(x) = \sum_{k=-\infty}^{\infty} a_k N_m(x-k) \qquad \big(\{a_k\} \in s(\mathbb{Z})\big)$$

defines a degree-$(m-1)$ *Cardinal spline for each sequence* $\mathbf{a} = \{a_k\}$ *in* $s(\mathbb{Z})$*; conversely, every* f *in* $\mathfrak{S}_{m-1}$ *has a unique representation in this form for some* $\mathbf{a} = \{a_k\}$ *in* $s(\mathbb{Z})$.

A proof of (4.8) is given in the notes and comments for this section. As a Cardinal spline is therefore completely determined by the numerical sequence $\{a_k\}$ of coefficients, it is not surprising that splines play a profound role in numerical mathematics. One part of the theory of wavelets is the search for other functions ϕ, defined in terms of B-splines of course, so that every f in $\mathfrak{S}_m$ can be written uniquely and more conveniently as $f(x) = \sum_k a_k \phi(x-k)$ for some choice of coefficients a_k.

Now in practice one doesn't always deal with polynomial splines of arbitrary degree, especially in visual settings. For the eye obviously 'sees' continuity and slope, as well as convexity, *i.e.*, derivatives of order 0, 1 and 2, but it cannot 'detect' derivatives of order 3 or higher. Consequently, when polynomial splines are used to generate fonts in computer type-setting for instance, only those polynomial splines imposing continuity of derivatives up to order 2 need be used. Hence, for many practical purposes, polynomial splines of degree 3, *i.e.*, *cubic splines*, usually suffice. For this reason, as well as for simplicity, we too will concentrate on cubic splines or splines of lesser degree. Since this entails knowledge of the properties of higher degree polynomial splines we shall first develop some aspects of the theory of splines in general.

In establishing properties of N_m such as those in (4.7) the definition in (4.5) is not particularly convenient, however. A recurrence formula which follows from (4.5) will be much more useful when used in conjunction with the known functions N_1 and N_2.

(4.9) LEMMA. *The forward* B*-spline* N_m *satisfies the recurrence relation*

$$N_m(x) = \frac{1}{m-1}\big(xN_{m-1}(x) + (m-x)N_{m-1}(x-1)\big)$$

for all $m = 1, 2, \ldots$.

Proof. We simply substitute (4.5) for $N_{m-1}(x)$ and then check that the right hand becomes $N_m(x)$ as defined by (4.5). Indeed, by (4.5),

$$\begin{aligned}
&\frac{1}{m-1}\big(xN_{m-1}(x) + (m-x)N_{m-1}(x-1)\big)\\
&\quad= \frac{1}{(m-1)!}\left(x\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(x-k)_+^{m-2} + (m-x)\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(x-k-1)_+^{m-2}\right)\\
&\quad= \frac{1}{(m-1)!}\left(x\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}(x-k)_+^{m-2} - (m-x)\sum_{k=1}^{m}(-1)^k\binom{m-1}{k-1}(x-k)_+^{m-2}\right) .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{m-1}\big(xN_{m-1}(x) + (m-x)N_{m-1}(x-1)\big)\\
&\quad= \frac{1}{(m-1)!}\Bigg(x_+^{m-1} + (-1)^m(x-m)_+^{m-1}\\
&\qquad + \sum_{k=1}^{m-1}(-1)^k(x-k)_+^{m-2}\left[x\binom{m-1}{k} - (m-k)\binom{m-1}{k-1}\right]\Bigg) .
\end{aligned}$$

But

$$\begin{aligned}
x\binom{m-1}{k} - (m-x)\binom{m-1}{k-1} &= \frac{x(m-1)!}{k!(m-k-1)!} - \frac{(m-x)(m-1)!}{(k-1)!(m-k)!}\\
&= \frac{(m-1)!}{k!(m-k)!}\big(x(m-k) - k(m-x)\big) = \frac{m!(x-k)}{k!(m-k)!} .
\end{aligned}$$

Thus

$$\frac{1}{m-1}\big(xN_{m-1}(x)+(m-x)N_{m-1}(x-1)\big) = \frac{1}{(m-1)!}\sum_{k=0}^{m}(-1)^m\binom{m}{k}(x-k)_+^{m-1} = N_m(x) ,$$

completing the proof. ∎

Properties of N_m and associated B-splines can now follow quickly.

(4.10) THEOREM. *For each $m = 1, 2, \ldots$, the forward B-spline N_m has the properties*

(i) $\operatorname{supp} N_m = [0, m]$, $\quad N_m > 0$ *on* $(0, m)$,

(ii) $\hat{N}_m(\xi) = \left[\dfrac{1 - e^{-2\pi i \xi}}{2\pi i \xi}\right]^m \qquad (x \in \mathbb{R})$,

(iii) $N_m(x) = N_m(m - x)$

Since the Fourier transform of the characteristic function χ_1 of $[0, 1)$ is given by

$$\hat{\chi}_1(\xi) = \frac{1 - e^{-\pi i \xi}}{2\pi i \xi} \ ,$$

property (4.10)(ii) provides an interesting alternative definition of N_m as the m-fold convolution product

$$N_m(x) = \chi_1 * \chi_1 * \cdots * \chi_1(x) \ . \tag{4.11}$$

Proof of (4.10). The proof is by induction on m.

(i) Suppose

$$\operatorname{supp} N_{m-1} = [0, m-1] \ , \qquad N_{m-1} > 0 \ \text{ on } \ (0, m-1) \ . \tag{4.12}$$

Then

$$x N_{m-1}(x) + (m - x) N_{m-1}(x - 1) > 0 \qquad (0 < x < m) \ ,$$

while

$$x N_{m-1}(x) + (m - x) N_{m-1}(x - 1) = 0 \qquad (x < 0 \ , \ x \geq m) \ ,$$

and so (4.12) holds with $m - 1$ replaced by m. But by (4.76), property (4.12) certainly holds for $m = 2$. Hence by induction, property (4.12) holds for all $m = 2, 3, \ldots$.

(ii) Being a continuous compactly supported function, each N_m belongs both to $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, and so the Fourier transform $\hat{N}_m$ of N_m will be the recurrence formula

$$\hat{N}_m(\xi) = e^{-2\pi i \xi} \hat{N}_{m-1}(\xi) - \frac{1}{m-1} \frac{1}{2\pi i}(1 - e^{-2\pi i \xi}) \frac{d}{d\xi} \hat{N}_{m-1} \tag{4.13}$$

obtained by taking the Fourier transform of (4.9). Suppose then that

$$\hat{N}_{m-1}(\xi) = \left[\frac{1 - e^{-2\pi i \xi}}{2\pi i \xi}\right]^{m-1} \ . \tag{4.14}$$

In view of (4.6), this is certainly true for $m=2$. But

$$\frac{d}{d\xi}\left[\frac{1-e^{-2\pi i\xi}}{2\pi i\xi}\right]^{m-1} = m-1\left(\frac{2\pi i\xi-(1-e^{-2\pi i\xi})}{\xi}\right)\left(\frac{1-e^{-2\pi i\xi}}{2\pi i\xi}\right)^{m-1}$$
$$= \frac{1}{\xi}(m-1)\big(2\pi i\xi-(1-e^{-2\pi i\xi})\big)\hat{N}_{m-1}(\xi)\ .$$

Consequently, by (4.13) and (4.14)

$$\hat{N}_m(\xi) = \left[\frac{1-e^{-2\pi i\xi}}{2\pi i\xi}\right]^m \quad .$$

Hence (4.14) holds for all $m=2,3,\ldots$.

(iii) Suppose

$$N_{m-1}(x) = N_{m-1}(m-1-x)\ . \tag{4.15}$$

Since

$$\chi_1(x)=\chi_1(1-x) \qquad (\chi_1=\chi_{[0,\infty)})\ ,$$

property (4.15) holds for $m=2$. But by (4.6) and (4.15),

$$N_m(x) = \frac{1}{m-1}\big(xN_{m-1}(x)+(m-x)N_{m-1}(x-1)\big)$$
$$= \frac{1}{m-1}\big(xN_{m-1}(m-1-x)+(m-x)N_{m-1}(m-x)\big)\ ,$$

while by (4.6) alone,

$$N_m(m-x) = \frac{1}{m-1}\big((m-x)N_{m-1}(m-x)+xN_{m-1}(m-x-1)\ .$$

Thus

$$N_m(x) = N_m(m-x)\ ,$$

and so (4.15) holds for all $m=2,3,\ldots$, completing the proof. ■

Property (4.10)(iii) shows that the forward B-spline N_m is symmetric about the line $x=\frac{1}{2}m$. This suggests that one might translate N_m so that the y-axis becomes the line of symmetry. Unless m is even however, the knots of this translated B-spline will occur at half-integers $\{k+\frac{1}{2}:k\in\mathbb{Z}\}$. All this is by way of motivation for the following definition.

(4.16) Definition. *For each $m = 1, 2, \ldots$ the polynomial spline given by*

$$M_{2m}(x) = N_{2m}(x + m)$$

is called the central degree $(2m-1)$ B-spline.

In view of (4.5) there is also a direct definition

$$M_{2m}(x) = \frac{1}{(2m-1)!} \sum_{k=-m}^{m} (-1)^{m+k} \binom{2m}{m+k} (x-k)_+^{2m-1}$$

of M_{2m} in terms of integer translates of x_+^{2m-1}. From this or from (4.6) we see that

(4.17)(i)
$$M_2(x) = \begin{cases} 1 - |x| & (|x| < 1)\ , \\ 0 & \text{(otherwise)}, \end{cases}$$

is the *even* tent function which we have used repeatedly so far; while

(4.17)(ii)
$$M_4(x) = \begin{cases} \frac{1}{6}\big((2-|x|)^3 - 4(1-|x|)^3\big) & (0 \le |x| < 1)\ , \\ \frac{1}{6}(2-|x|)^3 & (1 \le |x| < 2)\ , \\ 0 & \text{(otherwise)}. \end{cases}$$

is the central *cubic* B-spline which we shall use repeatedly in what follows. The fundamental properties of M_{2m} in general, however, can be read off conveniently from (4.8), (4.10) and (4.11).

(4.18) Theorem. *For each $m = 1, 2, \ldots$ the set of integer translates $\{M_{2m}(x-k) : k \in \mathbb{Z}\}$ of the central B-spline M_{2m} is linearly independent, and the function*

$$f(x) = \sum_k a_k M_{2m}(x-k) \qquad \big(\{a_k\} \in s(\mathbb{Z})\big)$$

defines a spline in $\mathcal{C}_{2m-1}$ for each $\mathbf{a} = \{a_k\}$ in $s(\mathbb{Z})$; conversely, every f in $\mathcal{C}_{2m-1}$ has a unique representation in this form for some $\mathbf{a} = \{a_k\}$ in $s(\mathbb{Z})$.

Since M_{2m} is just an *integer* translate of N_{2m}, theorem (4.18) follows immediately from (4.18). Simple such modifications to the statements of (4.10) and (4.11) lead easily to further properties of M_{2m}.

(4.19) THEOREM. *For each $m = 1, 2, \ldots$ the central B-spline M_{2m} has the properties*

(i) $\operatorname{supp} M_{2m} = [-m, m], \qquad M_{2m} > 0$ *on* $(-m, m)$,

(ii) $\hat{M}_{2m}(\xi) = \left(\dfrac{\sin \pi\xi}{\pi\xi}\right)^{2m} \qquad (\xi \in \mathbb{R})$,

(iii) M_{2m} *is the* $2m$*-fold convolution product* $\chi * \chi * \cdots * \chi$ *of the characteristic function* $\chi = \chi_{[-1/2, 1/2)}$ *of the interval* $[-\frac{1}{2}, \frac{1}{2})$.

We can now begin to construct spline wavelets. The characterization

$$\mathcal{C}_{2m-1} = \left\{ \sum_k a_k M_{2m}(x-k) : \{a_k\} \in s(\mathbb{Z}) \right\}$$

of all degree $(2m-1)$ polynomial splines is a good starting point because it is so reminiscent of the space

$$\mathcal{V} = \left\{ \sum_k a_k M_{2m}(x-k) : \{a_k\} \in \ell^2(\mathbb{Z}) \right\} \tag{4.20}$$

studied in so much detail in the preceding sections. Indeed, each function M_{2m} belongs to $L^2(\mathbb{R})$ since it is a continuous function having compact support, so $\mathcal{V}$ is a subspace of $L^2(\mathbb{R})$ and each spline which has a representation $\sum_k a_k M_{2m}(x-k)$ will be square integrable whenever $\{a_k\}$ lies in $\ell^2(\mathbb{Z})$; in particular, therefore, $\mathcal{V} \subseteq \mathcal{L}_{2m-1}$. To make further progress, however, we need to show that M_{2m} is a stable function in $\mathfrak{M}(\mathbb{R})$. This is one point at which the mathematics becomes particularly interesting.

(4.21) THEOREM. *For each $m = 1, 2, \ldots$ the central B-spline M_{2m} is a stable function in $\mathfrak{M}(\mathbb{R})$, and the corresponding closed subspace*

$$\mathcal{V} = \left\{ \sum_k a_k M_{2m}(x-k) : \{a_k\} \in \ell^2(\mathbb{Z}) \right\}$$

of $L^2(\mathbb{R})$ coincides with the space $\mathcal{L}_{2m-1}$ of all square-integrable cardinal splines.

Proof. We have seen already in example (5.8)(ii) of Part I that the tent function M_2 belongs to $\mathfrak{M}(\mathbb{R})$ and that its associated overlap function Φ_2 is given by

$$\Phi_2(\xi) = \left(\sum_k |M_2(\xi + k)|^2 \right)^{1/2} = \left(\tfrac{1}{3}(2 + \cos 2\pi\xi)\right)^{1/2}$$

since

$$\sum_k \frac{1}{(\xi+k)^4} = \left(\frac{\pi}{\sin \pi\xi}\right)^4 \left\{\tfrac{1}{3}(2+\cos 2\pi\xi)\right\} .$$

Thus $\Phi_2 \geq 1/\sqrt{3}$, and so M_2 is stable. But, more generally, in view of (4.19)(i), each M_{2m} is a compactly supported function having $(2m-2)$-derivatives; trivially, therefore,

$$|M_{2m}(x)| + |M''_{2m}(x)| \leq \frac{A}{1-x^2} \qquad (x \in \mathbb{R})$$

for each $m \geq 2$, and so theorem (5.6) in Part I guarantees that each such M_{2m} is a function in $\mathfrak{M}(\mathbb{R})$. On the other hand, by (4.19)(ii), the overlap function Φ_{2m} associated with M_{2m} is given by

$$\Phi_{2m}(\xi) = \left(\left(\frac{\sin\pi\xi}{\pi\xi}\right)^4 + \left(\frac{\sin\pi\xi}{\pi}\right)^{4m} \sum_k \frac{1}{(\xi+k)^4}\right)^{1/2} ,$$

i.e.,

$$\Phi_{2m}(\xi) \geq \left(\frac{\sin\pi\xi}{\pi\xi}\right)^{2m} \geq \left(\frac{2}{\pi}\right)^{2m} \qquad \left(-\tfrac{1}{2} \leq \xi \leq \tfrac{1}{2}\right) .$$

Thus Φ_{2m} is a continuous period 1 which is bounded below by $(2/\pi)^{2m}$ on $[-\frac{1}{2}, \frac{1}{2}]$, and so each M_{2m} must be stable. Hence, for each m, there exist positive constants A and B such that

$$A\left(\sum_k |a_k|^2\right)^{1/2} \leq \left\|\sum_k a_k M_{2m}(x-k)\right\|_{L^2} \leq B\left(\sum_k |a_k|^2\right)^{1/2} .$$

Now let f be a square integrable degree-$(2m-1)$ cardinal spline. By (4.18), it can be written as $\sum_k a_k M_{2m}(x-k)$ for some sequence $\{a_k\}$ in $s(\mathbb{Z})$; but then because of the stability of M_{2m},

$$\|f\|_{L^2} = \left\|\sum_k a_k M_{2m}(x-k)\right\|_{L^2} \geq A\left(\sum_k |a_k|^2\right)^{1/2} ,$$

which forces $\{a_k\}$ to be in $\ell^2(\mathbb{Z})$. Thus $\mathcal{L}_{2m-1} \subseteq \mathcal{V}$, completing the proof since we have shown already that $\mathcal{V} \subseteq \mathcal{L}_{2m-1}$. ∎

Just as the overlap function Φ_2 associated with M_2 has a simple closed form expression

(4.22) $$\Phi_2(\xi) = \left(\tfrac{1}{3}(2+\cos 2\pi\xi)\right)^{1/2} ,$$

so there are corresponding closed form expressions for the overlap function Φ_{2m} associated with M_{2m}, as well as many other interesting properties of M_{2m} which follow from applications of the Poisson summation formula. We begin with a conveniently chosen definition.

(4.23) Definition. *For each $m = 2, 3, \ldots$ set*

$$F_m(\xi) = (-1)^m \frac{1}{(m-1)!} \left(\frac{\sin \pi\xi}{\pi}\right)^m \left(\frac{d}{d\xi}\right)^{m-2} \left(\frac{\pi}{\sin \pi\xi}\right)^2 .$$

Clearly

$$F_2(\xi) \equiv 1 , \quad F_3(\xi) = \cos \pi\xi , \quad F_4(\xi) = \tfrac{1}{3}(2 + \cos 2\pi\xi) . \tag{4.23)(i}$$

In fact, since

$$F_{m+1}(\xi) = \cos \pi\xi F_m(\xi) - \frac{1}{m}\left(\frac{\sin \pi\xi}{\pi}\right) \frac{dF_m}{d\xi} , \tag{4.24}$$

an easy induction argument shows that

$$F_{2m}(\xi) = \sum_{k=0}^{m-1} a_k \cos 2\pi k\xi , \tag{4.25)(i}$$

for some choice of constants $a_0, \ldots, a_{m-1}$, while

$$F_{2m+1}(\xi) = \sum_{k=0}^{m-1} b_k \cos(2k+1)\pi\xi \tag{4.25)(ii}$$

for some other choice of constants $b_0, \ldots, b_{m-1}$. Recurrence formulae for these a_k and b_k can be derived, but they aren't very simple. On the other hand, by (4.17)(iii) and (4.23)(i),

$$F_4(\xi) = \tfrac{1}{3}(2 + \cos 2\pi\xi) = \tfrac{1}{3!} \sum_k M_4(k) e^{2\pi i k\xi}$$

suggesting that the coefficients in (4.25)(i) should be related to the values of M_{2m} at its knots.

(4.26) Theorem. *For each $m = 1, 2, \ldots,$*

$$F_{2m}(\xi) = \sum_k M_{2k}(k) e^{2\pi i k\xi} ,$$

i.e., F_{2m} is the Fourier series having the values $\{M_{2m}(k)\}$ of M_{2m} at its knots as Fourier coefficients.

Proof. As with many of the results in this section, the proof is essentially just an application of the Poisson summation formula. Indeed, since M_{2m} is in $\mathfrak{M}(\mathbb{R})$,

$$\sum_k M_{2k}(k) e^{2\pi i k\xi} = \sum_k \hat{M}_{2m}(k + \xi) = \left(\frac{\sin \pi\xi}{\pi}\right)^{2m} \sum_k \frac{1}{(\xi + k)^{2m}} .$$

But by example (5.8)(ii) in Part I,

$$\sum_k \frac{1}{(\xi+k)^{2m}} = \frac{1}{(2m-1)!}\left(\frac{d}{d\xi}\right)^{m-2} \sum_k \frac{1}{(\xi+k)^2} = \frac{1}{(2m-1)!}\left(\frac{d}{d\xi}\right)^{m-2} \left(\frac{\pi}{\sin \pi\xi}\right)^2 ,$$

and so

$$\sum_k M_{2m}(k) e^{2\pi i k\xi} = \frac{1}{(2m-1)!}\left(\frac{\sin \pi\xi}{\pi}\right)^{2m} \left(\frac{d}{d\xi}\right)^{m-2} \left(\frac{\pi}{\sin \pi\xi}\right)^2 ,$$

completing the proof. ∎

Since M_{2m} is an even function therefore,

$$F_{2m}(\xi) = M_{2m}(0) + 2\sum_{k=1}^{m-1} M_{2m}(k) \cos 2\pi k\xi \ , \tag{4.27}$$

which determines the coefficients a_k completely explicitly. In view of (4.16)(i), the actual value of $M_{2m}(k)$ is given by

$$\begin{aligned} M_{2m}(k) &= \frac{1}{(2m-1)!} \sum_{j=-m}^{k-1} (-1)^j \binom{2m}{m+j} (k-j)^{2m-1} \\ &= \frac{1}{(2m-1)!} \sum_{s=1}^{k+m} (-1)^{s-k} \binom{2m}{m+s-k} s^{2m-1} \ ; \end{aligned} \tag{4.28)(i}$$

alternatively, they can also be expressed in terms of the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ introduced in the notes and comments for this section:

$$M_{2m}(k) = \frac{1}{(2m-1)!} \left\langle {2m-1 \atop m+k-1} \right\rangle . \tag{4.28)(ii}$$

The explicit closed form expression for the overlap function

$$\Phi_{2m}(\xi) = \left(\sum_k |\hat{M}_{2m}(\xi+k)|^2\right)^{1/2} = \left(\sum_k \left(\frac{\sin \pi\xi}{\pi(\xi+k)}\right)^{4m}\right)^{1/2}$$

associated with M_{2m} now follows immediately. Indeed,

$$\begin{aligned} \sum_k \left(\frac{\sin \pi\xi}{\pi(\xi+k)}\right)^{4m} &= \left(\frac{\sin \pi\xi}{\pi}\right)^{4m} \sum_k \frac{1}{(\xi+k)^{4m}} \\ &= \frac{1}{(2m-1)!}\left(\frac{\sin \pi\xi}{\pi}\right)^{4m} \left(\frac{d}{d\xi}\right)^{4m-2} \sum_k \frac{1}{(\xi+k)^2} = F_{4m}(\xi) \ ; \end{aligned}$$

consequently,

$$\Phi_{2m}(\xi) = \big(F_{4m}(\xi)\big)^{1/2} = \left(\sum_k M_{4m}(k) e^{2\pi i k\xi}\right)^{1/2} . \tag{4.29}$$

Explicit formulations of the various cardinal splines associated with M_{2m} also follow easily.

(4.30) Definition. *For each $m = 1, 2, \ldots$ let O_{2m}, $\tilde{M}_{2m}$ and L_{2m} be the degree-$(2m-1)$ cardinal spline having the respective properties*

(i) *$\{O_{2m}(x+k) : k \in \mathbb{Z}\}$ is an orthonormal family,*

(ii) *$\{\tilde{M}_{2m}(x+k) : k \in \mathbb{Z}\}$ and $\{M_{2m}(x+k) : k \in \mathbb{Z}\}$ are bi-orthogonal families,*

(iii) $L_{2m}(0) = 1 \ , \quad L_{2m}(k) = 0 \quad (k \neq 0)$.

Each one of

$$\{O_{2m}(x+k) : k \in \mathbb{Z}\} \ , \quad \{\tilde{M}_{2m}(x+k) : k \in \mathbb{Z}\} \ , \quad \{L_{2m}(x+k) : k \in \mathbb{Z}\}$$

is a basis for L_{2m-1}, each having an important characteristic property: given f in L_{2m-1}, then

$$\text{(i)} \qquad f(x) = \sum_k a_k O_{2m}(x+k) \ , \qquad a_k = \int_{-\infty}^{\infty} f(y)\overline{O_{2m}(y+k)}\, dy \ ,$$

$$\text{(4.31)} \qquad \text{(ii)} \qquad f(x) = \sum_k b_k \tilde{M}_{2m}(x+k) \ , \qquad b_k = \int_{-\infty}^{\infty} f(y)\overline{M_{2m}(y+k)}\, dy \ ,$$

$$\text{(iii)} \qquad f(x) = \sum_k f(k) L_{2m}(x-k) \ .$$

Property (4.31)(iii) shows how useful the L_{2m} function is: *a square-integrable cardinal spline f is completely determined by its values $\{f(k)\}_{k\in\mathbb{Z}}$ at its knots and* (4.31)(iii) *provides an explicit reconstruction of the spline from these sample values.*

(4.32) Theorem. *The Fourier transforms of O_{2m}, $\tilde{M}_{2m}$ and L_{2m} are given by*

$$\hat{O}_{2m}(\xi) = \hat{M}_{2m}(\xi)/\big(F_{4m}(\xi)\big)^{1/2} \quad , \quad (\tilde{M}_{2m})^\wedge(\xi) = \hat{M}_{2m}(\xi)/F_{4m}(\xi) \ ,$$

and

$$\hat{L}_{2m}(\xi) = \hat{M}_{2m}(\xi)/F_{2m}(\xi)$$

respectively.

Expanding as Fourier series the period 1 functions in the denominators in (4.32) we obtain the corresponding expressions on the function side.

(4.33) Corollary. *The functions O_{2m}, $\tilde{M}_{2m}$ and L_{2m} are given as sums of integer translates of the central B-spline M_{2m} by*

$$O_{2m}(x) = \sum_k \alpha_k M_{2m}(x+k) , \quad \tilde{M}_{2m}(x) = \sum_k \beta_k M_{2m}(x+k) , \quad L_{2m}(x) = \sum_k \gamma_k M_{2m}(x+k)$$

where

$$\frac{1}{(F_{4m}(\xi))^{1/2}} = \sum_k \alpha_k e^{2\pi i k\xi} , \quad \frac{1}{F_{4m}(\xi)} = \sum_k \beta_k e^{2\pi i k\xi} , \quad \frac{1}{F_{2m}(\xi)} = \sum_k \gamma_k e^{2\pi i k\xi} .$$

After using some involved partial fractional arguments in the notes and comments for this section we have exhibited the precise values of the Fourier coefficients $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$. For computational purposes, however, it is more important perhaps to note simply that all three sets of Fourier coefficients have exponential decay and can be readily computed via numerical integration — in other words, efficient numerical techniques can be applied to each function O_{2m}, $\tilde{M}_{2m}$ and L_{2m}.

Proof of (4.32). The Fourier transform expressions for O_{2m} and M_{2m} follow immediately from (3.18) and (3.22) using (4.29). On the other hand,

$$\hat{L}_{2m}(\xi) = \hat{M}_{2m}(\xi)/F_{2m}(\xi) \tag{4.34}$$

certainly defines a function in $\mathfrak{L}_{2m-1}$ and $\mathfrak{M}(\mathbb{R})$ since, by the Wiener Tauberian theorem, $1/F_{2m}$ will have an absolutely convergent Fourier series.

We have to check that this function has the defining properties of the corresponding function in (4.30)(iii). Now, by the Poisson summation formula applied to (4.34),

$$\begin{aligned}\sum_k L_{2m}(k)e^{2\pi i k\xi} &= \sum_k \hat{L}_{2m}(\xi+k) = F_{2m}(\xi)\sum_k \hat{M}_{2m}(\xi+k) \\ &= F_{2m}(\xi)\Big(\sum_k M_{2m}(k)e^{2\pi i k\xi}\Big) = 1 \qquad (\xi \in \mathbb{R}) ,\end{aligned}$$

applying the Poisson summation formula to M_{2m} as well. But this will happen if and only if

$$L_{2m}(0) = 1 , \qquad L_{2m}(k) = 0 \qquad (k \neq 0) ,$$

completing the proof. ∎

Finally let us turn to multi-resolutions using splines and constructions of spline pre-wavelet mother functions. Yet again, splines seem to be perfectly designed for such constructions.

First we construct multi-resolutions using polynomial splines. For each integer j set

$$\text{(4.34)(i)} \qquad \mathcal{V}_j = \left\{ 2^{\frac{1}{2}j} \sum_k a_k M_{2m}(2^j x + k) : \{a_k\} \in \ell^2(\mathbb{Z}) \right\} .$$

The space $\mathcal{V}_0$ at a unit level of spatial resolution so to speak has been characterized already as the space $\mathfrak{L}_{2m-1}$ of square-integrable degree-$(2m-1)$ polynomial splines having knots at $\mathbb{Z}$. On the other hand, clearly f belongs to $\mathcal{V}_0$ if and only if $f(2^j x)$ to $\mathcal{V}_j$. Thus $\mathcal{V}_j$ will consist of all square-integrable degree-$(2m-1)$ polynomial splines having knots at $(1/2^j)\mathbb{Z}$, i.e., the mesh size is halved as we pass from splines in $\mathcal{V}_j$ to $\mathcal{V}_{j+1}$. This ensures that the $\mathcal{V}_j$ have the nested subspace property

$$\{0\} \subseteq \cdots \subseteq \mathcal{V}_{-j} \subseteq \cdots \subseteq \mathcal{V}_{-1} \subseteq \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_j \subseteq \cdots \subseteq L^2(\mathbb{R}) \ .$$

Furthermore,

$$\text{(4.34)(ii)} \qquad \bigcap_j \mathcal{V}_j = \{0\} \qquad , \qquad \overline{\bigcup_j \mathcal{V}_j} = L^2(\mathbb{R})$$

since $\hat{M}_{2m}(0) \neq 0$. Thus the $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ form a 2-scale multi-resolution of $L^2(\mathbb{R})$ because of the very nature of polynomial splines; in particular, therefore, M_{2m} must satisfy the 2-scale dilation equations

$$\text{(4.35)(i)} \qquad M_{2m}(x) = \sqrt{2} \sum_k h_k^{(2m)} M_{2k}(2x+k) \quad , \quad \hat{M}_{2m}(\xi) = H_{2m}\left(\tfrac{1}{2}\xi\right) \hat{M}_{2m}\left(\tfrac{1}{2}\xi\right)$$

where

$$\text{(4.35)(ii)} \qquad H_{2m}(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k^{(2m)} e^{2\pi i k \xi} \ .$$

But this can also be verified by direct calculation:

$$\hat{M}_{2m}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^{2m} = \left(\cos \tfrac{1}{2}\pi\xi\right)^{2m} \left(\frac{\sin \frac{1}{2}\pi\xi}{\frac{1}{2}\pi\xi}\right)^{2m} = \left(\cos \tfrac{1}{2}\pi\xi\right)^{2m} \hat{M}_{2m}\left(\tfrac{1}{2}\xi\right) .$$

Consequently,

$$(4.36)(\text{i}) \qquad H_m(\xi) = (\cos \pi\xi)^{2m} = \frac{1}{4^m} \sum_k \binom{2m}{m+k} e^{2\pi i k\xi}$$

and

$$(4.36)(\text{ii}) \qquad h_k^{(2m)} = \sqrt{2}\frac{1}{4^m}\binom{2m}{m+k} \ .$$

Notice that

$$(4.37)(\text{i}) \qquad H_{2m}\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right) = \left(\cos\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)\pi\right)^{2m} = \left(\sin \tfrac{1}{2}\pi\xi\right)^{2m} ,$$

while

$$(4.37)(\text{ii}) \qquad |H_{2m}\left(\tfrac{1}{2}\xi\right)|^2 + |H_{2m}\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2 = \left(\cos \tfrac{1}{2}\pi\xi\right)^{4m} + \left(\sin \tfrac{1}{2}\pi\xi\right)^{4m} .$$

Thus the central B-spline M_{2m} satisfies all the hypotheses of the following fundamental result constructing pre-wavelet mother functions, i.e., functions $\psi = \psi(x)$ in $\mathfrak{M}(\mathbb{R})$ such that $\{2^{m/2}\psi(2^m x + k) : k, m \in \mathbb{Z}\}$ is a complete stable family in $L^2(\mathbb{R})$ which is inter-scale orthonormal in the sense that

$$\int_{-\infty}^{\infty} \psi(2^m x + k)\overline{\psi(2^n x + \ell)}\, dx = 0$$

holds for all k, ℓ whenever $m \neq n$.

(4.38) Definition. *To each stable function ϕ in $\mathfrak{M}(\mathbb{R})$ there correspond tilde functions $\tilde{\phi}$ and $\underset{\sim}{\phi}$ defined by*

$$(\tilde{\phi})^\wedge(\xi) = \hat{\phi}(\xi)/\Phi^2(\xi) \quad , \quad (\underset{\sim}{\phi})^\wedge(\xi) = \Phi^2\left(\xi + \tfrac{1}{2}\right)\hat{\phi}(\xi) \ .$$

Of course, $\tilde{\phi}$ is just the bi-orthogonal mother function associated with ϕ.

(4.39) Theorem. *Let $\phi, \hat{\phi}(0) \neq 0$, be a stable function in $\mathfrak{M}(\mathbb{R})$ satisfying the 2-scale dilation equation*

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x + k) \ .$$

Then

$$\psi_1(x) = \sqrt{2}\sum_k (-1)^{k-1} h_{1-k}\tilde{\phi}(2x+k) \quad , \quad \psi_2(x) = \sqrt{2}\sum_k (-1)^{k-1} h_{1-k}\tilde{\phi}(2x+k)$$

define pre-wavelet mother functions which are as smooth as ϕ is.

Probably the simplest way to prove (4.39) is to deduce it from theorem (5.12) which constructs a wavelet from a function whose integer translates are orthonormal. Indeed, define ϕ_0 by

$$\hat{\phi}_0(\xi) = \hat{\phi}(\xi)/\Phi(\xi) \ ;$$

then $\{\phi_0(x+k) : k \in \mathbb{Z}\}$ is an orthonormal family and ϕ_0 satisfies the 2-scale dilation given by

$$\hat{\phi}_0(\xi) = \frac{H(\frac{1}{2}\xi)\Phi(\frac{1}{2}\xi)}{\Phi(\xi)}\ \hat{\phi}_0\big(\tfrac{1}{2}\xi\big) = H_0\big(\tfrac{1}{2}\xi\big)\hat{\phi}_0\big(\tfrac{1}{2}\xi\big)$$

where

$$\hat{\phi}(\xi) = H\big(\tfrac{1}{2}\xi\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big) \quad , \quad H(\xi) = \frac{1}{\sqrt{2}}\sum_k h_k e^{2\pi i k\xi} \ .$$

Thus the wavelet

$$\psi_0(x) = \sqrt{2}\sum_k (-1)^{k-1} h^{(0)}_{1-k}\phi_0(2x+k) \quad , \quad H_0(\xi) = \frac{1}{\sqrt{2}}\sum_k h^{(0)}_k e^{2\pi i k\xi} \ ,$$

associated with ϕ_0 has Fourier transform

$$\hat{\psi}_0(\xi) = e^{\pi i\xi} H_0\big(\tfrac{1}{2}\xi + \tfrac{1}{2}\big)\hat{\phi}_0\big(\tfrac{1}{2}\xi\big) = e^{\pi i\xi} H\big(\tfrac{1}{2}\xi + \tfrac{1}{2}\big)\frac{\Phi(\frac{1}{2}\xi + \frac{1}{2})}{\Phi(\xi)\Phi(\frac{1}{2}\xi)}\ \hat{\phi}\big(\tfrac{1}{2}\xi\big) \ .$$

Now let

$$\{0\} \subseteq \cdots \subseteq \mathcal{V}_{-1} \subseteq \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_j \subseteq \cdots \subseteq L^2(\mathbb{R})$$

be the multi-scale resolution of $L^2(\mathbb{R})$ generated by ϕ and let

$$L^2(\mathbb{R}) = \bigoplus_j \mathcal{W}_j = \cdots \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_j \oplus \cdots$$

be the corresponding orthogonal direct sum decomposition where

$$\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j \qquad (j \in \mathbb{Z}) \ .$$

Then $\{2^{\frac{1}{2}j}\phi_0(2^j x+k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{V}_j$, while $\{2^{\frac{1}{2}j}\psi_0(2^j x+k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{W}_j$. The proof in the pre-wavelet case can now proceed quickly.

Proof of (4.39). The Fourier transforms of ψ_1 and ψ_2 are given by

$$\hat{\psi}_1 = e^{\pi i\xi} H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)(\tilde{\phi})^\wedge\big(\tfrac{1}{2}\xi\big) = e^{\pi i\xi} H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)/\Phi^2\big(\tfrac{1}{2}\xi\big)$$

and

$$\hat{\psi}_2 = e^{\pi i\xi} H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)(\underset{\sim}{\phi})^\wedge\big(\tfrac{1}{2}\xi\big) = e^{\pi i\xi} H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)\Phi^2\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big) \ .$$

Thus

$$\hat{\psi}_1(\xi) = \big(1/\mu(\xi)\big)\hat{\psi}_0(\xi) \quad , \quad \hat{\psi}_2(\xi) = \mu(\xi)\hat{\psi}_0(\xi)$$

where

$$\mu(\xi) = \Phi(\xi)\Phi\big(\tfrac{1}{2}\xi\big)\Phi\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big) \ .$$

But $\mu = \mu(\xi)$ is a continuous period 1 function having absolutely convergent Fourier series; and because of the stability of ϕ, μ is a positive function. Hence by the Wiener Tauberian theorem the function $1/\mu(\xi)$ also is a positive, continuous period 1 function having absolutely convergent Fourier series. Thus each of

$$\big\{2^{\frac{1}{2}j}\psi_1(2^j x+k) : k \in \mathbb{Z}\big\} \quad , \quad \big\{2^{\frac{1}{2}j}\psi_2(2^j x+k) : k \in \mathbb{Z}\big\}$$

is a basis for $\mathcal{W}_j$, and so ψ_1 and ψ_2 are pre-wavelets. This completes the proof. ∎

We shall refer to ψ_1 and ψ_2 as the *pre-wavelets associated with* ϕ. The reconstruction identities linking these pre-wavelets to ϕ have a particularly simple form because of the identity

$$(4.40) \qquad |H\big(\tfrac{1}{2}\xi\big)|^2\Phi^2\big(\tfrac{1}{2}\xi\big) + |H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)|^2\Phi^2\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big) = \Phi^2(\xi) \ .$$

Set

$$(4.41)(\mathrm{i}) \qquad \begin{aligned} A^{(1)}(\xi) &= \Big(\Phi^2\big(\tfrac{1}{2}\xi\big)\overline{H\big(\tfrac{1}{2}\xi\big)} + \Phi^2\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)}\Big)\,\Phi^{1/2}(\xi) \ , \\ B^{(1)}(\xi) &= e^{-\pi i\xi}\,\big(H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big) - H\big(\tfrac{1}{2}\xi\big)\big)\,\frac{\Phi^2\big(\frac{1}{2}\xi\big)\Phi^2\big(\frac{1}{2}\xi+\frac{1}{2}\big)}{\Phi^2(\xi)} \ , \end{aligned}$$

and

$$(4.41)(\text{ii}) \qquad \begin{aligned} C^{(1)}(\xi) &= e^{\pi i\xi}\left(\Phi^2\left(\tfrac{1}{2}\xi\right)\overline{H\left(\tfrac{1}{2}\xi\right)} - \Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\overline{H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)}\right)\Phi^{1/2}(\xi)\ , \\ D^{(1)}(\xi) &= \left(H\left(\tfrac{1}{2}\xi\right)+H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\right)\frac{\Phi^2\left(\tfrac{1}{2}\xi\right)\Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)}{\Phi^2(\xi)}\ . \end{aligned}$$

These are all continuous period 1 functions having absolutely convergent Fourier series

$$(4.42) \qquad \begin{aligned} A^{(1)}(\xi) &= \sum_k \alpha_k^{(1)} e^{2\pi ik\xi} \quad , \quad B^{(1)}(\xi) = \sum_k \beta_k^{(1)} e^{2\pi ik\xi}\ , \\ C^{(1)}(\xi) &= \sum_k \gamma_k^{(1)} e^{2\pi ik\xi} \quad , \quad D^{(1)}(\xi) = \sum_k \delta_k^{(1)} e^{2\pi ik\xi}\ . \end{aligned}$$

Then, as an easy calculation shows, the reconstruction identities linking ψ_1 to ϕ are given by

$$(4.43) \qquad \begin{aligned} \phi(2x) &= \sum_k \alpha_k^{(1)}\phi(x+k) + \sum_k \beta_k^{(1)}\psi_1(x+k)\ , \\ \phi(2x+1) &= \sum_k \gamma_k^{(1)}\phi(x+k) + \sum_k \delta_k^{(1)}\psi_1(x+k)\ . \end{aligned}$$

Similarly, if

$$(4.44)(\text{i}) \qquad \begin{aligned} A^{(2)}(\xi) &= \left(\Phi^2\left(\tfrac{1}{2}\xi\right)\overline{H\left(\tfrac{1}{2}\xi\right)} + \Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\overline{H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)}\right)\frac{1}{\Phi^2(\xi)} \\ B^{(2)}(\xi) &= e^{-\pi i\xi}\left(H\left(\tfrac{1}{2}\xi\right)-H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\right)\frac{1}{\Phi^2(\xi)} \end{aligned}$$

and

$$(4.44)(\text{ii}) \qquad \begin{aligned} C^{(2)}(\xi) &= e^{\pi i\xi}\left(\Phi^2\left(\tfrac{1}{2}\xi\right)\overline{H\left(\tfrac{1}{2}\xi\right)} - \Phi^2\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\overline{H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)}\right)\frac{1}{\Phi^2(\xi)} \\ D^{(2)}(\xi) &= \left(H\left(\tfrac{1}{2}\xi\right)+H\left(\tfrac{1}{2}\xi+\tfrac{1}{2}\right)\right)\frac{1}{\Phi^2(\xi)} \end{aligned}$$

are the corresponding continuous period 1 functions for the second pre-wavelet and

$$(4.45) \qquad \begin{aligned} A^{(2)}(\xi) &= \sum_k \alpha_k^{(2)} e^{2\pi ik\xi} \quad , \quad B^{(2)}(\xi) = \sum_k \beta_k^{(2)} e^{2\pi ik\xi}\ , \\ C^{(2)}(\xi) &= \sum_k \gamma_k^{(2)} e^{2\pi ik\xi} \quad , \quad D^{(2)}(\xi) = \sum_k \delta_k^{(2)} e^{2\pi ik\xi} \end{aligned}$$

are their absolutely convergent Fourier series, then the reconstruction identities linking ψ_2 to ϕ are given by

$$
(4.46)\qquad
\begin{aligned}
\phi(2x) &= \sum_k \alpha_k^{(2)} \phi(x+k) + \sum_k \beta_k^{(2)} \psi_2(x+k) \\
\phi(2x+1) &= \sum_k \gamma_k^{(2)} \phi(x+k) + \sum_k \delta_k^{(2)} \psi_2(x+k)\ .
\end{aligned}
$$

The pre-wavelet ψ_2 is particularly interesting. For suppose that both

$$\Phi^2(\xi) = \sum_k a_k e^{2\pi i k\xi} \quad , \quad H(\xi) = \frac{1}{\sqrt{2}} \sum_\ell h_\ell e^{2\pi i \ell \xi}$$

are trigonometric polynomials, i.e., only finitely a_k and h_ℓ are non-zero. Then the product

$$\overline{H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)} \Phi^2\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right) = \frac{1}{\sqrt{2}} \sum_k (-1)^k \left(\sum_\ell a_{k+\ell} h_\ell \right) e^{\pi i k \xi}$$

is itself a trigonometric polynomial. Hence, in this case

$$\psi_2(x) = \sqrt{2} \sum_k (-1)^{k-1} \left(\sum_\ell a_{k+\ell-1} h_\ell \right) \phi(2x+k)\ ,$$

i.e., ψ_2 is the sum of *finitely* many integer translates of ϕ. Consequently, ψ_2 will be a compactly supported pre-wavelet if ϕ has compact support and if both of Φ^2 and H are trigonometric polynomials. All of these conditions are met when ϕ is the central B-spline M_{2m}. For then

$$\Phi_{2m}^2(\xi) = F_{4m}(\xi) = \sum_{k=-2m}^{2m} M_{4m}(k) e^{2\pi i k \xi}\ ,$$

while

$$H(\xi) = (\cos \pi\xi)^{2m} = \frac{1}{4^m} \sum_{k=-m}^{m} \binom{2m}{m+k} e^{2\pi i k \xi}\ .$$

What is most remarkable about the pre-wavelets associated with M_{2m} is that both of them have a natural explicit realization in terms of familiar splines. This is what makes the Chui theory so attractive.

5. Daubechies construction. A second family of wavelets was constructed by Ingrid Daubechies in 1988. In her construction the emphasis is placed on the coefficients specifying the two-scale dilation property of ϕ by contrast with the Chin construction where the emphasis is on the simplicity and natural occurrence of ϕ. But the Daubechies construction produces compactly supported wavelet mother functions $\psi = \psi(x)$ which have a specified degree of smoothness; furthermore, the set of integer translates and dilates

$$\left\{2^{\frac{1}{2}m}\psi(2^m x + k) : k, m \in \mathbb{Z}\right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. In every respect, therefore, except possibly for simplicity and natural occurrence, such ψ are ideal wavelet mothers.

To begin the construction we choose any ϕ in $\mathfrak{M}(\mathbb{R})$ such that

$$\text{(5.1)(i)} \qquad \left\{\phi(x+k) : k \in \mathbb{Z}\right\} \textit{ is orthonormal in } L^2(\mathbb{R}) \ ,$$

$$\text{(5.1)(ii)} \qquad \phi(x) = \sqrt{2}\ \sum_k h_k \phi(2x+k) \textit{ for some } h_i \textit{ in } \mathbb{R} \ .$$

Using the properties of $\mathfrak{M}(\mathbb{R})$-functions together with the Wiener Tauberian theorem, it is not hard to show that in (5.1)(ii) the coefficients $\{h_k\}$ belong to $\ell^1(\mathbb{Z})$, hence also to $\ell^2(\mathbb{Z})$; in fact, by the orthonormality of the $\phi(x+k)$, $k \in \mathbb{Z}$,

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} |\phi(x)|^2\,dx &= 2\sum_{j,k} h_j h_k \int_{-\infty}^{\infty} \phi(2x+j)\overline{\phi(2x+k)}\,dx \\ &= \sum_{j,k} h_j h_k \int_{-\infty}^{\infty} \phi(x+j)\overline{\phi(x+k)}\,dx = \sum_k |h_k|^2 \ , \end{aligned}$$

while

$$\int_{-\infty}^{\infty} \phi(x)\,dx = \sqrt{2}\sum_k h_k \int_{-\infty}^{\infty} \phi(2x+k)\,dx = \frac{1}{\sqrt{2}}\Big(\sum_k h_k\Big)\int_{-\infty}^{\infty} \phi(x)\,dx \ .$$

This proves the first of a sequence of basic lemmas.

(5.2) LEMMA. *The coefficients $\{h_k\}$ in the two-scale dilation equation for a function ϕ in $\mathfrak{M}(\mathbb{R})$ satisfy*

$$\text{(i)} \quad \sum_k |h_k| < \infty \ , \qquad \text{(ii)} \quad \sum_k |h_k|^2 = 1 \ , \qquad \text{(iii)} \quad \sum_k h_k = \sqrt{2}$$

whenever $\{\phi(x+k) : k \in \mathbb{Z}\}$ *is orthonormal in* $L^2(\mathbb{R})$ *and* $\hat{\phi}(0) \neq 0$.

In particular, therefore, the function

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{2\pi i k \xi} \tag{5.3)(i}$$

defines a continuous, period 1 function having absolutely convergent Fourier series; furthermore, because of (5.2)(iii),

$$H(0) = 1 \; . \tag{5.3)(ii}$$

This function $H = H(\xi)$ is often referred to as the *symbol* of ϕ. The coefficients $\{h_k\}$ can also be computed by exploiting the orthonormality of the integer translates $\phi(x+k)$, $k \in \mathbb{Z}$:

$$\sqrt{2} \int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x+j)}\, dx = 2 \sum_k h_k \int_{-\infty}^{\infty} \phi(2x+k)\overline{\phi(2x+j)}\, dx = h_j \; . \tag{5.4}$$

This leads immediately to the second property of the coefficients.

(5.5) LEMMA. *If the function* ϕ *in* $\mathfrak{M}(\mathbb{R})$ *satisfying* (5.1) *has compact support, then only finitely many of the* h_k *are non-zero; in particular, the symbol*

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{2\pi i k \xi}$$

of ϕ *is then a trigonometric polynomial.*

Now we come to the crucial definition.

(5.6) DEFINITION. *For each* ϕ *in* $\mathfrak{M}(\mathbb{R})$ *satisfying* (5.1) *define the associated wavelet mother function by*

$$\psi(x) = \sqrt{2} \sum_k (-1)^{k-1} h_{1-k} \phi(2x+k) \; .$$

One explanation for the precise form of this definition of the wavelet mother will emerge. To continue the sequence of basic lemmas note that ψ belongs to $\mathfrak{M}(\mathbb{R})$, and hence to $L^2(\mathbb{R})$, since $\{h_k\}$ is a sequence in $\ell^1(\mathbb{Z})$; in fact,

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 2\sum_{j,k}(-1)^{j+k}h_{1-j}h_{1-k}\int_{-\infty}^{\infty}\phi(2x+j)\overline{\phi(2x+k)}\,dx$$

$$= \sum_k |h_k|^2 = 1$$

because of (5.2)(ii). Consequently, ψ belongs both to $\mathfrak{M}(\mathbb{R})$ and to the closed subspace $\mathcal{V}_1$ of $L^2(\mathbb{R})$ having $\{\sqrt{2}\,\phi(2x+k) : k \in \mathbb{Z}\}$ as basis. It has many other fundamental properties.

(5.7) LEMMA. *When $H = H(\xi)$ is defined by* (5.3)(i), *the Fourier transforms of ϕ and the associated mother function ψ satisfy*

$$\text{(i)}\quad \hat{\phi}(\xi) = H\big(\tfrac{1}{2}\xi\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ , \qquad \text{(ii)}\quad \hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)}\,\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ .$$

Applying these factorizations again we see that

$$\hat{\phi}(\xi) = H\big(\tfrac{1}{2}\xi\big)H\big(\tfrac{1}{4}\xi\big)\hat{\phi}\big(\tfrac{1}{4}\xi\big)$$

while

$$\hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)}\,H\big(\tfrac{1}{4}\xi\big)\,\hat{\phi}\big(\tfrac{1}{4}\xi\big)\ ;$$

and so after successive iterations we obtain the following descriptions of ϕ and ψ.

(5.8) COROLLARY. *The symbol $H = H(\xi)$ completely determines ϕ and ψ in the sense that*

$$\hat{\phi}(\xi) = \Big(\prod_{m=1}^{\infty} H\big(\tfrac{1}{2}\xi\big)\Big)\hat{\phi}(0) \quad , \quad \hat{\psi}(\xi) = e^{\pi i\xi}\,\overline{H\big(\tfrac{1}{2}\xi+\tfrac{1}{2}\big)}\Big(\prod_{m=2}^{\infty} H\big(\tfrac{1}{2}\xi\big)\Big)\,\hat{\phi}(0)\ .$$

Proof of (5.7). The proof consists of two routine calculations. By (5.1)(ii),

$$\hat{\phi}(\xi) = \sqrt{2}\int_{-\infty}^{\infty}\Big(\sum_k h_k\phi(2x+k)\Big)e^{-2\pi ix\xi} = \frac{1}{\sqrt{2}}\sum_k h_k\int_{-\infty}^{\infty}\phi(x+k)e^{-\pi ix\xi}\,dx$$

$$= \frac{1}{\sqrt{2}}\Big(\sum_k h_k e^{\pi ik\xi}\Big)\hat{\phi}\big(\tfrac{1}{2}\xi\big) = H\big(\tfrac{1}{2}\xi\big)\hat{\phi}\big(\tfrac{1}{2}\xi\big)\ .$$

On the other hand, by the same arguments starting with the defining relation (5.6),

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\left(\sum_k (-1)^{k-1} h_{1-k} e^{\pi i k \xi}\right)\hat{\phi}\left(\tfrac{1}{2}\xi\right)$$

$$= \frac{1}{\sqrt{2}} e^{\pi i \xi}\left(\sum_k e^{\pi i k} h_{1-k} e^{\pi i (k-1)\xi}\right)\hat{\phi}\left(\tfrac{1}{2}\xi\right)$$

$$= \frac{1}{\sqrt{2}} e^{\pi i \xi}\left(\sum_k h_{1-k} e^{\pi i (k-1)(\xi+1)}\right)\hat{\phi}\left(\tfrac{1}{2}\xi\right) .$$

But

$$\overline{H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)} = \sum_k h_k e^{-\pi i k(\xi+1)} = \sum_k h_{1-k} e^{-\pi i (k-1)(\xi+1)}$$

since the h_k are assume to be real; consequently,

$$\hat{\psi}(\xi) = e^{\pi i \xi}\, \overline{H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)}\, \hat{\phi}\left(\tfrac{1}{2}\xi\right) ,$$

completing the proof. ∎

Now let $\Phi = \Phi(\xi)$ and $\Psi = \Psi(\xi)$ be the overlap functions corresponding to ϕ and ψ respectively. Of course, we know in advance that $\Phi(\xi) = 1$, but let's ignore that for the moment and use solely (5.7). Then

$$\Phi^2(\xi) = \sum_k |\hat{\phi}(\xi + k)|^2 = \sum_k \left\{ |\hat{\phi}(\xi + 2k)|^2 + |\hat{\phi}(\xi + 2k + 1)|^2 \right\}$$

$$= \sum_k \left\{ |H\left(\tfrac{1}{2}\xi + k\right)|^2 |\hat{\phi}\left(\tfrac{1}{2}\xi + k\right)|^2 + |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2} + k\right)|^2 |\hat{\phi}\left(\tfrac{1}{2}\xi + \tfrac{1}{2} + k\right)|^2 \right\} .$$

But H has period 1, so

$$\Phi^2(\xi) = |H\left(\tfrac{1}{2}\xi\right)|^2 \sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi + k\right)|^2 + |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2 \sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi + \tfrac{1}{2} + k\right)|^2 .$$

Consequently,

$$\text{(5.8)(i)} \qquad \Phi^2(\xi) = |H\left(\tfrac{1}{2}\xi\right)|^2 \Phi^2\left(\tfrac{1}{2}\xi\right) + |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2 \Phi^2\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right) .$$

Similarly,

$$\Psi^2(\xi) = |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2 \sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi + k\right)|^2 + |H\left(\tfrac{1}{2}\xi\right)|^2 \sum_k |\hat{\phi}\left(\tfrac{1}{2}\xi + \tfrac{1}{2} + k\right)|^2 ,$$

and so

$$\Psi^2(\xi) = |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2\Phi^2\left(\tfrac{1}{2}\xi\right) + |H\left(\tfrac{1}{2}\xi\right)|^2\Phi^2\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right) . \tag{5.8)(ii}$$

Hence the symbol of ϕ satisfies the fundamental equation

$$|H\left(\tfrac{1}{2}\xi\right)|^2 + |H^2\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2 = 1 \qquad (\xi \in \mathbb{R}) \tag{5.9}$$

and in turn, the overlap function of ψ satisfies

$$\Psi(\xi) = \left(|H\left(\tfrac{1}{2}\xi\right)|^2 + |H\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\right)|^2\right)^{1/2} = 1 \qquad (\xi \in \mathbb{R})$$

as a *consequence of the integer translates* $\{\phi(x+k) : k \in \mathbb{Z}\}$ *being assumed to be an orthonormal family*. In particular, the set of integer translates $\{\psi(x+k) : k \in \mathbb{Z}\}$ is an orthonormal family. Still more is true along these lines.

(5.10) LEMMA. *When ϕ is a function in $\mathfrak{M}(\mathbb{R})$ satisfying* (5.1) *and ψ is the associated wavelet mother function, then each integer translate $\phi(x+\ell)$ of ϕ is orthogonal to each integer translate $\psi(x+m)$ of ψ in the sense that*

$$\int_{-\infty}^{\infty} \phi(x+\ell)\overline{\psi(x+m)}\, dx = 0 \qquad (\ell, m \in \mathbb{Z}) .$$

To appreciate the significance of this last lemma, let

$$\mathcal{V}_0 = \left\{\sum_k a_k\phi(x+k) : \{a_k\} \in \ell^2(\mathbb{Z})\right\} ,$$

be the closed subspace of $L^2(\mathbb{R})$ having $\{\phi(x+k) : k \in \mathbb{Z}\}$ as orthonormal basis, and let

$$\mathcal{W}_0 = \left\{\sum_k b_k\psi(x+k) : \{b_k\} \in \ell^2(\mathbb{Z})\right\}$$

together with

$$\mathcal{V}_1 = \left\{\sum_k \sqrt{2}\, c_k\phi(2x+k) : \{c_k\} \in \ell^2(\mathbb{Z})\right\}$$

be the corresponding subspaces having $\{\psi(x+k) : k \in \mathbb{Z}\}$ and $\{\sqrt{2}\,\phi(2x+k) : k \in \mathbb{Z}\}$ respectively as orthonormal bases. Then by (5.1) and (5.6) both $\mathcal{V}_0$ and $\mathcal{W}_0$ are closed subspaces of $\mathcal{V}_1$, while lemma (5.10) ensures that $\mathcal{V}_0$ is orthogonal to $\mathcal{W}_0$. Thus $\mathcal{V}_0$ and $\mathcal{W}_0$ are orthogonal subspaces of $\mathcal{V}_1$. In fact, the Laplacian pyramid idea of Burt and Adelson shows that $\mathcal{V}_1$ is the sum of these two orthogonal subspaces.

(5.11) LEMMA. *When $\{h_k\}$ is the sequence of coefficients in the two-scale dilation equation for a function ϕ in $\mathfrak{M}(\mathbb{R})$ satisfying (5.1), then*

(i) $$\phi(2x) = \frac{1}{\sqrt{2}}\left\{\sum_k h_{2k}\phi(x-k) - \sum_k h_{2k+1}\psi(x+k)\right\} ,$$

while

(ii) $$\phi(2x+1) = \frac{1}{\sqrt{2}}\left\{\sum_k h_{2k+1}\phi(x-k) + \sum_k h_{2k}\psi(x+k)\right\} .$$

In particular, each

(iii) $$f(x) = \sqrt{2}\ \sum_k a_k^{(1)}\phi(2x+k) \qquad (f \in \mathcal{V}_1)$$

in $\mathcal{V}_1$ has an orthogonal sum decomposition

(iv) $$f(x) = \sum_k a_k^{(0)}\phi(x-k) + \sum_k b_k^{(0)}\psi(x+k) = f_0 + g_0 \ .$$

with f_0 in $\mathcal{V}_0$, g_0 in $\mathcal{W}_0$, and

(v) $$a_k^{(0)} = \sum_j a_{j-2k}^{(1)} h_j \quad , \quad b_k^{(0)} = \sum_j (-1)^j a_{2k+1-j}^{(1)} h_j$$

Proof of (5.10). Because of the two-scale dilation property of ϕ and the defining property of ψ,

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(x+\ell)\overline{\psi(x+m)}\,dx &= 2\sum_{j,k}(-1)^{k-1}h_j h_{1-k}\int_{-\infty}^{\infty}\phi(2x+2\ell+j)\overline{\phi(2x+2m+k)}\,dx \\ &= \sum_{j,k}(-1)^{k-1}h_j h_{1-k}\int_{-\infty}^{\infty}\phi(x+2\ell+j)\phi(x+2m+k)\,dx \\ &= \sum_k (-1)^{k-1}h_{k+2m-2\ell}h_{1-k} \ .\end{aligned}$$

Now change the index of summation by setting $n = 1-k-2m+2\ell$. Then

$$\int_{-\infty}^{\infty} \phi(x+\ell)\overline{\psi(x+m)}\,dx = \sum_n (-1)^n h_{1-n}h_{n+2m-2\ell} = -\int_{-\infty}^{\infty}\phi(x+\ell)\overline{\psi(x+m)}\,dx \ ,$$

and so each integer translate $\phi(x+\ell)$ is orthogonal to each integer translate $\psi(x+m)$, completing the proof. ■

Proof of (5.11). Suppose

$$\phi(2x) = \sum_j \alpha_j \phi(x+j) + \sum_j \beta_j \psi(x+j)$$

for some choice of sequences $\{\alpha_j\}$ and $\{\beta_j\}$. Then by orthonormality,

$$\alpha_k = \int_{-\infty}^{\infty} \phi(x)\overline{\phi(x+k)}\,dx \quad , \quad \beta_k = \int_{-\infty}^{\infty} \phi(2x)\overline{\psi(x+k)}\,dx \ .$$

Thus

$$\alpha_k = \sqrt{2}\ \sum_j h_j \int_{-\infty}^{\infty} \phi(2x)\overline{\phi(2x+2j+k)}\,dx = \frac{1}{\sqrt{2}}h_{-2k} \ .$$

Similarly,

$$\beta_k = \sqrt{2}\ \sum_j (-1)^{j-1} h_{1-j} \int_{-\infty}^{\infty} \phi(2x)\overline{\phi(2x+2j+k)}\,dx = -\frac{1}{\sqrt{2}}h_{1+2k} \ .$$

Entirely analogous arguments starting from the decomposition

$$\phi(2x+1) = \sum_j \gamma_j \phi(x+j) + \sum_j \delta_j \psi(x+j)$$

show that

$$\gamma_k = \frac{1}{\sqrt{2}}h_{1-2k} \quad , \quad \delta_k = \frac{1}{\sqrt{2}}h_{2k} \ .$$

This establishes (i) and (ii). Now f has a representation

$$f(x) = \sqrt{2}\ \sum_k a_k^{(1)} \phi(2x+k)$$

for some sequence $\{a_k^{(1)}\}$ in $\ell^2(\mathbb{Z})$. Then by (i) and (ii),

$$f(x) = \sqrt{2}\Big\{\sum_k a_{2k}^{(1)} \phi(2x+2k) + \sum_k a_{2k+1}^{(1)} \phi(2x+2k+1)\Big\}$$

$$= \sum_{j,k} \{a_{2k}^{(1)} h_{2j} \phi(x+k-j) - a_{2k}^{(1)} h_{2j+1} \psi(x+k+j)\}$$

$$+ \sum_{j,k} \{a_{2k+1}^{(1)} h_{2j+1} \phi(x+k-j) + a_{2k+1}^{(1)} h_{2j} \psi(x+k+j)\} \ .$$

Thus

$$f(x) = \sum_k \left(\sum_j a^{(1)}_{2j-2k} h_{2j} + a^{(1)}_{2j+1-2k} h_{2j+1} \right) \phi(x-k)$$

$$+ \sum_k \left(\sum_j a^{(1)}_{2k-2j+1} h_{2j} - a^{(1)}_{2k-2j} h_{2j+1} \right) \psi(x+k) \ .$$

Properties (iv) and (v) now follow easily, completing the proof. ∎

The precise form of decomposition (5.11)(iv) can be established also using (5.9). All the pieces have now been assembled so that the third of the general wavelet decomposition results can be stated and proved.

(5.12) THEOREM. *Let ϕ be a function in $\mathfrak{M}(\mathbb{R})$ such that*

$$\text{(i)} \quad \{\phi(x+k) : k \in \mathbb{Z}\} \ \ \textit{is orthonormal in} \ \ L^2(\mathbb{R}) \ ,$$

$$\text{(ii)} \quad \phi(x) = \sqrt{2} \ \sum_k h_k \phi(2x+k) \ , \qquad \text{(iii)} \quad \hat{\phi}(0) \neq 0 \ ,$$

and define $\psi = \psi(x)$ by

$$\psi(x) = \sqrt{2} \ \sum_k (-1)^{k-1} h_{1-k} \phi(2x+k) \ .$$

Then $\{2^{\frac{1}{2}m}\psi(2^m x + k) : k, m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Thus each f in $L^2(\mathbb{R})$ admits an orthonormal wavelet-type decomposition

$$f(x) = \sum_{k,m} 2^m \left(\int_{-\infty}^{\infty} f(y) \overline{\psi(2^m y + k)} \, dy \right) \psi(2^m x + k)$$

in which the orthonormal basis elements are generated from a single wavelet mother function ψ by translation and dilation. More remarkable still perhaps is the fact that everything in theorem (5.12) and the results leading up to it depend solely on the coefficients $\{h_k\}$ in the two-scale; dilation equation for ϕ. The basic idea underlying the Daubechies construction is that this sole dependence on $\{h_k\}$ means that one should start with a sequence $\{h_k\}$, *then* construct ϕ and ψ, and hence obtain orthonormal wavelet-type decompositions. But first we prove (5.12).

(5.13) THEOREM. *Given $\varphi \in \mathfrak{M}(\mathbb{R})$ such that*

(i) $\{\varphi(x+k) : k \in \mathbb{Z}\}$ *is orthonormal*

(ii) $\varphi(x) = \sqrt{2}\sum_k h_k \varphi(2x+k)$, $\{h_k\} \in \ell^1(\mathbb{Z})$

(iii) $\hat{\varphi}(0) \neq 0$.

Define $\psi(x) = \sqrt{2}\sum(-1)^{k-1}h_{1-k}\varphi(2x+k)$. *Then* $\{2^{m/2}\psi(2^m x+k) : k, m \in \mathbb{Z}\}$ *is a complete orthonormal basis of* $L^2(\mathbb{R})$.

To prove this theorem, we first require the following

LEMMA 1. *Given φ satisfying the hypotheses of Theorem 5.12, set*

$$\mathcal{V}_0 = \mathcal{V}_\varphi = \left\{\sum_k a_k \varphi(x+k) : \{a_k\} \in \ell^2(\mathbb{Z})\right\},$$

and in general

$$\mathcal{V}_m = \left\{\sum_k a_k 2^{m/2}\varphi(2^m x+k) : \{a_k\} \in \ell^2(\mathbb{Z})\right\} \quad \text{for} \quad m \in \mathbb{Z}\ .$$

Then $\bigcup_{m\in\mathbb{Z}} \mathcal{V}_m$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}$.

Proof of Lemma 1. First we show that if $\mathcal{V} = \bigcup_{m\in\mathbb{Z}} \mathcal{V}_m$, then $\mathcal{V}$ is a dense subspace of $L^2(\mathbb{R})$. To show this we first show that translates of φ by dyadic rationals belong to $\mathcal{V}$. Set $r = \ell/2^m$ for some $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. From (ii) it follows that $\varphi(x) = \sum_k b_k 2^{m/2}\varphi(2^m x+k)$ with $\{b_k\} \in \ell^2(\mathbb{Z})$. Hence

$$\begin{aligned}\varphi\Big(x+\frac{\ell}{2^m}\Big) &= \sum_k b_k 2^{m/2}\varphi\Big(2^m\Big(x+\frac{\ell}{2^m}\Big)-k\Big)\\ &= \sum_k b_k 2^{m/2}\varphi(2^m x+\ell-k)\ .\end{aligned}$$

This latter sum converges in $\mathcal{V}_m \subseteq \mathcal{V}$.

Now suppose that $f \in \mathcal{V}^\perp$. Then if r is any dyadic rational, we have

$$\begin{aligned}f*\varphi^*(r) &= \int f(r-x)\varphi^*(x)\,dx\\ &= \int f(y-r)\bar{\varphi}(y)\,dy \quad (\text{where } y=-x)\\ &= \int f(y)\bar{\varphi}(y+r)\,dy = 0\end{aligned}$$

since $\varphi(y+r) \in \mathcal{V}$.

Now $\varphi \in \mathfrak{M}(\mathbb{R})$ implies $\varphi^* \in \mathfrak{M}(\mathbb{R})$ so that $f * \varphi^*$ is a continuous function which vanishes at every dyadic rational r. It follows that $f * \varphi^* \equiv 0$. On the other hand, since $\mathcal{V}$ is invariant under dilation by factor 2^m, $m \in \mathbb{Z}$, it follows that $\mathcal{V}^\perp$ is also invariant under dilation by 2^m, and this shows that $f * \varphi^*_{1/2^m} \equiv 0$. Now applying Lemma 4.10, it follows that $\lim_{m\to\infty} \|f * \varphi^*_{1/2^m} - \hat{\varphi}^*(0) f\|_{L^2} = 0$. Since $\hat{\varphi}^*(0) \neq 0$, we conclude that $f = 0$. This shows that $\mathcal{V}^\perp = \{0\}$. In other words, $\mathcal{V}$ is dense in $L^2(\mathbb{R})$.

We next show that $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}$. Since $\mathcal{V}_m \subseteq \mathcal{V}_{m+1}$ (this follows from (ii)), it suffices to show that the operators P_m of projection onto $\mathcal{V}_m$ converge weakly to 0 as $m \to -\infty$. That is, it suffices to show that

$$\forall\ f \in L^2(\mathbb{R})\ , \qquad \lim_{m\to-\infty} P_m f = 0 \ \text{ in } \ L^2(\mathbb{R})\ .$$

Fix $f \in L^2(\mathbb{R})$, and $\varphi > 0$, and write $f = \sum_{n=1}^{n_0} c_n \chi_{In} + g$, where $\{c_n\} \in \ell^2(\mathbb{Z})$, χ_{In} is the characteristic function of the dyadic interval In, and $g \in L^2(\mathbb{R})$ with $\|g\|_{L^2} < \varepsilon$. Then it clearly suffices to show that for each dyadic interval I, $P_m(\chi_I) \to 0$ as $m \to \infty$. In fact, (due to the translation and dilation structure of the $\mathcal{V}_m$'s) it suffices to show that $P_m(\chi_{[0,1]}) \to 0$ as $m \to \infty$. Now

$$\begin{aligned}
\|P_m(\chi_{[0,1]})\|^2_{L^2} &= \sum_{k\in\mathbb{Z}} |\langle \chi_{[0,1]}, 2^{m/2}\varphi(2^m x + k)\rangle|^2 \\
&= 2^m \sum_{k\in\mathbb{Z}} \left| \int_0^1 \overline{\varphi(2^m x + k)}\, dx \right|^2 \\
&= 2^{-m} \sum_{k\in\mathbb{Z}} \left| \int_0^{2^m} \overline{\varphi(x + k)}\, dx \right|^2 \\
&\leq \sum_{k\in\mathbb{Z}} \int_0^{2^m} |\varphi(x+k)|^2\, dx\ .
\end{aligned}$$

Now since $\varphi \in \mathfrak{M}(\mathbb{R})$, it follows that $\sum_{k\in\mathbb{Z}} |\varphi(x+k)|^2 = \mu(x)$ is an integrable function on $[0,1]$. Now a basic result of measure theory shows that $\lim_{m\to-\infty} \mu([0,2^m]) = \mu(\lim_{m\to-\infty}[0,2^m]) = 0$. This completes the proof of the lemma. ∎

LEMMA 2. *Set*

$$\mathcal{W}_0 = \mathcal{V}_\psi = \left\{ \sum_k b_k \psi(x+k) : \{b_k\} \in \ell^2(\mathbb{Z}) \right\},$$

and more generally

$$\mathcal{W}_m = \left\{ \sum_k b_k 2^{m/2} \psi(2^m x + k) : \{b_k\} \in \ell^2(\mathbb{Z}) \right\}.$$

Then $\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$, and more generally, $\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m$.

Proof of Lemma 2. We've already seen that (i) and (ii) along with the definition of ψ show that $\{\psi(x+k)\}_{k\in\mathbb{Z}}$ is an orthonormal family, and that $\int_{-\infty}^{\infty} \varphi(x+k)\overline{\psi(x+\ell)}\,dx = 0$, $(k,\ell \in \mathbb{Z})$. Hence $\mathcal{V}_0 \perp \mathcal{W}_0$.

Next, we note that by its definition, $\psi(x) \in \mathcal{V}_1$. Now by Lemma 5.11, it follows that $\varphi(2x), \varphi(2x+1) \in \mathcal{V}_0 + \mathcal{W}_0$. This shows that in fact $\mathcal{V}_1 = \mathcal{V}_0 \oplus \mathcal{W}_0$.

To show that $\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m$, we change scales. Simple integration shows that $\{2^{m/2}\varphi(2^m x + k) : k \in \mathbb{Z}\}$ and $\{2^{m/2}\psi(2^m x + k) : k \in \mathbb{Z}\}$ are orthonormal families for m fixed, and furthermore that these families of functions are orthogonal to each other. Furthermore,

$$\varphi(2^{m+1}x) = \frac{1}{\sqrt{2}}\left\{ \sum_k h_{2k}\varphi(2^m x - k) - \sum_k h_{2k+1}\psi(2^m x_k) \right\}$$

and

$$\varphi(2^{m+1}x + 1) = \frac{1}{\sqrt{2}}\left\{ \sum_k h_{2k+1}\varphi(2^m x - k) + \sum_k h_{2k}\psi(2^m x + k) \right\},$$

as in Lemma 5.11. This shows that $\mathcal{V}_{m+1} \subseteq \mathcal{V}_m + \mathcal{W}_m$, and we conclude that $\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m$ as claimed. ∎

LEMMA 3. $\mathcal{W}_{m+1} \perp \mathcal{W}_m$.

Proof. $\mathcal{W}_m \subseteq \mathcal{V}_{m+1} \perp \mathcal{W}_{m+1}$. ∎

Proof of Theorem 5.13. It is an immediate consequence of Lemma 2 and 3, and the fact that $2^{m/2}\psi(2^m x + k)$ has norm 1, that the family $\{2^{m/2}\psi(2^m x + k) : k, m \in \mathbb{Z}\}$ is an orthonormal family. We just need to check that the family is complete. From Lemma 1, it suffices to show that $\mathcal{V} = \oplus_{m\in\mathbb{Z}}\mathcal{W}_m$. But

$$\mathcal{V}_m = \mathcal{W}_{m-1} \oplus \mathcal{V}_{m-1} = \mathcal{W}_{m-1} \oplus \mathcal{W}_{m-2} \oplus \mathcal{V}_{m-2} = \cdots = \bigoplus_{-\infty}^{m-1} \mathcal{V}_n$$

since

$$\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \lim_{m\to-\infty} \mathcal{V}_m = \{0\} \ .$$

But $f \in \mathcal{V} \Rightarrow f \in \mathcal{V}_m$ for m sufficiently large, so $\mathcal{V} \subseteq \oplus_{m\in\mathbb{Z}}\mathcal{W}_m$. Since the direct sum is a closed subspace of $L^2(\mathbb{R})$, it follows from Lemma 1 that $L^2(\mathbb{R}) \oplus_{m\in\mathbb{Z}} \mathcal{W}_m$. This shows that $\{2^{m/2}\psi(2^m x + k) : m, k \in \mathbb{Z}\}$ is a complete orthonormal basis of $L^2(\mathbb{R})$. This completes the proof of Theorem 5.13. ∎

Fourth Major Result

Theorem (Daubechies). *Let $H(\xi)$ be a trigonometric polynomial of degree at most N satisfying*

(a) $H(0) = 1$

(b) $|H(\xi/2)|^2 + |H(\xi/1+1/2)| = 1$ *Suppose furthermore that* $H(\xi) = \left[\frac{1}{2}(1+e^{2\pi i\xi})\right]^2 P(\xi)$ *where* $|P(\xi)| \le B < 2$. *Then*

$$\hat{\varphi}(\xi) = \prod_{m=1}^{\infty} H(\xi/2^m)$$

$$\hat{\varphi}(0) \swarrow$$

defines a function $\varphi \in \mathfrak{M}(\mathbb{R})$ satisfying (i) and (ii) of Theorem 5.13. Furthermore, φ has compact support in $[-N, N]$.

The proof of this theorem is a bit involved and we shall require several lemmas.

Lemma 1. *The product $\prod_{m=1}^{\infty} H(\xi/2^m)$ converges (absolutely and uniformly on compact sets) if and only if the sum $\sum_{m=1}^{\infty} |H(\xi/2^m) - 1|$ converges (uniformly on compact sets).*

`[Give a reference, ~ any standard complex variable text. The proof is a bit involved -- should it be involved?]`

LEMMA 2. *For $H(\xi)$ as in the theorem, the infinite product*

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{m=1}^{\infty} H(x/2^m)$$

converges pointwise and uniformly on compact sets. In fact, there is a $C > 0$ such that

$$|\hat{\varphi}(\xi)| \leq C(1+|\xi|)\frac{\log B}{\log 2} - 2$$

where

$$B = \sup_{\xi \in \mathbb{R}} |P(\xi)|$$

Using "Viete's formula," that is,

$$\prod_{m=1}^{\infty} \cos\left(\frac{\pi x}{2^m}\right) = \frac{\sin \pi x}{\pi x} ,$$

we obtain

$$\begin{aligned}\left[\prod_{m=1}^{\infty}\left(\tfrac{1}{2}(1+e^{2\pi i(\xi/2^m)})\right)\right]^2 &= \left[\prod_{m=1}^{\infty} e^{\pi i(\xi/2^m)} \cos\left(\frac{\pi\xi}{2^m}\right)\right]^2 \\ &= \left[\prod_{m=1}^{\infty} e^{\pi i(\xi/2^m)}\right]^2 \left(\frac{\sin \pi x}{\pi x}\right)^2 = e^{2\pi i\xi}\left(\frac{\sin \pi x}{\pi x}\right)^2 .\end{aligned}$$

Hence

$$\begin{aligned}\prod_{m=1}^{\infty} H(\xi/2^m) &= \prod_{m=1}^{\infty} \left[\tfrac{1}{2}(1+e^{2\pi i(\xi/2^m)})\right]^2 \prod_{m=1}^{\infty} P(\xi/2^m) \\ &= e^{2\pi i\xi}\left(\frac{\sin \pi\xi}{\pi\xi}\right)^2 \prod_{m=1}^{\infty} P(\xi/2^m) ,\end{aligned}$$

so it suffices to check the convergence of $\prod_{m=1}^{\infty} P(\xi/2^m)$. Now write $P(\xi) = \sum_{-N}^{N} f(n)e^{2\pi i n\xi}$.

Since $H(0) = 1$ it follows that $P(0) = \sum_{-N}^{N} f(n) = 1$, so

$$
\begin{aligned}
|P(\xi) - 1| &= \Big| \sum_{-N}^{N} f(n)(e^{2\pi i n\xi} - 1) \Big| \\
&\le 2 \sum_{-N}^{N} |f(n)| \, |\sin \pi n \xi| \\
&\le 2 \sum_{|n| \le N} |f(n)| \, |\pi n \xi| \\
&= 2\pi |\xi| \sum_{|n| \le N} |f(n)| \, |n| = C_0 |\xi| \ .
\end{aligned}
$$

This shows that

$$\sum_{m=1}^{\infty} |P(\xi/2^m) - 1| \le C_0 \sum_{m=1}^{\infty} |\xi/2^m| = C_0 |\xi| \ .$$

We conclude from Lemma 1 that $P(\xi)$ converges uniformly on compact sets. To improve the size estimate we proceed as follows. If $|\xi| \le 1$ then the size estimate above is ???. If $|\xi| > 1$, then choose m_0 so that $2^{-m_0}|\xi| < 1 \le 2^{1-m_0}|\xi|$. In other words, $\log|\xi|/\log 2 < m_0 \le 1 + \log|\xi|/log 2$. Then

$$
\begin{aligned}
\Big| \prod_{m=1}^{\infty} P(\xi/2^m) \Big| &= \Big| \prod_{m=1}^{m_0} P(\xi/2^m) \Big| \, \Big| \prod_{m=m_0+1}^{\infty} P(\xi/2^m) \Big| \\
&\le B^{m_0} \Big| \prod_{m=1}^{\infty} P(2^{-m-m_0}\xi) \Big| \\
&\le B^{m_0} \prod_{m=1}^{\infty} \big(1 + C_0 2^{-m}(|\xi/2^{m_0}|\big) \ .
\end{aligned}
$$

Since $|\xi/2^{m_0}| < 1$, the last product is handled by an absolute constant. This shows that

$$
\begin{aligned}
\Big| \prod_{m=1}^{\infty} P(\xi/2^m) \Big| &\le C B^{m_0(\xi)} \le C B^{1 + \log|\xi|/\log 2} \\
&= C' \exp\left(\log B \cdot \frac{\log|\xi|}{\log 2} \right) \ .
\end{aligned}
$$

Thus

$$|\hat{\varphi}(\xi) = C\Big|\prod_{m=1}^{\infty} H(\xi/2^m)\Big| \le C\left(\frac{\sin\pi\xi}{\pi\xi}\right)^2 \exp\left(\log B\cdot\frac{\log|\xi|}{\log 2}\right)$$

$$\le C'(1+|\xi|)\frac{\log B}{\log 2} - 2\ .$$

This proves the lemma.

Proof of Theorem. First we show that φ satisfies condition (ii) of Theorem 5.12. The decay estimate on $\hat{\varphi}$ shows that $\hat{\varphi} \in L^1 \cap L^2(\mathbb{R}_\xi)$. Hence $\varphi \in L^2 \cap C_0(\mathbb{R}_x)$. Now $\hat{\varphi}(2\xi) = \prod_{m=1}^{\infty} H(2\xi/2^m) = H(\xi)\prod_{m=1}^{\infty} H(\xi/2^m) = H(\xi)\hat{\varphi}(\xi)$, and this is equivalent to the statement that φ satisfies (ii).

It remains to show that φ has compact support. This can be done using the classical Paley-??? theorem. We follow instead the clever (but elementary) approach given in Daubechies [ ].

Consider the operator T_H defined by

$$T_H(f)(x) = \sqrt{2}\sum h(n)\varphi(2x+k)\ .$$

To show that φ has compact support, we find a sequence of compactly supported functions which have supports in a fixed compact set (in fact, this set can be taken to be $[-N,N]$) and which converge to φ in $L^2(\mathbb{R})$.

Set $f_0 = \chi_{[-\frac{1}{2},\frac{1}{2}]}$, so that $\hat{f}_0(\xi) = \sin\pi\xi/\pi\xi$, and define $f_n(x) = T_H(f_{n-1}(x))$, $n = 1, 2, \ldots$. We claim that $f_n \to \varphi$ in $L^2(\mathbb{R})$. That is, we show that $\hat{f}_n \to \hat{\varphi}$ in $L^2(\mathbb{R}_\xi)$. First, note that

$$\hat{f}_n(\xi) = \left(\prod_{m=1}^{n} H(\xi/2^m)\right)\left(\frac{\sin(xi/2^n)/2^n}{\pi(\xi/2^n)}\right)\ .$$

thus $\hat{f}_n(\xi) \to \hat{\varphi}(\xi)$ uniformly on compact sets, and in turn this implies that $\hat{f}_n(\xi) \to \hat{\varphi}(\xi)$ locally in $L^2(\mathbb{R}_\xi)$. That is, given $\varphi > 0$, and $R > 0$, there exists n_0 large enough that $n \ge n_0$ implies

$$\int_{|\xi|<R} |\hat{f}_n(\xi) - \hat{\varphi}(\xi)|^2 < \varepsilon^2/9\ .$$

On the other hand, since $\hat{\varphi} \in L^2(\mathbb{R}_\xi)$, we may choose R large enough (for fixed ε) so that $\int_{|\xi|>R} |\hat{\varphi}(\xi)|^2 < \varepsilon^2/9$. Thus to prove L^2 converges, it suffices to show that taking n_0 large enough we have for all $n \geq n_0$,

$$\int_{|\xi|>R} |\hat{f}_n(\xi)|^2 < \varepsilon^2/9 \ .$$

We proceed by estimating the integral

$$\int_{|\xi|>R} |P_n(\xi)|^2 \left(\frac{\sin \pi\xi/2^n}{\pi\xi/2^n}\right)^2$$

where $p_n(\xi) = \prod_{m=1}^n H(\xi/2^m)$. We break up the integral into two parts corresponding $|\xi| > 2^n$ and to $R \geq |\xi| \leq 2^n$. We note that since $|H(\xi)| \leq 1$, one has $|P_n(\xi| \leq 1$. Furthermore,

$$\begin{aligned}
|P_n(\xi)| &\leq \left[\prod_{m=1}^{n} |\cos(\pi\xi/2^m)|\right]^2 \prod_{m=1}^{n} |P(\xi/2^m)| \\
&\leq C\left[2^{-n}\frac{\sin \pi\xi}{\sin \pi\xi/2^n}\right]^2 (1+|\xi|)^{\beta} \ , \quad \text{where} \quad \beta = \frac{\log B}{\log 2}
\end{aligned}$$

with B as above.

We also note that P_n is periodic of period 2^n. Using the periodicity, we first estimate

$$\begin{aligned}
\int_{|\xi|>2^n} |P_n(\xi)|^2 \left|\frac{\sin \pi\xi/2^n}{\pi\xi/2^n}\right|^2 &= \sum_{k\neq 0} \int_{|\xi|<2^n} |P_n(\xi)|^2 \frac{|\sin \pi\xi/2^n|^2}{|\frac{\pi\xi}{2^n} + k\pi|^2} \\
&\leq C \int_{|\xi|<2^n} |P_n(\xi)|^2 |\sin \pi\xi/2^n|^2
\end{aligned}$$

Now choose $\lambda = 2^{-\alpha n}$ for some $\alpha \in (0,1)$ to be fixed later. so

$$\int_{|\xi|<2^n} |P_n(\xi)|^2 |\sin \pi\xi/2^n|^2 \leq \lambda^2 \int_{|\xi|\leq 2^n\lambda} |P_n(\xi)|^2 \, d\xi + \int_{2^n\lambda\leq|\xi|\leq 2^n} |P_n(\xi)|^2 \, d\xi \ .$$

Now

$$\begin{aligned}
\int_{|\xi|\leq 2^n\lambda} |P_n(\xi)|^2 \, d\xi &\leq \int_{|\xi|\leq 1} |P_n(\xi)|^2 \, d\xi + C\int_{1\leq|\xi|\leq 2^n\lambda} (1+|\xi|)^{2\beta}[2^n \sin \pi\xi/2^n]^{-4} \\
&\leq 1 + C\int_1^\infty (1+|\xi|)^{2\beta}|\xi|^{-4} \leq C_1 \quad \text{since} \quad \beta < 1 \ .
\end{aligned}$$

On the other hand,

$$\int_{2^n\lambda\le|\xi|\le 2^n} |P_n(\xi)|^2\,d\xi \le C2^{-4n}(1+2^n)^{2\beta}\cdot 2^n\int_\lambda^1 |\sin\pi\xi|^{-4}\,d\xi$$

$$\le C_2(2^{-3n+2\beta})\lambda^4 \ .$$

In order to get $\int_{|\xi|>2^n} |\hat{f}_n(\xi)|^2\,d\xi < \varepsilon^2/9$, we need to choose λ so that $\max(\lambda^2, 2^{-3n+2\beta n}\lambda^{-4}) \to 0$ as $n\to\infty$. Thus is suffices to take $\lambda = 2^{-n\alpha}$ where $\alpha < 3-2\beta/4$.

We still need to evaluate

$$\int_{R\le|\xi|le 2^n} |P_k(\xi)|^2\left|\frac{\sin\pi\xi/2^n}{\pi\xi/2^n}\right|^2\,d\xi \ .$$

Using the estimate $\sin\pi\xi \ge 2|\xi|$ for $|\xi|\le\frac{1}{2}$, this integral is handled by $4C\int_{R\le|\xi|\le 2^n}(1+|\xi|)^{2\beta}|\xi|^{-4}\,d\xi$. Since $\beta<1$ this term tends to 0 as $R\to\infty$. Thus we've shown that if n_0 is sufficiently large then we have $\int_{|\xi|>R}|\hat{f}_n(\xi)|^2\,d\xi < \varepsilon^3/9$. Combining our estimates shows that $\hat{f}_n\to\hat{\varphi}$ in L^2, which in turn shows that $f_n\to\varphi$ in L^2.

Now to show that φ has compact support, it suffices to show that f_n has compact support contained in a fixed compact set independent of n. From the definition of f_n, it follows that each f_n has support contained in a compact set $[-N_n, N_n]$, so we just need to check that N_n remained bounded. We have $\operatorname{supp} f_0 \subseteq [-\frac{1}{2},\frac{1}{2}]$. Since $f_1(x) = \sum_{-N}^N h(k)f_0(2x+k)$, it follows that $f_1(x)=0$ if $2x+k\notin[-\frac{1}{2},\frac{1}{2}]$ for any $-N\le k\le N$. In other words, $\operatorname{supp} f_1\subseteq[(-N-\frac{1}{2}/2),(N+\frac{1}{2}/2)]$. In general, $\operatorname{supp} f_n\subseteq[-N_n,N_n]$ where $N_n = (N/2)+(N_{n-1}/2)$. Thus (by induction), $\operatorname{supp} f_n\subseteq[-N,N]$ for all $n=1,2,\ldots$ so we conclude that $\operatorname{supp}\varphi\subseteq[-N,N]$. Since $\hat{\varphi}\in L^1(\mathbb{R}_\xi)$ and satisfies Wiener's condition, we may also conclude that $\varphi\in\mathfrak{M}(\mathbb{R})$. This completes the proof of the theorem.