KNOTS

Cameron McA. Gordon

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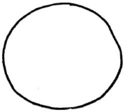
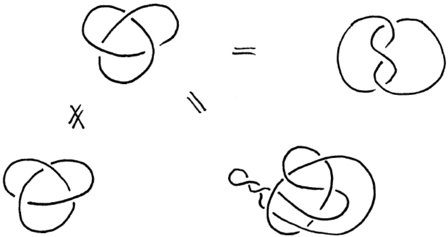
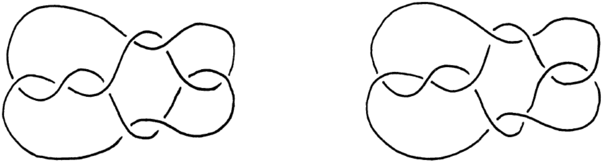


unknot





$c(K)$ = crossing number of K = minimum number of crossings in any diagram of K.

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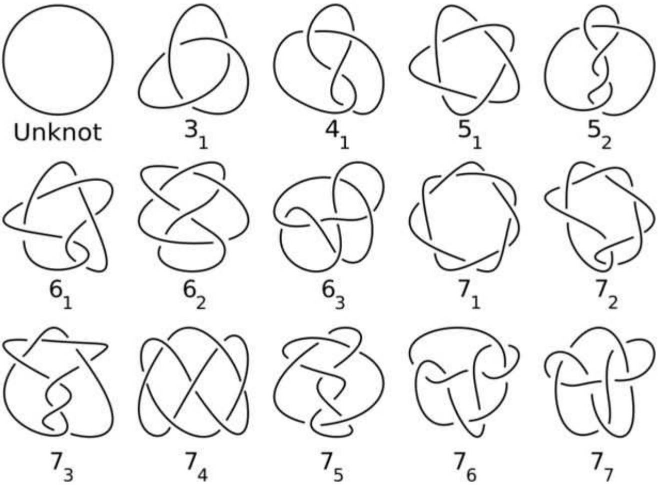
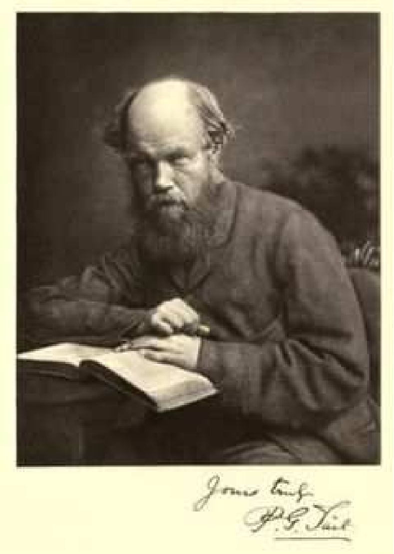
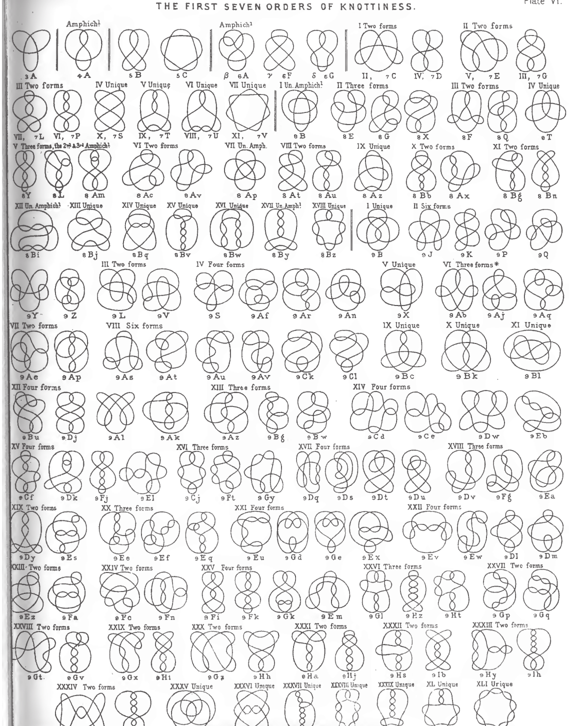


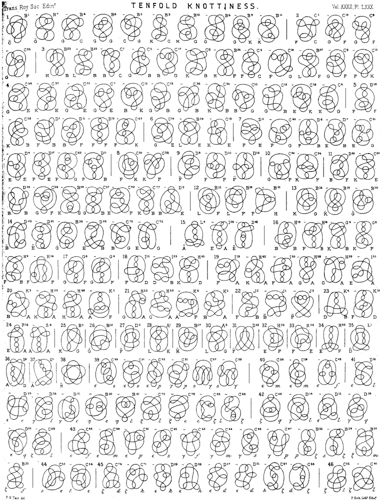


Peter Guthrie Tait (1831-1901)

Vortex Atoms
(Lord Kelvin, 1867)

$c(K)$ = order of knottiness of K


THE FIRST SEVEN ORDERS OF KNOTTINESS.
Plate VI.
Amphich1
Amphich1
I Two forms
II Two forms
3 A
4 A
5 B
5 C
β 6 A
γ 6 F
δ 6 G
II, 7 C
IV, 7 D
V, 7 E
III, 7 G
III Two forms
IV Unique
V Unique
VI Unique
VII Unique
I Un. Amphich1
II Three forms
III Two forms
IV Unique
VII, 7 L
VI, 7 P
X, 7 S
IX, 7 T
VIII, 7 U
XI, 7 V
8 B
8 E
8 G
8 X
8 F
8 Q
8 T
V Three forms, the 2nd & 3rd Amphich1
VI Two forms
VII Un. Amph.
VIII Two forms
IX Unique
X Two forms
XI Two forms
8 Y
8 L
8 Am
8 Ac
8 Av
8 Ap
8 At
8 Au
8 Az
8 Bb
8 Ax
8 Bg
8 Bn
XII Un. Amphich1
XIII Unique
XIV Unique
XV Unique
XVI Unique
XVII Un. Amph1
XVIII Unique
I Unique
II Six forms
8 Bi
8 Bj
8 Bq
8 Bv
8 Bw
8 By
8 Bz
9 B
9 J
9 K
9 P
9 Q
III Two forms
IV Four forms
V Unique
VI Three forms*
9 Y
9 Z
9 L
9 V
9 S
9 Af
9 Ar
9 An
9 X
9 Ab
9 Aj
9 Aq
VII Two forms
VIII Six forms
IX Unique
X Unique
XI Unique
9 Ae
9 Ap
9 As
9 At
9 Au
9 Av
9 Ck
9 Cl
9 Bc
9 Bk
9 Bl
XII Four forms
XIII Three forms
XIV Four forms
9 Bu
9 Dj
9 Al
9 Ak
9 Az
9 Bg
9 Bw
9 Cd
9 Ce
9 Dw
9 Eb
XV Four forms
XVI Three forms
XVII Four forms
XVIII Three forms
9 Cf
9 Dk
9 Fj
9 El
9 Cj
9 Ft
9 Gy
9 Dq
9 Ds
9 Dt
9 Du
9 Dv
9 Fg
9 Ea
XIX Two forms
XX Three forms
XXI Four forms
XXII Four forms
9 Dy
9 Es
9 Ee
9 Ef
9 Eq
9 Eu
9 Gd
9 Ge
9 Ex
9 Ev
9 Ew
9 Dl
9 Dm
XXIII Two forms
XXIV Two forms
XXV Four forms
XXVI Three forms
XXVII Two forms
9 Ez
9 Fa
9 Fc
9 Fn
9 Fi
9 Fk
9 Gk
9 Em
9 Gl
9 Hz
9 Ht
9 Gp
9 Gq
XXVIII Two forms
XXIX Two forms
XXX Two forms
XXXI Two forms
XXXII Two forms
XXXIII Two forms
9 Gt
9 Gv
9 Gx
9 Hi
9 Gz
9 Hh
9 Ha
9 Hj
9 Hs
9 Ib
9 Hy
9 Ih
XXXIV Two forms
XXXV Unique
XXXVI Unique
XXXVII Unique
XXXVIII Unique
XXXIX Unique
XL Unique
XLI Unique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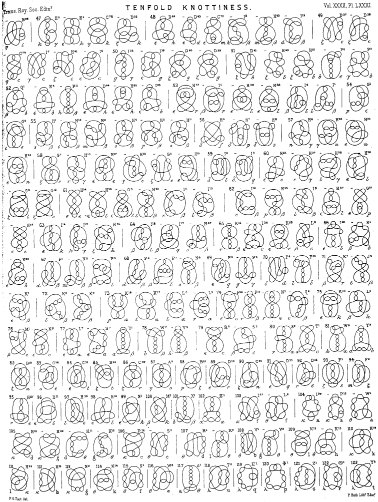
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$c(K)$	# of knots
3	1
4	1
5	2
6	3
7	7
8	21
9	49
10	165

$c(K)$	# of knots
11	552
12	2, 176
13	9, 988
14	46, 972
15	253, 293
16	1, 388, 705
17	8, 053, 378

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9, 755, 313 prime knots with $c(K) \leq 17$.

How can you prove that two knots are different?

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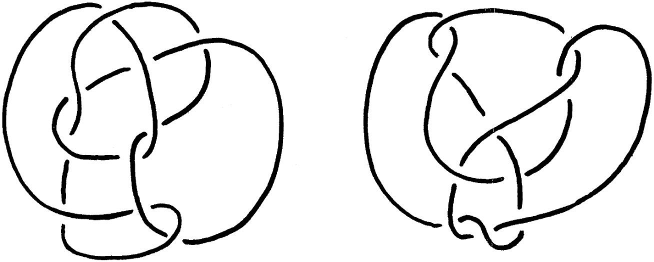
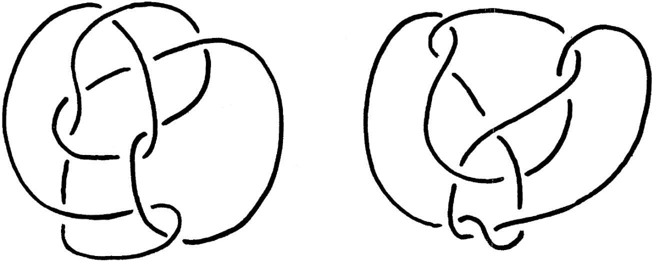


The Perko Pair

How can you prove that two knots are different?



The Perko Pair

How can you determine whether a given knot is the unknot or not?

There are many knot invariants that help with these questions.

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Example. Alexander polynomial (1928)

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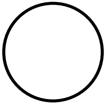
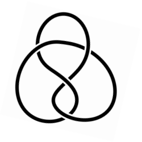
Can associate to K a polynomial $\Delta(t) = \Delta_K(t)$.



$\Delta(t) = 1$

$\Delta(t) = 1 - t + t^2$



$\Delta(t) = 1 - 3t + t^2$

James Waddell Alexander
(1888-1971)

If $\Delta_K(t) \neq \Delta_{K'}(t)$ then $K \neq K'$.

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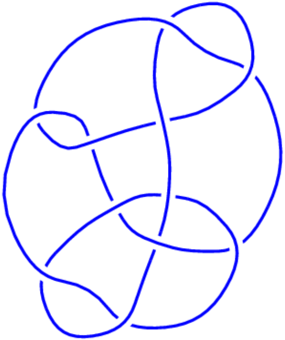
E.g.



$\Delta(t) = 1$
But $K \neq$ unknot

Is there an algorithm (systematic procedure, computer program, Turing machine, ...) to decide whether or not any given knot is the unknot?

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Fundamental dichotomy in mathematics:

solvable/decidable unsolvable/undecidable

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There is an algorithm to decide:

Yes iff W_1 and W_2 each have the same number of A's and same number of B's.

Example 2

Same, but with words in A, B, C, D, E, and substitution rules

$$AB = BA, \quad AD = DA, \quad CB = BC, \quad CD = DC$$

$$DAE = ED, \quad BCE = EB, \quad DBAD = EDBD$$

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There is no algorithm to decide.

"…A similar problem which might well be unsolvable is the one concerning knots …"

(Turing, 1954)

Alan Turing (1912-1954)

Wolfgang Haken (1928-)

There is an algorithm to decide whether or not a given knot is the unknot.

(Haken, 1957)

Used 3-dimensional topology.

There are now other proofs.

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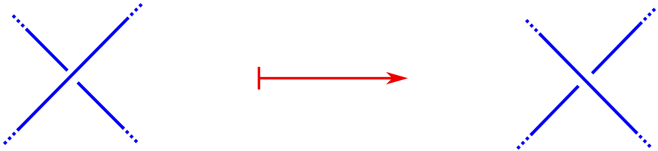

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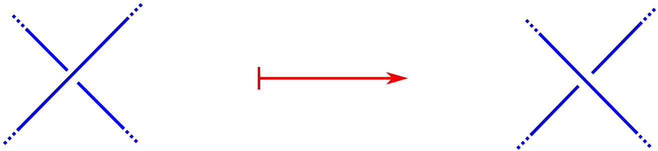

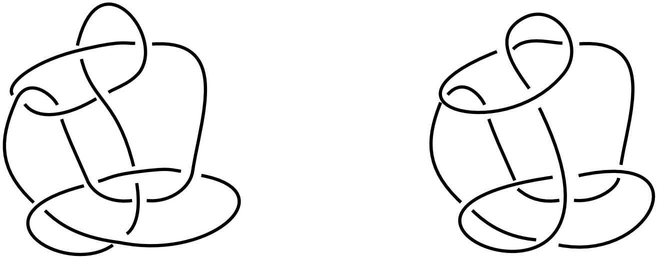

The unknotting number of K, $u(K)$, is the minimum number of such pass moves needed to unknot K.

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"In what follows the term Beknottedness will be used to signify the peculiar property in which knots, even when of the same order of knottiness, may thus differ: and we may define it, at least provisionally, as the smallest number of changes of sign which will render all the crossings in a given scheme nugatory. The question is, as we shall soon see, a delicate and difficult one."
(Tait, 1877)

$u(K) = 0$ iff K is the unknot

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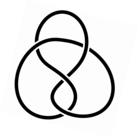
$u(K) = 1$



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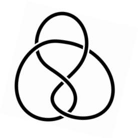

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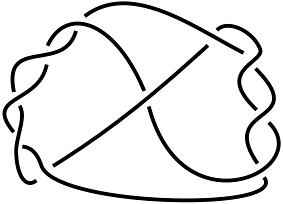

$u(K) = 2$

$u(K) = 0$ iff K is the unknot

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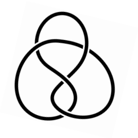

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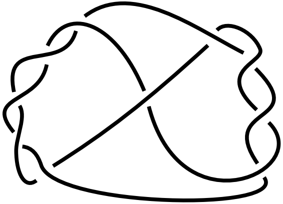
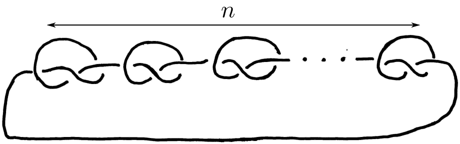


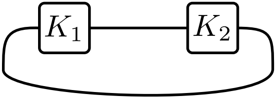
$u(K) = 2$



$u(K) = n$

Sum of knots:

$$K_1 + K_2 =$$



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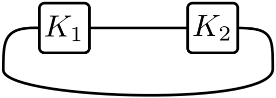
$K_1 + K_2 =$ 

Is $u(K_1 + K_2) = u(K_1) + u(K_2)$?

(Certainly $\leq$)

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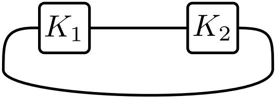

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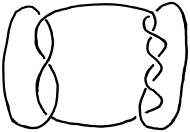
Example:

K_1



K_2

$K = K_1 + K_2$

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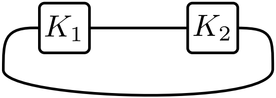

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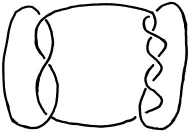


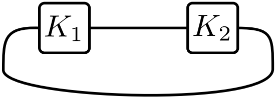
K_2

$K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Sum of knots:

$$K_1 + K_2 =$$



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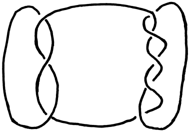

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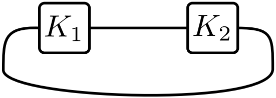
$K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Also, $\quad u(K) > 1$ (hard!)

Sum of knots:

$$K_1 + K_2 =$$



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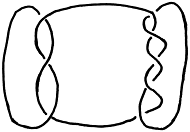
K_1  K_2 $\quad K = K_1 + K_2$

$$u(K_1) = 1, u(K_2) = 2, \text{ so } u(K) \leq 3$$

Also, $\quad u(K) > 1$ (hard!)

Is $u(K) = 2$ or 3?

Is there an algorithm to compute $u(K)$?

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Is there an algorithm to decide whether or not $u(K) = 1$?

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(Haken's theorem says there is an algorithm to decide whether or not $u(K) = 0$!)

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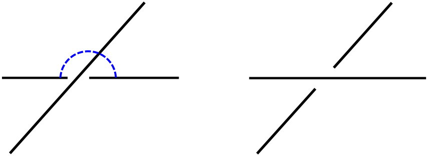
Is $c(K_1 + K_2) = c(K_1) + c(K_2)$?

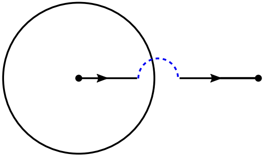
(Certainly $\leq$)

Felix Klein (1849-1925)

Every knot can be unknotted in 4 dimensions.



Analog in 3 dimensions.



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Consequently, 4-dimensional topological methods give information about $u(K)$.

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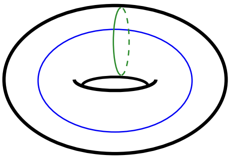
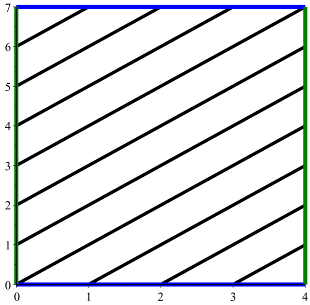
p, q coprime integers
$T_{p,q} = (p, q)$-torus knot,
lying on torus



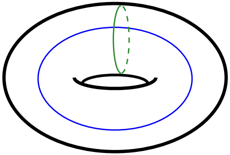
$\longleftarrow$

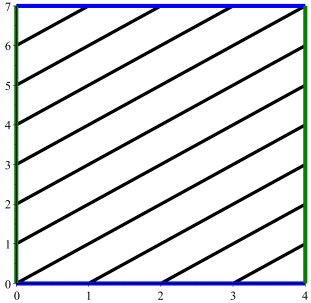


So $u(K)$ = minimum # of "jumps" into the 4th dimension needed to unknot K.

Consequently, 4-dimensional topological methods give information about $u(K)$.

p, q coprime integers
$T_{p,q} = (p, q)$-torus knot,
lying on torus

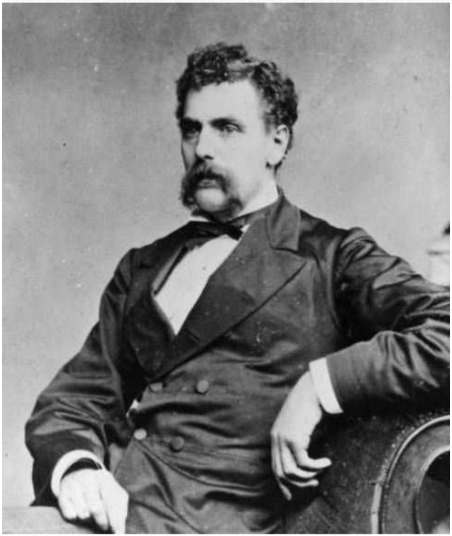




$\boxed{u(T_{p,q}) = \frac{(p-1)(q-1)}{2}}$ (Milnor Conjecture; proved in 1993)

My soul's an amphicheiral knot
Upon a liquid vortex wrought
By Intellect in the Unseen residing,
While thou dost like a convict sit
With marlinspike untwisting it
Only to find my knottiness abiding,
Since all the tools for my untying
In four-dimensioned space are lying,
Where playful fancy intersperces,
Whole avenues of universes;
Where Klein and Clifford fill the void
With one unbounded, finite homaloid,
Whereby the Infinite is hopelessly
destroyed.

James Clerk Maxwell
(1831-1879)



Henry Slade (1835-1905)

Arthur Conan Doyle
1918.

TRANSCENDENTAL PHYSICS.

An Account of Experimental Investigations.

From the Scientific Treatises

Shows once for all that Slade was a true medium.

OF

JOHANN CARL FRIEDRICH ZÖLLNER,

Professor of Physical Astronomy at the University of Leipsic;
Member of the Royal Saxon Society of Sciences;
Foreign Member of the Royal Astronomical Society of London;
of the Imperial Academy of Natural Philosophers at Moscow;
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and of the "British National Association of Spiritualists," at London.

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CHARLES CARLETON MASSEY,

OF LINCOLN'S INN, BARRISTER-AT-LAW.

SECOND EDITION.

LONDON:
W. H. HARRISON, 33 MUSEUM STREET, W.C.
1882.

Johann Zöllner (1834-1882)

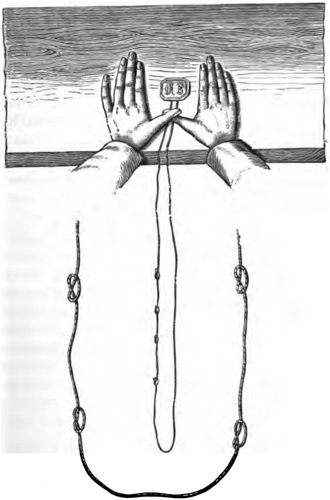

PLATE I.

"If a single cord has its ends tied together and sealed, an intelligent being, having the power voluntarily to produce on this cord four-dimensional bendings and movements, must be able, *without* loosening the seal, to tie one or more knots in this endless cord."

(Zöllner, 1879)

G Grey

THE

UNSEEN UNIVERSE

OR

PHYSICAL SPECULATIONS

ON A

FUTURE STATE

BY

B. STEWART, AND P. G. TAIT.

—— the things which are seen are temporal, but the things which are not seen are eternal.

FIFTH EDITION.
(*Revised and Enlarged.*)

London
MACMILLAN AND CO.
1876.

"There must be some very simple method of determining the amount of beknottedness for any given knot; but I have not hit upon it."

(Tait, 1877)