$L$-spaces and left-orderability

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Geometry and Topology Down Under

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Left Orderability

A group $G \neq 1$ is \textit{left-orderable} (LO) if $\exists$ strict total order $<$ on $G$ such that $g < h \Rightarrow fg < fh \ \forall \ f \in G$

- $\mathbb{R}$ is LO
- $G$ LO, $1 \neq H < G \Rightarrow H$ LO
- $G \ni g$ ($\neq 1$) finite order $\Rightarrow G$ not LO
- $G$ locally indicable $\Rightarrow G$ LO
- $G, H$ LO $\Rightarrow G \ast H$ LO (Vinogradov, 1949)
- $G$ (countable) LO $\iff \exists$ embedding $G \subset \text{Homeo}_+(\mathbb{R})$
- braid group $B_n$ is LO (Dehornoy, 1994)
• $M$ compact, orientable, prime 3-manifold (poss. with boundary)

Then $\pi_1(M)$ is LO $\iff$ $\pi_1(M)$ has an LO quotient

(Boyer-Rolfsen-Wiest, 2005)

Hence $\beta_1(M) > 0 \Rightarrow \pi_1(M)$ LO

So interesting case is when

$M$ is a $\mathbb{Q}$-homology 3-sphere (QHS)
Suppose $M$ has a co-orientable taut foliation $\mathcal{F}$

$\pi_1(M)$ acts on leaf space $\mathcal{L}$ of universal covering of $M$

If $\mathcal{L} \cong \mathbb{R}$ ($\mathcal{F}$ is $\mathbb{R}$-covered) then we get non-trivial homomorphism

$\pi_1(M) \to \text{Homeo}_+(\mathbb{R})$ \quad \therefore \quad \pi_1(M)$ is LO

**Theorem (BRW, 2005)**

*M a Seifert fibered QHS. Then $\pi_1(M)$ is LO $\iff$ $M$ has base orbifold $S^2(a_1, \ldots, a_n)$ and admits a horizontal foliation.*
Theorem (Calegari-Dunfield, 2003)

\[ M \text{ a prime, atoroidal QHS with a co-orientable taut foliation, } \tilde{M} \text{ the universal abelian cover of } M. \text{ Then } \pi_1(\tilde{M}) \text{ is LO.} \]

Thurston’s universal circle construction gives

\[ \rho : \pi_1(M) \subset \text{Homeo}_+(S^1) \]

Central extension

\[ 1 \to \mathbb{Z} \to \text{Homeo}_+(S^1) \to \text{Homeo}_+(S^1) \to 1 \]

Restriction of \( \rho \) to \( \pi_1(\tilde{M}) \) lifts to \( \text{Homeo}_+(S^1) \subset \text{Homeo}_+(\mathbb{R}) \)
**Heegaard Floer Homology** (Ozsváth-Szabó)

$M$ a QHS

$\widehat{HF}(M)$: finite dimensional $\mathbb{Z}_2$-vector space

$$\dim \widehat{HF}(M) \geq |H_1(M)|$$

$M$ is an **$L$-space** if equality holds

E.g. lens spaces are $L$-spaces

Is there a “topological” characterization of $L$-spaces?

**Conjecture**

$M$ a prime QHS. Then

$M$ is an $L$-space $\iff \pi_1(M)$ is not LO
E.g.

\[ \pi_1(M) \text{ finite} \quad \overset{\longrightarrow}{\iff} \quad \pi_1(M) \text{ not LO} \]

\[ M \text{ is an } L\text{-space} \]

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**Theorem (OS, 2004)**

*If $M$ is an $L$-space then $M$ does not admit a co-orientable taut foliation.*

So Conjecture $\Rightarrow$ : if $M$ has a co-orientable taut foliation then

$\pi_1(M)$ is LO \hspace{1cm} (virtually true by Calegari-Dunfield)

$M$ ZHS graph manifold admits a taut foliation, horizontal in each Seifert piece. Hence $M$ not an $L$-space, $\pi_1(M)$ LO

(Boileau-Boyer, 2011)
(A) Seifert manifolds

Theorem

The Conjecture is true if $M$ is Seifert fibered.

Base orbifold is either

$S^2(a_1, \ldots, a_n)$:

$M$ an $L$-space $\Leftrightarrow$ $M$ does not admit a horizontal foliation

(Lisca-Stipsicz, 2007)

$\Leftrightarrow$ $\pi_1(M)$ not LO (BRW, 2005)

(also observed by Peters)

$P^2(a_1, \ldots, a_n)$: $\pi_1(M)$ not LO (BRW, 2005)
Show $M$ is an $L$-space by inductive surgery argument using:

$N$ compact, orientable 3-manifold, $\partial N$ a torus; $\alpha, \beta \subset \partial N$, $\alpha \cdot \beta = 1$, such that

$$|H_1(N(\alpha + \beta))| = |H_1(N(\alpha))| + |H_1(N(\beta))|$$

Then $N(\alpha), N(\beta)$ $L$-spaces $\Rightarrow N(\alpha + \beta)$ $L$-space  \hspace{1cm} (\ast)

(OS, 2005)

(uses $\widehat{HF}$ surgery exact sequence of a triad)
(B) Sol manifolds

$N = \text{twisted } I\text{-bundle/Klein bottle}$

$N$ has two Seifert structures:

- base Möbius band; fiber $\varphi_0$
- base $D^2(2, 2)$; fiber $\varphi_1$

$\varphi_0 \cdot \varphi_1 = 1$ on $\partial N$

$f : \partial N \rightarrow \partial N$ homeomorphism, $M = N \cup_f N$

Assume $M$ a QHS \quad $(f(\varphi_0) \neq \pm \varphi_0)$

$M$ Seifert $\Leftrightarrow f(\varphi_i) = \pm \varphi_j$ (some $i, j \in \{0, 1\}$)

Otherwise, $M$ is a Sol manifold
\[ \pi_1(M) \text{ is not LO} \quad (\text{BRW, 2005}) \]

**Theorem**

*\(M\) is an L-space*

\[ f_* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (c \neq 0) \text{ with respect to basis } \varphi_0, \varphi_1 \]

(1) True if \[ f_* = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \]

\[ f(\varphi_1) = \varphi_0, \text{ so } M \text{ Seifert} \]

(2) True if \[ f_* = \begin{bmatrix} a & b \\ 1 & d \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} (t_0)^d \]

where \(t_0 : \partial N \to \partial N\) is Dehn twist along \(\varphi_0\)
Write \( W(f) = N \cup_f N \)

Bordered \( \widehat{HF} \) calculation shows \( \widehat{HF}(W(f)) \cong \widehat{HF}(W(f \circ t_0)) \)

So reduced to case (1)

\((3)\) In general, induct on \(|c|\) : do surgery on suitable simple closed curves \( \subset \partial N \) and use \((*)\)
(C) Dehn surgery

Theorem (OS, 2005)

$K$ a hyperbolic alternating knot. Then $K(r)$ is not an L-space $\forall \; r \in \mathbb{Q}$

Theorem (Roberts, 1995)

$K$ an alternating knot.

(1) If $K$ is not special alternating then $K(r)$ has a taut foliation $\forall \; r \in \mathbb{Q}$.

(2) If $K$ is special alternating then $K(r)$ has a taut foliation either $\forall \; r > 0$ or $\forall \; r < 0$. 
$K(1/q)$ is a ZHS .: foliation is co-orientable

$K(1/q)$ atoroidal .: $\pi_1(K(1/q)) \subset \text{Homeo}_+(S^1)$

$H^2(\pi_1(K(1/q))) = 0$; so lifts to $\pi_1(K(1/q)) \subset \text{Homeo}_+(\mathbb{R})$

.: $\pi_1(K(1/q))$ is LO (\forall q \neq 0 \text{ in (1)}, \forall q > 0 \text{ or } \forall q < 0 \text{ in (2))}

**Theorem**

*Let $K$ be the figure eight knot. Then $\pi_1(K(r))$ is LO for $-4 < r < 4$.*

Uses representations $\rho : \pi_1(S^3 \setminus K) \to PSL_2(\mathbb{R})$

(Also true for $r = \pm 4$ (Clay-Lidman-Watson, 2011))
(D) 2-fold branched covers

\[ L \text{ a link in } S^3 \]

\[ \Sigma(L) = 2\text{-fold branched cover of } L \]

**Theorem (OS, 2005)**

*If \( L \) is a non-split alternating link then \( \Sigma(L) \) is an L-space.*

(uses (*) ;

\[ \begin{array}{ccc}
\times & \bowtie & ) ( \\
L & L_0 & L_\infty
\end{array} \]

\[ \implies \Sigma(L), \Sigma(L_0), \Sigma(L_\infty) \text{ a surgery triad} \]

with \( \det L = \det L_0 + \det L_\infty \)

**Theorem**

*If \( L \) is a non-split alternating link then \( \pi_1(\Sigma(L)) \) is not LO.*

(Also proofs by Greene, Ito)
$L$ a link in $S^3$, $D$ a diagram of $L$

Define group $\pi(D)$:

- generators $a_1, \ldots, a_n \leftrightarrow$ arcs of $D$
- relations $\leftrightarrow$ crossings of $D$

\[ a_j^{-1} a_i a_j^{-1} a_k \]

Theorem (Wada, 1992)

\[ \pi(D) \cong \pi_1(\Sigma(L)) \ast \mathbb{Z} \]

\[ \therefore \pi(D) \text{ LO} \leftrightarrow \pi_1(\Sigma(L)) \text{ LO} \quad \text{(if } L \neq \text{ unknot)} \]
Suppose $\pi(D)$ LO

$$a_j^{-1}a_i a_j^{-1}a_k = 1 \iff a_j^{-1}a_i = a_k^{-1}a_j$$

$$a_i < a_j \iff a_j^{-1}a_i < 1$$

∴ at each crossing either

$$a_i < a_j < a_k$$

or $$a_i > a_j > a_k$$

or $$a_i = a_j = a_k$$

Shade complementary regions of $D$ alternately Black/White

Define graph $\Gamma(D) \subset S^2$:

vertices $\leftrightarrow$ $B$-regions

edges $\leftrightarrow$ crossings
Assume $D$ connected, alternating

We want to show $\pi_1(\Sigma(L))$ not LO

True if $L = \text{unknot}$; so assume $L \neq \text{unknot}$

Then $\pi_1(\Sigma(L)) \text{ LO } \iff \pi(D) \text{ LO}$

So assume $\pi(D) \text{ LO}$

Orient edges of $\Gamma(D)$

\[ a_i < a_j < a_k \quad a_i > a_j > a_k \quad a_i = a_j = a_k \]
Γ a connected, **semi-oriented** graph ⊂ $S^2$

cycle  sink  source

where, in each case, there is at least one oriented edge
Lemma

Let $\Gamma \subset S^2$ be a connected semi-oriented graph with at least one oriented edge. Then $\Gamma$ has a sink, source or cycle.

Let $\Gamma = \Gamma(D)$
cycle:

\[ a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_r} \leq a_{i_1} \]

\[ \therefore \quad a_1 = a_2 = \cdots = a_r \]

a contradiction, since at least one oriented edge
sink:

\[ a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_r} \leq a_{i_1} \]

\[ \therefore a_1 = a_2 = \cdots = a_r \]

a contradiction, since at least one oriented edge
Similarly for a source

\[ a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_r} \geq a_{i_1}, \text{ contradiction} \]

\[ \therefore \text{ by Lemma, all edges of } \Gamma(D) \text{ are unoriented} \]

\[ \therefore \text{ (since } D \text{ connected)} \quad a_1 = a_2 = \cdots = a_n \]

\[ \therefore \quad \pi(D) \cong \mathbb{Z} \]

\[ \therefore \quad \pi_1(\Sigma(K)) = 1 \]

\[ \therefore \quad L = \text{ unknot, contradiction} \]
$L$ quasi-alternating $\implies \Sigma(L)$ an $L$-space

**Question**

Does $L$ quasi-alternating $\implies \pi_1(\Sigma(L))$ not LO?
HAPPY BIRTHDAY, HYAM!