Surface subgroups of Coxeter and Artin groups.

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Abstract

We prove that any Coxeter group that is not virtually free contains a surface group. In particular if the Coxeter group is word hyperbolic and not virtually free this establishes the existence of a hyperbolic surface group, and answers in the affirmative a question of Gromov in this setting. We also discuss when Artin groups contain hyperbolic surface groups.

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1 Introduction

Throughout this paper by a **surface group** we shall mean the fundamental group of a closed orientable surface of genus at least 1; if the genus is at least 2 we shall use the term **hyperbolic surface group**. An intriguing question, due to Gromov ([3]), asks whether every 1-ended word hyperbolic group contains a surface group. In the present paper we answer this question affirmatively for word hyperbolic Coxeter groups. This follows immediately from our main result, which gives a characterisation of those Coxeter groups that do not contain surface groups: they are precisely the Coxeter groups that can be built up from the finite Coxeter groups by a sequence of successive free products with amalgamation along finite Coxeter subgroups. More formally, let $\mathcal{F}$ be the set of all finite Coxeter groups (including the trivial group); these are well-known (see [6] and [14]). Let $\mathcal{G}$ be the smallest class of Coxeter groups such that:

(1) $\mathcal{F} \subset \mathcal{G}$*; and*

(2) if $G_1, G_2 \in \mathcal{G}$ *and* $G_0 \in \mathcal{F}$*, and* $G = G_1 *_{G_0} G_2$*, where the inclusion of* G_0 *in* G_i *is as a special subgroup,* $i = 1, 2$*, then* $G \in \mathcal{G}$*.*

Recall that a group is called **virtually free** if it contains a free subgroup of finite index. (We note that finite groups are virtually free.) If A and B are virtually free and C is finite then $A *_C B$ is also virtually free ([26], pp 191–192). It follows easily that if $G \in \mathcal{G}$ then G is virtually free. In particular, if $G \in \mathcal{G}$ then G does not contain a surface group. Our main result is that $\mathcal{G}$ is precisely the class of Coxeter groups that do not contain surface groups.

Theorem 1.1 *Let* G *be a Coxeter group. Then the following are equivalent.*

(1) G *is virtually free.*

(2) $G \in \mathcal{G}$*.*

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(3) G does not contain a surface group.

Since a word hyperbolic group cannot contain $\mathbf{Z} \oplus \mathbf{Z}$, we get

Corollary 1.2 *Let G be a word hyperbolic Coxeter group. Then either G is virtually free, or G contains a hyperbolic surface group.*

We remark that Moussong [24] has shown that for Coxeter groups, being word hyperbolic is equivalent to not having a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup.

If a group G is virtually free, then the number of ends of G is either 0, 2 or ∞, according as the free subgroup of finite index has rank 0, 1 or ≥ 2. Hence we can answer Gromov's question for Coxeter groups.

Corollary 1.3 *Let G be a 1-ended word hyperbolic Coxeter group. Then G contains a hyperbolic surface group.*

These results further strengthen the similarities between the theory of Coxeter groups and 3-manifold groups (see [9], [13] and [22]).

A class of groups closely related to Coxeter groups are Artin groups (see §3 for a definition). Except in the trivial case when it is a free group, an Artin group always contains a $\mathbf{Z} \oplus \mathbf{Z}$. However, we can ask when an Artin group contains a hyperbolic surface group. We give some results on this question in §3.

In Theorem 1.1, clearly (1) implies (3), and, as noted earlier (2) implies (1); hence it remains to prove that (3) implies (2). The proof of this uses a fact about finite graphs due to G. A. Dirac, [12]. We include a proof below (see §2.6) for completeness. To state this, we consider the class of graphs that can be built up from complete graphs by successive amalgamations along complete subgraphs. We fix some notation. All graphs will be finite, without loops and without multiple edges. The vertex set of the graph Γ will be denoted $V(\Gamma)$. Given a subset $V \subset V(\Gamma)$, set $\mathrm{Sp}(V)$ to be the **span** of V, that is to say the subgraph of Γ consisting of all edges of Γ with both vertices in V. A subgraph $\Gamma_0 \subset \Gamma$ is **full** if $\mathrm{Sp}(V(\Gamma_0)) = \Gamma_0$, that is to say, any edge of Γ with both endpoints in Γ_0 is contained in Γ_0. For $n \geq 3$, we define an **n-cycle**, to be the graph P_n with vertices $\{v_1, v_2, \ldots, v_n\}$ and edge set E given by: v_i and v_j are connected if and only $j = i \pm 1 \mod n$. Let K_n be the complete graph on n vertices, $n \geq 0$. (By convention K_0 is the empty graph.) Let $\mathcal{C}$ be the smallest class of finite graphs such that:

(1) $K_n \in \mathcal{C}$ $\forall n \geq 0$; and

(2) if $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$ where $\Gamma_0 \cong K_n$ for some $n \geq 0$, and $\Gamma_1, \Gamma_2 \in \mathcal{C}$, then $\Gamma \in \mathcal{C}$.

It is easy to see that if $\Gamma \in \mathcal{C}$ then Γ does not contain a full n-cycle for any $n \geq 4$. In fact the converse also holds, as is proved in [12] (see §2.6 for a proof):

Theorem 1.4 *(Dirac) Let Γ be a finite graph. Then either $\Gamma \in \mathcal{C}$ or Γ contains a full n-cycle for some $n \geq 4$.*

The plan of the proof (3) implies (2) in Theorem 1.1 is the following. As is recalled in §2, associated to a Coxeter group G is a labelled finite graph Γ (not the Coxeter diagram for the group), where all labels are finite. By Theorem 1.4, it is enough to analyze the cases when Γ contains a full n-cycle, $n \geq 4$, and when Γ is a complete graph. In the first case we show directly that the corresponding special subgroup contains a surface group. In the second case, we use standard results from the theory of Coxeter groups to show that G is either finite or contains a surface group.

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2 Coxeter groups

Good references for the material on Coxeter groups covered here are [5], [6], [14] and [10].

2.1

Let Γ be a graph (as in §1) with vertex set $V(\Gamma) = \{s_1, s_2, \ldots, s_n\}$, and edge set $E(\Gamma)$. A **labelling** of Γ is a function $\mathbf{m} : E(\Gamma) \to \{2, 3, \ldots\}$. If $e \in E(\Gamma)$ has endpoints s_i and s_j, we write $\mathbf{m}(e) = m_{ij}$, and by convention we set $m_{ii} = 1$. Then the **Coxeter group** $C(\Gamma, \mathbf{m})$ associated to the labelled graph $(\Gamma, \mathbf{m})$ is the group with presentation

$$< s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 > .$$

Thus there is a relation for every pair $\{s_i, s_j\}$ such that there is an edge $e \in E(\Gamma)$ with endpoints s_i and s_j, together with the relations $s_i^2 = 1$, $1 \le i \le n$.

Caution: The associated labelled graph we have defined **is not** what is usually called the Coxeter diagram associated to the Coxeter group. In the case of the Coxeter diagram, vertices are left unconnected if the product has order 2, and vertices that are not connected in our graph are connected by edges labelled ∞. The convention we use follows [10] and is more convenient for our purposes.

2.2

That the notation used is convenient for us is illustrated in the following theorem. We denote by **2** the labelling $\mathbf{m}$ such that $\mathbf{m}(e) = 2$ for all $e \in E(\Gamma)$.

Theorem 2.1 *Let $G = C(P_n, \mathbf{m})$ be the Coxeter group associated to some labelling $\mathbf{m}$ of an n-cycle P_n. If $n \ge 4$, then G contains a surface group of finite index, which is hyperbolic unless $n = 4$ and $\mathbf{m} = \mathbf{2}$.*

Proof: We construct an n-gon in $\mathbf{E}^2$ or $\mathbf{H}^2$ such that the group generated by reflections in the faces of this n-gon is isomorphic to the Coxeter group. The result will then follow, for then a surface group is obtained from a torsion-free subgroup of finite index. Let $\mathbf{m}(s_i, s_{i+1}) = p_i < \infty$. Assume first that $n \ge 4$ and if $n = 4$, not all the $p_i = 2$. In this case we can construct a hyperbolic polygon $\mathcal{P}$ whose interior angles are consecutively π/p_i (see [2] p. 155 for example). The group generated by reflections in the sides of $\mathcal{P}$ is discrete by Poincaré's polygon theorem, and is easily seen to be isomorphic to the Coxeter group $C(P_n, \mathbf{m})$. In this case the surface group is hyperbolic.

If $n = 4$ and all labels are 2 we construct a Euclidean crystallographic group S as the group generated by reflections in the faces of a square in $\mathbf{E}^2$. Note that this Euclidean crystallographic group contains $\mathbf{Z} \oplus \mathbf{Z}$ of finite index, and no other surface groups. □

2.3

We say that $(\Gamma', \mathbf{m}')$ is a **full subgraph** of the labelled graph $(\Gamma, \mathbf{m})$, if Γ' is a full subgraph of Γ and $\mathbf{m}'$ is the restriction of $\mathbf{m}$ to the edges of Γ'. The following lemma is a standard consequence of properties of Coxeter groups (see [14], Theorem 5.5 p.113).

Lemma 2.2 *Let $(\Gamma', \mathbf{m}')$ be a full subgraph of $(\Gamma, \mathbf{m})$. Then the induced homomorphism of Coxeter groups $C(\Gamma', \mathbf{m}') \to C(\Gamma, \mathbf{m})$ is injective.*

The image of $C(\Gamma', \mathbf{m}')$ in $C(\Gamma, \mathbf{m})$ is called a **special subgroup** of $C(\Gamma, \mathbf{m})$.

We shall make use of the following theorem (see [5] and [6] p. 62). Recall that an **irreducible** Coxeter group is a Coxeter group G such that G does not decompose as a direct product $G_1 \times G_2$ with G_i a Coxeter group for $i = 1, 2$.

Theorem 2.3 *Let G be a Coxeter group. Then every proper special subgroup of G is finite if and only if one of the following occurs:*

(i) G is finite;

(ii) G is an irreducible Euclidean reflection group;

(iii) G is a cocompact reflection group acting on $\mathbf{H}^n$ for some n, whose fundamental domain is a simplex contained entirely in $\mathbf{H}^n$.

Case (iii) of Theorem 2.3 can be analyzed further. In dimension 2 these groups G correspond exactly to cocompact hyperbolic triangle groups. For, as in the proof of 2.1, we may construct a triangle with sides S_1, S_2 and S_3 corresponding to the vertices v_1, v_2, and v_3 of the Coxeter diagram, and with angles $(\frac{\pi}{p_1}, \frac{\pi}{p_2}, \frac{\pi}{p_3})$ between pairs of sides (S_1, S_2), (S_2, S_3) and (S_1, S_3) (corresponding to the labelling). The group generated by reflections in the faces is the (p_1, p_2, p_3) triangle group which will contain a hyperbolic surface group if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$$

In dimension ≥ 3, there are only a finite number of examples (see [14] p. 141), namely 9 in dimension 3 and 5 in dimension 4. In dimension 3 it is well known that these reflection groups all contain surface groups (see for example [23], [9] or [18]). For example taking a face F of a simplex, the centralizer of the reflection in the hyperbolic plane spanned by F contains a hyperbolic surface group of finite index. In dimension 4, one can argue in a similar fashion to the above: given a 2-dimensional face F of one of these simplices, let $\mathcal{H}$ denote the 2-dimensional hyperbolic plane in $\mathbf{H}^4$ spanned by F. Then the subgroup of the Coxeter group G (obtained as the group generated by reflections in the faces of the simplex) stabilizing $\mathcal{H}$ contains a surface group of finite index (since the image of $\mathcal{H}$ in $\mathbf{H}^4/G$ is a closed subset of a compact set).

In fact, it follows from standard arithmetic constructions (see [20] for example) that those simplex groups that are arithmetic groups of hyperbolic isometries, contain infinitely many commensurability classes of hyperbolic surface groups.

Note also that in case (ii) of Theorem 2.3, by definition G acts on $\mathbf{E}^n$ as a discrete group of isometries. As such it has a normal subgroup of finite index that is free abelian of rank $\leq n$. If the rank of this free abelian subgroup is ≥ 2, then we may deduce that G contains a surface group. If the rank is 0, G is finite and if the rank is 1 a simple geometric argument shows that G must be generated by exactly two reflections, and hence is the infinite dihedral group.

We summarize all of this discussion in the following:

Corollary 2.4 *Let G be an infinite Coxeter group in which every proper special subgroup of G is finite, but G is not the infinite dihedral group. Then G contains a surface subgroup.* $\square$

2.4

We can now prove Theorem 1.1. We begin with some simple observations.

Lemma 2.5 *Suppose that a graph Γ contains a full n-cycle for some $n \geq 4$. Then for any labelling $\mathbf{m}$ of Γ the Coxeter group $G = C(\Gamma, \mathbf{m})$ contains a surface group.*

Proof: The full n-cycle gives rise to a special subgroup of G which injects in G by Lemma 2.2. The labelled graph associated to the full n-cycle is isomorphic to (a labelled) P_n, and so by Theorem 2.1 the special subgroup contains a surface group of finite index. □

Lemma 2.6 *Let K_n be the complete graph on n vertices. Then for any labelling $\mathbf{m}$ of K_n, the Coxeter group $G = C(K_n, \mathbf{m})$ either is finite or contains a surface group.*

Proof: If $n = 0$ or 1, then G is finite. So suppose that $n \geq 2$, and assume by induction that the result is true for $n - 1$. Consider the proper special subgroups of G corresponding to the n K_{n-1} subgraphs of K_n. If these are all finite then, by Corollary 2.4, either G is finite or G contains a surface group (note that the infinite dihedral group case cannot occur here as the graph is complete). If one of them is infinite, then by induction it, and hence G, contains a surface group. □

Proof of Theorem 1.1
Let $G = C(\Gamma, \mathbf{m})$ be a Coxeter group. We need to prove that (3) implies (2). If Γ contains a full n-cycle for some $n \geq 4$, then G contains a surface group by Lemma 2.5. Hence by Theorem 1.4, we may assume that $\Gamma \in \mathcal{C}$. We proceed by induction on the number of vertices of Γ. If $\Gamma \cong K_n$, for some $n \geq 0$, then the result follows from Lemma 2.6. If not, then by definition of $\mathcal{C}$, $\Gamma = \Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where $\Gamma_0 \cong K_n$ and each of Γ_1 and Γ_2 has fewer vertices than Γ. This induces a decomposition $G_1 *_{G_0} G_2$, where G_i is the special subgroup of G corresponding to Γ_i, $i = 0, 1, 2$. By Lemma 2.6, G_0 is either finite or contains a surface group. Hence we may assume that G_0 is finite. Then $G_0 \in \mathcal{F}$, and $G_1, G_2 \in \mathcal{G}$ by induction, implying that $G \in \mathcal{G}$. □

2.5

Although we know exactly which Coxeter groups contain surface groups, we can ask, which Coxeter groups contain hyperbolic surface groups? The above proof shows that if $(\Gamma, \mathbf{m})$ contains either a full n-cycle for some $n \geq 4$ other than $(\diamond, \mathbf{2})$ (that is the square with edges labelled 2), or a labelled complete graph $(K_n, \mathbf{k})$ such that $C(K_n, \mathbf{k})$ is a cocompact hyperbolic Coxeter group (Theorem 2.3(iii)), then $C(\Gamma, \mathbf{m})$ contains a hyperbolic surface group. However, it is easy to show that the converse does not hold. That is, there are Coxeter groups $C(\Gamma, \mathbf{m})$ which contain a hyperbolic surface group but where $(\Gamma, \mathbf{m})$ contains no full n-cycles for $n \geq 4$, except possibly $(\diamond, \mathbf{2})$, and no complete subgraphs $(K_n, \mathbf{k})$ with $C(K_n, \mathbf{k})$ cocompact hyperbolic.

A family of examples of this type can be described as follows. There are precisely 23 non-compact tetrahedra in $\mathbf{H}^3$ with dihedral angles being integer submultiples of π, corresponding to the 23 hyperbolic Coxeter groups that act non-cocompactly but with finite co-volume on $\mathbf{H}^3$ (these are listed on p. 142 of [14]). For each of these Coxeter groups, the associated labelled graph is K_4 with all edge labels in the set $\{2, 3, 4, 5, 6\}$. All the proper special subgroups of these Coxeter groups are either finite or Euclidean Coxeter groups. In particular, we are in the situation described above. That these contain hyperbolic surface subgroups can be deduced from [8]. For, on passing to a torsion-free subgroup of finite index in these Coxeter groups, we obtain a subgroup which is the fundamental group of cusped hyperbolic 3-manifold of finite volume. The main result of [8] now applies to produce a hyperbolic surface subgroup.

Similar hyperbolic simplex groups exist in all dimensions up through dimension 9. That is, there are labelings $\mathbf{k}$ of the complete graph K_n where $5 \leq n \leq 10$ for which the associated Coxeter group $C(K_n, \mathbf{k})$ acts non-cocompactly but with finite co-volume on $\mathbf{H}^{n-1}$. Apart from one example in dimension 5, all these groups are arithmetic and it is easy to see using standard properties of non-cocompact arithmetic subgroups that they contain hyperbolic surface subgroups.

The final example can be handled as follows. The associated labelled K_6 in this case is given by the labeling:

$$m_{ij} = \begin{cases} 4 & \{i\ j\} = \{1\ 2\} \\ 3 & \{i\ j\} = \{n\ n+1\} \bmod 6,\ 2 \le n \le 6 \\ 2 & \text{otherwise} \end{cases} \cdot$$

Notice that this labeling gives rise to two Euclidean special subgroups, and that the other proper special subgroups are finite. From this it follows that the simplex σ in $\mathbf{H}^5$ has two ideal vertices, the remaining 4 being interior vertices. If we denote by G the group generated by reflections in the faces of σ, then G acts with finite co-volume on $\mathbf{H}^5$, having two cusp ends. Now each top dimensional face of σ spans a co-dimension 1 totally geodesic submanifold of $\mathbf{H}^5$. Moreover, since σ is a simplex any pair of these top dimensional faces has non-empty co-dimension 2 totally geodesic intersection. Now since there are only two ideal vertices, we can find a pair of these co-dimension 1 geodesic submanifolds, say $\mathcal{H}_1$ and $\mathcal{H}_2$, that intersect in a co-dimension 2 geodesic submanifold $\mathcal{H}$ which does not contain either of the two ideal vertices.

The argument is completed as follows. First note that the subgroups of G stabilizing $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, act with finite co-volume on the hyperbolic 4-spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (these stabilizers are just the centralizers of the reflections in each of $\mathcal{H}_i$, $i = 1, 2$). To see this, pass to a normal torsion-free subgroup Δ of finite index in G. Then reflection in any co-dimension 1 face of σ descends to an orientation-reversing involution on the manifold $\mathbf{H}^5/\Delta$. The fixed-point set is well-known to be a co-dimension 1 embedded totally geodesic submanifold. It therefore follows that the projections of $\mathcal{H}_i$ for $i = 1, 2$ into $\mathbf{H}^5/G$ are immersed finite volume totally geodesic non-compact hyperbolic 4-orbifolds. Furthermore, by the remarks above, these will meet (in general position) along the projection of $\mathcal{H}$ into $\mathbf{H}^5/G$, an immersed, closed (by choice of $\mathcal{H}_i$, $i = 1, 2$), totally geodesic 3-orbifold, denoted below by Σ. Again since σ is a simplex, we may choose another co-dimension 1 face of σ so that its projection into $\mathbf{H}^5/G$ meets Σ in a closed totally geodesic 2-orbifold. The corresponding subgroup of G is virtually a hyperbolic surface group.

¿From the above discussions, an interesting case of infinite Coxeter groups to consider are those with associated graphs K_n for some $n \ge 11$, and labelled by integers in $\{2, 3, 4, 5, 6\}$.

Another interesting case is the following. Recall that an **all right** Coxeter group is one for which the associated labelling $\mathbf{m} = \mathbf{2}$. We can ask whether the converse mentioned above holds for all right Coxeter groups. Since $C(K_n, \mathbf{2}) \cong (\mathbf{Z}/2\mathbf{Z})^n$ is finite, this becomes:

Question 2.7 *Does an all right Coxeter group $C(\Gamma, \mathbf{2})$ contain a hyperbolic surface group if and only if Γ contains a full n-cycle for some $n \ge 5$?*

2.6

Here we give the proof of Theorem 1.4. We induct on $|V(\Gamma)|$. If $|V(\Gamma)| = 0$ or 1, the theorem is trivially true, so assume $|V(\Gamma)| \ge 2$.

If Γ is a complete graph we are done. If not, then Γ has a pair of distinct vertices a and b that are not joined by an edge. Let $\Gamma' = \mathrm{Sp}(V(\Gamma) \setminus \{a, b\})$. Then Γ' is a full subgraph of Γ, and is **separating** in the sense that $\Gamma \setminus \Gamma'$ has at least two components. Let Γ_0 be a (possibly empty) full separating subgraph of Γ which is minimal (with respect to inclusion).

Let $X_1, \ldots X_m$, $m \ge 2$ be the components of $\Gamma \setminus \Gamma_0$. Let $\overline{X}_i$ be the closure of X_i in Γ, so that $\overline{X}_i = X_i \cup V_i$, for some $V_i \subset V(\Gamma)$. Since Γ_0 is full, $\mathrm{Sp}(V_i)$ is a subgraph of Γ_0. Since $\Gamma \setminus \mathrm{Sp}(V_i)$ has at least two components (one of which is X_i), the minimality of Γ_0 implies that $V_i = V(\Gamma_0)$, $1 \le i \le m$.

If Γ_0 is not a complete graph, there are distinct vertices $u, v \in V(\Gamma_0)$ such that there is no edge of Γ_0, and hence no edge of Γ, connecting u and v. Consider two complementary components, X_1 and X_2 say. Since $V_i = V(\Gamma_0)$, there is a path in $X_i \cup u \cup v$ running from u to v, $i = 1, 2$; let γ_i be such a path of minimal length. Since γ_i contains at least two edges, $\gamma = \gamma_1 \cup \gamma_2$ is a full n-cycle in Γ of length at least 4.

So we may suppose that Γ_0 is a complete graph. Then we can express Γ as $\Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are subgraphs with $\Gamma_1 \cap \Gamma_2 = \Gamma_0$, and $|V(\Gamma_i)| < |V(\Gamma)|$, $i = 1, 2$. By induction, either Γ_1 or Γ_2, and hence Γ, contains a full n-cycle for some $n \geq 4$, or Γ_1 and Γ_2, and hence Γ, belong to the class $\mathcal{C}$. □

3 Artin groups

A family of groups closely related to Coxeter groups are **Artin groups**. These are defined in a similar way to Coxeter groups as follows.

Given a labelled graph $(\Gamma, \mathbf{m})$, with $V(\Gamma) = \{s_1, s_2, \ldots, s_n\}$, we define the **Artin group** $A(\Gamma, \mathbf{m})$ to be the group with presentation

$$< s_1, s_2, \ldots, s_n \mid \text{prod}(s_i, s_j; m_{ij}) = \text{prod}(s_j, s_i; m_{ij}) >,$$

where $\text{prod}(u, v; m) = uvuv \ldots$ (m letters). Thus there is a relation for every edge $e \in E(\Gamma)$. We say that $A(\Gamma, \mathbf{m})$ is the Artin group associated with the Coxeter group $C(\Gamma, \mathbf{m})$. Note that if $E(\Gamma) \neq \emptyset$ then $A(\Gamma, \mathbf{m})$ contains a $\mathbf{Z} \oplus \mathbf{Z}$.

Question 3.1 *Which Artin groups contain a hyperbolic surface group?*

In this section we shall give some partial answers to this question.

3.1

As for Coxeter groups, an Artin group is said to be **all right** if it is of the form $A(\Gamma, \mathbf{2})$.

Theorem 3.2 *Let A be an all right Artin group for which the associated Coxeter group contains a hyperbolic surface group. Then A contains a hyperbolic surface group.*

Proof: This follows easily from [11], but needs some of the set-up of [11]. Let $G = C(\Gamma, \mathbf{2})$ be an all right Coxeter group. We construct a new graph Γ' as follows. The vertex set of Γ' is $V(\Gamma) \times \{0, 1\}$. Vertices $(v_i, 1) \leftrightarrow (v_j, 1)$ if and only if $v_i \leftrightarrow v_j$ in Γ. Vertices $(v_i, 0) \leftrightarrow (v_j, 0)$ if and only if $i \neq j$. Finally $(v_i, 0) \leftrightarrow (v_j, 1)$ if and only if $i \neq j$. Let G' be the all right Coxeter group $C(\Gamma', \mathbf{2})$. Then the main result of [11] is that the Artin group $A(\Gamma, \mathbf{2})$ is a subgroup of index $2^{|V(\Gamma)|}$ in G'. Now note that by construction of Γ', G is a special subgroup of G', and therefore if G contains a hyperbolic surface group then G', and hence $A(\Gamma, \mathbf{2})$, does also. □

The only all right Coxeter groups which are shown to contain a hyperbolic surface group by the proof of Theorem 1.1 are those whose graphs contain a full n-cycle for some $n \geq 5$. In this case, the fact that the corresponding Artin group contains a hyperbolic surface group is a result of Servatius, Droms and Servatius [27]; more specifically they show that it contains the fundamental group of a surface of genus $1 + (n-4)2^{n-3}$. We therefore ask (cf. Question 2.7):

Question 3.3 *Does an all right Artin group $A(\Gamma, \mathbf{2})$ contain a hyperbolic surface group if and only if Γ contains a full n-cycle for some $n \geq 5$?*

Theorem 3.2 also suggests the following question:

Question 3.4 *If a Coxeter group contains a hyperbolic surface group, does the corresponding Artin group also contain a hyperbolic surface group?*

3.2

We note that there are many instances in which an Artin group contains a hyperbolic surface group but the corresponding Coxeter group does not. In fact a finite or Euclidean Coxeter group clearly does not contain a hyperbolic surface group, but the corresponding Artin groups (ie those of finite or Euclidean type) usually do. To state this precisely, recall that every Artin (resp. Coxeter) group is a direct product of irreducible Artin (resp. Coxeter) groups, and note that such a direct product will contain a hyperbolic surface group if and only if one of the irreducible factors does.

For Artin groups of finite type, we have the following theorem. The notion of "type" for the underlying Coxeter group is that of [14] page 32 and conforms to the more traditional Coxeter diagram and not that used here.

Theorem 3.5 *Let A be an irreducible Artin group of finite type.*

(1) If A is of type A_1 or $I_2(m)$, then A does not contain a hyperbolic surface group.

(2) If A is not of type A_1, $I_2(m)$ or H_3, then A contains a hyperbolic surface group.

This prompts:

Question 3.6 *Does the Artin group of type H_3 contain a hyperbolic surface group?*

Note that the graph corresponding to the Artin group of type H_3 in our notation is a 3-cycle labelled $(2, 3, 5)$.

For Artin groups of Euclidean type we have (the notation is that of [14] page 34).

Theorem 3.7 *Let A be an irreducible Artin group of Euclidean type. If A is not of type $\tilde{A}_1$, $\tilde{B}_2$ or $\tilde{G}_2$ then A contains a hyperbolic surface group.*

The Artin group $\tilde{A}_1$ is $\mathbf{Z} * \mathbf{Z}$, which does not contain a surface group.

Question 3.8 *Does the Artin group of type $\tilde{B}_2$ or $\tilde{G}_2$ contain a hyperbolic surface group?*

Note that in our notation the graphs corresponding to these groups are both 3-cycles with labellings $(2, 4, 4)$ and $(2, 3, 6)$ respectively.

Recall that n-string braid group $\mathcal{B}_n$ is the Artin group of finite type A_{n-1}, $n \geq 2$, the corresponding Coxeter group being the symmetric group S_n. The main ingredient in the proof of Theorems 3.5 and 3.7 are the following two lemmas.

Lemma 3.9 *The braid group $\mathcal{B}_4$ contains a hyperbolic surface group.*

Proof: The figure eight knot complement M has a description as a once-punctured torus bundle. Using this, it was shown in [21] that $\pi_1(M)$ admits a faithful representation ι into $\mathcal{B}_4/Z(\mathcal{B}_4)$, where $Z(\mathcal{B}_4)$ denotes the center. Now $\pi_1(M)$ contains a hyperbolic surface group (see for instance [8] or [19]), and it is easy to see, using the fact that $Z(\mathcal{B}_4) \cap [\mathcal{B}_4, \mathcal{B}_4] = 1$, that if Σ is such a surface subgroup of $\pi_1(M)$, then the representation ι restricted to Σ can be lifted to a faithful representation of Σ into $\mathcal{B}_4$. $\square$

Lemma 3.10 *(1) The Artin group of type B_3 contains a hyperbolic surface group.*

(2) The Artin group of type $\tilde{A}_2$ contains a hyperbolic surface group.

Proof:(1) That $A(B_3)$ contains a surface group follows from Lemma 3.9 as can be seen as follows. $A(B_n)$ is isomorphic to the subgroup $\mathcal{B}_{1,n} < \mathcal{B}_{n+1}$ consisting of all braids which fix the first string, [16]. Since $\mathcal{B}_{1,n}$ has finite index in $\mathcal{B}_{n+1}$ (it contains the pure braid group as a subgroup of finite index), $A(B_3) \cong \mathcal{B}_{1,3}$ contains a hyperbolic surface group by Lemma 3.9.

(2) It is shown in [15] that the Artin group $A(B_n)$ is isomorphic to a semi-direct product $A(\tilde{A}_{n-1}) \rtimes_\phi \mathbf{Z}$, where ϕ has finite order. Hence $A(\tilde{A}_{n-1}) \times \mathbf{Z}$ injects in $A(B_n)$ as a subgroup of finite index. It follows from (1) that $A(\tilde{A}_2) \times \mathbf{Z}$, and hence $A(\tilde{A}_2)$, contains a hyperbolic surface group. □

Proof of Theorem 3.5:(1) The Artin group $A(A_1) \cong \mathbf{Z}$. The Artin group $A(I_2(m)) = A(m)$, say, has a non-trivial center which we denote by $Z(m)$. If $\Sigma < A(m)$ is a surface subroup of genus ≥ 2, then $\Sigma \cap Z(m) = 1$, so that Σ injects in $A(m)/Z(m)$. However, $A(m)/Z(m)$ is a free product of two cyclic groups, and this cannot contain a surface group.

(2) The graphs of the irreducible Artin groups of finite type, other than those of type A_1, $I_2(m)$, B_3, F_4 and H_3 all contain as a subgraph, the graph of type A_3. Since the analogue of Lemma 2.2 holds in the context of Artin groups (this is proved in [17], see also [25]), Lemma 3.9 shows the Artin groups associated to these finite groups contain a hyperbolic surface group.

Since the graph F_4 contains the graph B_3 as a subgraph, the above argument holds on applying Lemma 3.10(1). □

Proof of Theorem 3.7: Let A be an irreducible Artin group of Euclidean type, other than $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_2$ or $\tilde{G}_2$. Then the graph of A, except in the case of type $\tilde{C}_3$, contains the graph of finite type A_3 as a subgraph, while the graph of $\tilde{C}_3$ contains the graph of type B_3. Since the Artin groups of type A_3 and B_3 contain a hyperbolic surface group by Theorem 3.5, the theorem is proved. □

3.3

A class of Artin groups that do not contain a hyperbolic surface group is given by the following theorem. (Here, Γ denotes the graph associated with an Artin group according to **our** convention).

Theorem 3.11 *Let Γ be a tree. Then for any labelling $\mathbf{m}$ of Γ, the Artin group $A(\Gamma, \mathbf{m})$ does not contain a hyperbolic surface group.*

The proof of this theorem requires the following lemma.

Lemma 3.12 *Let X be the exterior of the $(2, m)$-torus link, and let $A \subset \partial X$ be a disjoint union of meridional annuli. Let F be a compact, connected, orientable surface with non-empty boundary, not a disk, and let $f : (F, \partial F) \to (X, A)$ be a map such that $f_* : \pi_1(F) \to \pi_1(X)$ is injective. Then F is an annulus.*

Deferring the proof of this lemma for the moment, we complete the proof of Theorem 3.11.

Proof: The Artin group associated to the labelling of K_2 by an integer $m \geq 2$, is the group of the $(2, m)$ torus link (which has one or two components according as m is odd or even), with the vertices of the graph corresponding to meridians of the link. It follows that if Γ is a tree then $G = A(\Gamma, \mathbf{m})$ is the group of a link L which is a connected sum of $(2, m)$ torus links; see [7].

Let M be the exterior of L. Then there is a disjoint union $\mathcal{A}$ of properly embedded annuli in M, such that M cut along $\mathcal{A}$ is a disjoint union $\cup_1^n X_i$, where X_i is the exterior of a $(2, m_i)$-torus link, n is the number of edges in Γ, and where the components of $\mathcal{A}$ are meridional in the boundaries of the X_i's.

Suppose $A(\Gamma, \mathbf{m}) \cong \pi_1(M)$ contains a hyperbolic surface group. Then there is a closed, connected, orientable surface S of genus ≥ 2 and a map $f : S \to M$ such that $f_* : \pi_1(S) \to \pi_1(M)$ is injective. Homotop f so that it is transverse to $\mathcal{A}$ and so that the number of components of $f^{-1}(\mathcal{A})$ is minimal. Since $\pi_1(X_i)$ does not contain a hyperbolic surface group by Theorem 3.5(1), $f^{-1}(\mathcal{A})$ is non-empty. Also, by minimality, no component of S cut along $f^{-1}(\mathcal{A})$ is a disk. Therefore, by Lemma 3.12, each component of S cut along $f^{-1}(\mathcal{A})$ is an annulus. But this would give $\chi(S) = 0$, a contradiction. □

Proof of Lemma 3.12 This is similar to the proof of Proposition 1.2 of [1]. Let the boundary components of X be $\partial_1 X$ and $\partial_2 X$. (We carry out the proof in the case where ∂X has two components; the proof in the case where ∂X has a single component is obtained by simply omitting all reference to $\partial_2 X$.) X is a fiber bundle over S^1 with fiber B, say, where $\partial B \cap \partial_i X$ is connected, $i = 1, 2$, and with monodromy of finite order n. Hence there is an n-fold cyclic covering $p : \tilde{X} \to X$, where $\tilde{X} \cong B \times S^1$. Also, there exists a connected, k-sheeted covering $p' : \tilde{F} \to F$, for some k, and a map $\tilde{f} : \tilde{F} \to \tilde{X}$ such that $p\tilde{f} = fp'$. Note that $\tilde{f}_* : \pi_1(\tilde{F}) \to \pi_1(\tilde{X})$ is injective. Let $\tilde{Z}$ be the infinite cyclic central subgroup of $\pi_1(\tilde{X})$ generated by $[\{\mathrm{pt}\} \times S^1]$. Then we may assume that $\tilde{f}_*(\pi_1(\tilde{F})) \cap \tilde{Z} = 1$, for otherwise $\pi_1(\tilde{F})$ would have a non-trivial center, implying that $\tilde{F}$, and hence F, is an annulus. Hence if $q : \tilde{X} \to B$ is projection onto the first factor, then $(q\tilde{f})_* : \pi_1(\tilde{F}) \to \pi_1(B)$ is injective. Since we may assume that $\tilde{F}$ is not an annulus, it follows by a well-known argument that $q\tilde{f} : \tilde{F} \to B$ is homotopic rel ∂ to a covering projection.

For $i = 1, 2$, let $\partial_i \tilde{X} = p^{-1}(\partial_i X)$, $\partial_i F = f^{-1}(\partial_i X)$, and $\partial_i \tilde{F} = p'^{-1}(\partial_i F)$. Let $\mu_i \subset \partial_i X$ be an oriented meridian, and let $\tilde{\mu}_i = p^{-1}(\mu_i)$, with orientation induced from μ_i. Orient F; this induces orientations on ∂F, $\tilde{F}$ and $\partial \tilde{F}$. Since $q\tilde{f}$ has non-zero degree, $\tilde{f}_*([\partial_i \tilde{F}])$ is a non-zero multiple of $[\tilde{\mu}_i]$ in $H_1(\partial_i \tilde{X})$. Therefore $p_* \tilde{f}_*([\partial_i \tilde{F}]) = n_i[\mu_i]$ in $H_1(X)$, for some $n_i \neq 0$. But

$$p_* \tilde{f}_*([\partial_i \tilde{F}]) = f_* p'_*([\partial_i \tilde{F}]) = k f_*([\partial_i F]).$$

Since $f_*([\partial F]) = 0$ in $H_1(X)$, we then get $n_1[\mu_1] + n_2[\mu_2] = 0$ in $H_1(X)$. Since $\{[\mu_1], [\mu_2]\}$ is a basis for $H_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$, this is impossible. □

We remark that there are other examples. For example, if we consider the labelled graph $(\diamond, \mathbf{2})$, the Artin group in this case is isomorphic to $F_2 \times F_2$ (where F_2 denotes the free group of rank 2), and so does not contain a hyperbolic surface group.

4 Final comments

We close with some other comments. Coxeter groups are well-known to be linear groups, and so a corollary of [11] is that all right Artin groups are linear groups. A recent result of Bigelow [4] shows that braid groups are linear. Thus the results of this paper, and a lack of counterexamples prompts a "surface group version" of the Tits Alternative. Recall that the Tits Alternative for linear groups says that a finitely generated linear group is either solvable-by-finite or contains a free non-cyclic subgroup. As with the motivating question of Gromov, we will restrict our attention to 1-ended groups.

Question 4.1 *Let G be a finitely generated 1-ended linear group that is not solvable-by-finite. Does G contain a surface group?*

Note that one cannot drop the solvable-by-finite hypothesis since there are linear groups (for example the Baumslag-Solitar group $BS(1, n)$) that contain no $\mathbf{Z} \oplus \mathbf{Z}$, and contain no infinite cyclic subgroup of finite index. Also, one cannot weaken this to ask for a hyperbolic surface group, as there are many examples of 1-ended linear groups which are not solvable-by-finite, but for which the only surface subgroups are $\mathbf{Z} \oplus \mathbf{Z}$. Finally, we point out that, this question is not yet resolved for the fundamental groups of hyperbolic 3-manifolds, but is believed to be true in this case.

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