DECISION PROBLEMS ABOUT
HIGHER-DIMENSIONAL
KNOT GROUPS

joint with

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n-knot: $K \subset S^{n+2}$, $K \cong S^n$ (PL, locally flat)

e.g. $n = 1$:

\[ K_1 \cong K_2 \iff (S^{n+2}, K_1) \cong (S^{n+2}, K_2) \]

trivial knot (unknot): standard inclusion

$S^n \subset S^{n+2}$
$$G(K) = \pi_1(S^{n+2} - K), \text{ the group of } K$$

$$K_1 \sim K_2 \implies G(K_1) \cong G(K_2)$$

$$G(\text{unknot}) \cong \mathbb{Z}$$

$$n=1: \quad G(K) \cong \mathbb{Z} \implies K = \text{unknot}$$

$$K_1 \text{ prime, } G(K_1) \cong G(K_2) \implies K_1 \sim K_2$$
Spinning (Arnold, 1925)

$n$-knot $K \subset S^{n+2} \rightarrow K_0 \subset \mathbb{R}^{n+2}$, $K_0 \cong \mathbb{D}^n$, $\partial K_0 \subset \mathbb{R}^{n+1}$

Rotate $\mathbb{R}^{n+2}$ about $\mathbb{R}^{n+1}$; sweeps out $\mathbb{R}^{n+3}$

$K_0$ sweeps out an $(n+1)$-sphere; the spin of $K \subset \mathbb{R}^{n+3}$ ($\subset S^{n+3}$)

$\text{Spin}(K)$
\[ G(\text{spin } K) \cong G(K) \]

So there exist non-trivial \( n \)-knots \( \forall n > 1 \).

\( n=2 \) : \( G(K) \cong \mathbb{Z} \Rightarrow K \text{ Topologically } \sim \text{ unknot} \) (Freedman, 1982)

Q. Is \( K \text{ PL } \sim \text{ unknot} \)?

\exists \text{ (TOP) prime } 2 \text{-knots } K_1, K_2 \text{ with } G(K_1) \cong G(K_2),

\[ K_1 \not\sim K_2 \]

\( n \geq 3 \) : \( \exists \) non-trivial \( n \)-knots with \( G(K) \cong \mathbb{Z} \)
2-knot $K \subset S^4$ (Fox, 1962)

$G(K) = \langle a, t : t^{-1}at = a^{-1}, a^3 = 1 \rangle$
Let $K_n = \{n$-knot groups $\}$

$K_0 = \{\mathbb{Z}\}$; spinning shows $K_n \subset K_{n+1}$ $\forall n \geq 0$.

If $G$ a group; $Y$ complex with $\pi_1(Y) \cong G$, $\pi_n(Y) = 0$, $n \neq 1$.

Then $H_k(G) = H_k(Y)$

$K \subset S^{n+2}$ an $n$-knot

$X = S^{n+2} - K$

$G = G(K) = \pi_1(X)$
$X$ finite complex $\Rightarrow G \in G = \{ \text{finitely presented groups} \}$

$H_\ast(X) \cong H_\ast(S^1)$

$H_1(X) \cong \mathbb{Z} \Rightarrow H_1(G) \cong G/[G,G] \cong \mathbb{Z}$

$H_2(X) = 0 \Rightarrow H_2(G) = 0$

Van Kampen $\Rightarrow$

$G/\langle \langle m \rangle \rangle = \pi_1(S^{n+2}) = 1$

($G$ has weight 1)

$m = \text{meridian of } K$
Thm (Kervaire, 1965) For $n > 3$, $G \in K_n$ iff
(1) $G \in \mathbb{S}$;  (2) $H_1(G) \cong \mathbb{Z}$;  (3) $H_2(G) = 0$;
(4) $G$ has weight $1$.

So $\mathbb{Z}^2 = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq K_3 = K_n$, $n > 3$

$K_0 \neq K_1$: $\quad$ $G = \langle x, y : x^2 = y^3 \rangle \neq \mathbb{Z}$

$K_1 \neq K_2$: $\quad \langle t, a : t^{-1}at = a^2, a^3 = 1 \rangle \in K_2 - K_1$

$K_2 \neq K_3$: $\quad \langle t, a : t^{-1}at = a^2, a^5 = 1 \rangle \in K_3 - K_2$
Q. Is there a "nice" characterization of $K_2$?

\[ S_n = \{ \pi_1(S^{n+2} - N) : N \text{ conn., closed, orientable } n\text{-mfld} \} \]
\[ M_n = \{ \pi_1(M^{n+2} - N) : M \text{ closed, } 1\text{-conn. } (n+2)\text{-mfld} \} \]
\[ N \ldots ? \]

\[ S_0 = M_0 = K_0 \quad ; \quad S_1 = M_1 = K_1 \]
\[ S_2 = S_n, \ n \geq 2 \quad (= S) \]
\[ M_2 = M_n, \ n \geq 2 \quad (= M) \]

So, $S = \{ \pi_1(S^4 - N) : N \text{ conn. closed, orientable surface} \}$
(Simon, 1980) \( G \in S \) iff 
(1) \( G \in \mathfrak{g} \); 
(2) \( H_1(G) \cong \mathbb{Z} \); 
(3) \( \exists t \in G \) such that \( \langle t \rangle = G \times t \wedge C_t = H_2(G) \).

\( C_t = \text{centralizer of } t \); \( \wedge = \text{Pontryagin product} \)

\( G \in \mathcal{M} \) iff
(1) \( G \in \mathfrak{g} \), \( x \)
(2) \( G \) has weight 1.

\[ \{ \mathbb{Z} \} = K_0 \subset K_1 \subset K_2 \subset K_3 \subset \mathcal{S} \subset \mathcal{M} \subset \mathfrak{g} \]

All inclusions proper: \( A_5 \times \mathbb{Z} \in \mathcal{S} - K_3 \):

\[ \langle x, y, s : x^2 = y^3 = (xy)^7, s^{-1}xs = x^2 \rangle \in \mathcal{M} - \mathcal{S} \]

\( \mathbb{Z} \times \mathbb{Z} \in \mathfrak{g} - \mathcal{M} \)
Let $B \triangleleft A \leq G$, where

\{ finite presentations of groups $\in A \}$ is recursively enumerable.

The recognition problem $\text{Rec}(A, B)$ is solvable if there is an algorithm to decide, given a finite presentation of a group $A \in A$, whether or not $A \in B$.

Thm (Adjan, Rabin, 1958)

$\text{Rec}(\mathbb{Z}, \{1\})$ is unsolvable.

i.e. there is no algorithm to decide whether or not the group defined by a given finite presentation is trivial.
Based on

Thm (Novikov, Boone, 1955-56). \exists G \forall g \text{ with unsolvable word problem.}

i.e. there is no algorithm to decide, given a word \( w \) in a set of generators of \( G \), whether or not \( [w] = 1 \in G \).
It has recently been announced from Russia that the ‘word problem in groups’ is not solvable. This is a decision problem not unlike the ‘word problem in semi-groups’, but very much more important, having applications in topology: attempts were being made to solve this decision problem before any such problems had been proved unsolvable.

...Another problem which mathematicians are very anxious to settle is known as ‘the decision problem of the equivalence of manifolds’.... It is probably unsolvable, but has never been proved to be so. A similar decision problem which might well be unsolvable is the one concerning knots which has already been mentioned.

(Turing, 1954)
Thm (Haken, 1962) There is an algorithm to decide whether or not a given 1-knot \( K \subset S^3 \) is trivial.

Since \( K \) trivial \( \iff G(K) \cong \mathbb{Z} \):

Cor. Rec. \( (K_0, K_1) \) is solvable.

\[
G = \{ K_0, K_1, K_2, K_3, S, M, \mathcal{G} \}
\]

If \( A \in G \) then the finite presentations of groups \( \in A \) is recursively enumerable.
Thm (Gonzalez-Acuña - G. - Simon). Let \( A, B \in G \), \( B \neq A \), \( A > K_3 \). Then \( \text{Rec}(A,B) \) is unsolvable.

Q. Does this also hold for \( A = K_2 \)?

Most cases proved by using unsolvability of \( \text{Rec}(G, E_13) \).

E.g.

Prop. \( \exists \) effective procedure which takes a finite presentation of a group \( G \) and produces a finite presentation of a group \( H \) such that

1. \( H \leq K_3 \)
2. \( G = 1 \Rightarrow H \cong \mathbb{Z} \)
3. \( G+1 \Rightarrow H \neq K_2 \).
Cor.  If $K_0 < B < K_2$ then $\text{Rec} (K_3, B)$ is unsolvable.
(a sol'n. to $\text{Rec} (K_3, B)$ wd. $\rightarrow$ sol'n. to $\text{Rec} (G, \aleph_1)$.)


Cor. $\exists H \in K_3$ with unsolvable word problem.

Cor. $\exists H \in K_3$ which contains a copy of every $G \in G$.

Q. Does every $G \in G$ embed in some $H \in K_2$?
n-knot triviality problem

\{ n-knots \} is recursively enumerable. So can ask:

"given an n-knot, is it trivial?"

Solvable if n=1.

Thm (G-A-G-S.) If \exists \ G \in K_n with unsolvable word problem
then the n-knot triviality problem is unsolvable.

Cor (Nabutovsky-Weinberger, 1996) If n\geq3 the
n-knot triviality problem is unsolvable.

Q. Is there a G \in K_n with unsolvable word problem?
Sketch proof. \((S^{n+2}, K)\) \(n\)-knot, \(n \neq 2\), with
\(G = G(K)\) having unsolvable word problem.

Surgery on \(K\):
\[ M^{n+2} = (S^{n+2} - N(K)) \cup (D^{n+1} \times S^1) \]
\[ \cup \]
\[ S^n \times D^2 \]

\[ C = \ast \times S^1 \subset D^{n+1} \times S^1 \subset M \]

\[ \pi_1(M) \cong G ; \quad [C] = m = \text{meridian of } K \]

Let \( \Sigma \) be a trivial \( n \)-sphere \( \subset M \)

\[ \pi_1(M - \Sigma) \cong G \times \mathbb{Z}(\sigma) , \quad \sigma = \text{meridian of } \Sigma. \]
w word in the generators of G, \([w] \in [G, G]\).

γ_w loop in \(M - \Sigma\) such that \([\gamma_w] = o^{-w} o w m \in \pi_1(M - \Sigma)\)

\([\gamma_w] = m \in \pi_1(M)\)

\(\gamma_w\) isotopic to \(C\) in \(M\)

Do surgery on \(\gamma_w\):

\(M \rightarrow S^{n+2}\)

\(\Sigma \rightarrow n\text{-sphere } \Sigma_w \subset S^{n+2}\)
\[ [\omega] = 1 \in G : \quad [\gamma] = m \in \pi_1(M - \Xi) \]
\[ \therefore \gamma \text{ isotopic to } C \text{ in } M - \Xi \]
\[ \therefore (S^{n+2}, \Xi, \omega) \text{ trivial} \]

\[ [\omega] \neq 1 \in G : \quad \pi_1(S^{n+2} - \Xi, \omega) = \langle G, \sigma : \sigma^{-1} \omega \sigma = \omega^m \rangle \]
\[ = \text{HNH extension of } G \quad \therefore \neq \mathbb{Z} \]
\[ \therefore (S^{n+2}, \Xi, \omega) \text{ non-trivial} \]

\[ \therefore \text{a solution to the } n \text{-knot triviality problem would give a solution to the word problem for } G. \]
Simple connectivity of 3-manifolds problem:
"given a connected, closed 3-mfld. $M$, is $\pi_1(M) = 1$?" (*)

Let $M_3 = \{ \pi_1(M) : M$ closed 3-mfld. $\}$

Finite presentations of groups in $M_3$ is rec. enumerable.

Can show

If $G \in M_3$ then $G = 1 \iff H_1(G) = H_3(G) = 0$

(*) is solvable: (1) $\pi_1(M) = 1 \iff M \cong S^3$ (Perelman, 2003)

(2) I algorithm to decide whether or not a given

3-mfld. $M$ is $\cong S^3$ (Rubinstein-Thompson, 1994)
Given $G \in \mathcal{G}$, $H_2(G)$ finitely generated abelian groups $(H_n(G), k \geq 3$, not nec. fin. gen. (Stallings, 1963)).

Clearly I algorithm to compute $H_1(G)$ for $G \in \mathcal{G}$.

Thm (G, 1980) There is no algorithm to decide, given $G \in \mathcal{G}$, whether or not $H_2(G) = 0$.

More generally:

Thm (G-A-G-S). Let $I \in \mathbb{N}$, $I \neq \emptyset$, $\{1\}$. Then "given $G \in \mathcal{G}$, is $H_k(G) = 0 \ \forall k \in I"$ is unsolvable.