1. **Question 8 (20 points):** Use the degree 4 Taylor polynomial centered at the origin for \( f(x) = e^{-x^2/2} \approx T_4(x) \) to estimate the integral
\[
I = \int_0^1 f(x) \, dx \approx \int_0^1 T_4(x) \, dx.
\]

What is the value of that approximation of \( I \)? What estimate does the Taylor’s Inequality provide for the error between \( I \) and its computed approximation?

2. **Question 9 (10 points):** Evaluate the limit or show the limit does not exist.
\[
\lim_{(x,y) \to (0,0)} \frac{\cos x - \cos y}{x^2 - y^2}
\]

**Answer:**

**Question 8:** The Maclaurin series for \( e^{-y} \):
\[
e^{-y} = \sum_{k=0}^{\infty} \frac{(-1)^k y^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k y^k}{k!} + R_n(y),
\]

Since \( \frac{d}{dy^{n+1}} e^{-y} = |(-1)^{n+1} e^{-y}| \leq 1 \), for \( y \geq 0 \). So according to Taylor’s inequality
\[
R_n(y) \leq \frac{y^{n+1}}{(n+1)!} \quad \text{for} \quad y \geq 0.
\]

Setting \( y = x^2/2 \) yields
\[
e^{-x^2/2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^k k!} + R_n \left( \frac{x^2}{2} \right), \quad \text{where} \quad R_n \left( \frac{x^2}{2} \right) \leq \frac{x^{2n+2}}{2^{n+1}(n+1)!}
\]

Therefore, the degree 4 Taylor polynomial \((n = 2)\) centered at the origin for \( f(x) \) is
\[
T_4(x) = \sum_{k=0}^{2} (-1)^k \frac{x^{2k}}{2^k k!} = 1 - \frac{x^2}{2} + \frac{x^4}{8},
\]

The approximation for \( I \) is
\[
I \approx \int_0^1 T_4(x) \, dx = \left( x - \frac{x^3}{6} + \frac{x^5}{40} \right) \bigg|_0^1 = 1 - \frac{1}{6} + \frac{1}{40} = \frac{103}{120}
\]

And the error between \( I \) and its computed approximation is
\[
|\text{error}| \leq \left| \int_0^1 \frac{x^{2n+2}}{2^{n+1}(n+1)!} \, dx \right| = \frac{1}{(2n + 3)2^{n+1}(n+1)!} = \frac{1}{(7)2^33!} = \frac{1}{336}
\]
Question 9:

\[
\lim_{(x, y) \to (0, 0)} \frac{\cos x - \cos y}{x^2 - y^2} = \lim_{(x, y) \to (0, 0)} \frac{-2 \sin \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)}{(x + y)(x - y)}
\]

\[= -\frac{1}{2} \lim_{(x, y) \to (0, 0)} \left( \frac{\sin \left(\frac{x + y}{2}\right)}{(x + y)/2} \right) \left( \frac{\sin \left(\frac{x - y}{2}\right)}{(x - y)/2} \right) \]

When \((x, y) \to (0, 0)\), we have \(u = \frac{x + y}{2} \to 0\) and \(v = \frac{x - y}{2} \to 0\). We know that

\[\lim_{z \to 0} \frac{\sin z}{z} = 1.\]

Therefore

\[\lim_{(x, y) \to (0, 0)} \left( \frac{\sin \left(\frac{x + y}{2}\right)}{(x + y)/2} \right) \left( \frac{\sin \left(\frac{x - y}{2}\right)}{(x - y)/2} \right) = 1 \]

So

\[\lim_{(x, y) \to (0, 0)} \frac{\cos x - \cos y}{x^2 - y^2} = -\frac{1}{2} \]