A tank contains 1000L of brine with 10kg of dissolved salt. Brine that contains 0.03kg of salt/L of water enters the tank at a rate of 10L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 20 minutes? What happens in the long run?

Note: please write down step by step to get credits.

Solution: Let \( y(t) \) be the amount of salt (kg) after \( t \) minutes. We are given that \( y(0) = 10 \). The rate of change of the amount of salt is

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out} \left( \frac{\text{kg}}{\text{min}} \right),
\]

where

\[
\text{Rate in} = (\text{concentration of salt}) \times (\text{brine flows in rate}) = \left( 0.03 \frac{\text{kg}}{\text{L}} \right) \times \left( 10 \frac{\text{L}}{\text{min}} \right) = 0.3 \frac{\text{kg}}{\text{min}},
\]

and

\[
\text{Rate out} = (\text{concentration of salt}) \times (\text{brine flows out rate}) = \left( \frac{y}{1000} \frac{\text{kg}}{\text{L}} \right) \times \left( 10 \frac{\text{L}}{\text{min}} \right) = \frac{y(t)}{100} \frac{\text{kg}}{\text{min}}.
\]

Therefore

\[
\frac{dy}{dt} = 0.3 - \frac{y}{100} = \frac{30 - y}{100}
\]

\[
\int \frac{dy}{30 - y} = - \int \frac{dt}{100}
\]

\[- \ln |30 - y| = \frac{t}{100} + c \]

\[30 - y = ke^{-\frac{t}{100}}
\]

\[y = 30 - ke^{-\frac{t}{100}}.
\]

Since \( y(0) = 10 \), we have at time \( t = 0 \)

\[10 = 30 - k \Rightarrow k = 20.
\]

So

\[y = 30 - 20e^{-\frac{t}{100}} \quad \text{(2 points)}
\]

\[y(20) = 30 - 20e^{-1/5}. \quad \text{(1 point)}
\]

When \( t \to \infty \), \( y \) goes to 30.
The air in a room with volume 180 m³ contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room with a rate of 2 m³ and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

Note: please write down step by step to get credits.

Solution: Let \( y(t) \) be the amount of carbon dioxide (m³) after \( t \) minutes. Then
\[
y(0) = 0.0015(180) = 0.27 m³
\]
The rate of change of the amount of carbon dioxide with respect to time is
\[
\frac{dy}{dt} = \text{rate in} - \text{rate out} \left( \frac{m³}{min} \right),
\]
where
\[
\text{Rate in} = (0.0005) \left( 2 \frac{m³}{min} \right) = 0.001 \frac{m³}{min}
\]
and
\[
\text{Rate out} = \frac{y(t)}{180} \left( 2 \frac{m³}{min} \right) = \frac{y(t)}{90} \left( \frac{m³}{min} \right)
\]
Therefore
\[
\frac{dy}{dt} = 0.001 - \frac{y}{90} = \frac{0.09 - y}{90}
\]
\[
\int \frac{dy}{0.09 - y} = \int \frac{dt}{90}
\]
\[
-\ln |0.09 - y| = \frac{t}{90} + c
\]
\[
0.09 - y = ke^{-\frac{t}{90}}
\]
\[
y = 0.09 - ke^{-\frac{t}{90}}.
\]
Since \( y(0) = 0.27 \), we have at time \( t = 0 \)
\[
0.27 = 0.09 - k \Rightarrow k = -0.18.
\]
So
\[
y = 0.09 + 0.18 e^{-\frac{t}{90}}
\]
The percentage of carbon dioxide in the room is
\[
\frac{y}{180} \times 100 = \frac{(0.09 + 0.18 e^{-\frac{t}{90}})(100)}{180} = \frac{9 + 18 e^{-\frac{t}{90}}}{180} = \frac{1}{20} + \frac{1}{10} e^{-\frac{t}{90}}
\]
When \( t \to \infty \), the percentage of carbon dioxide goes to \( \frac{1}{20} \).

Note: Grading scheme for Section 52795 is the same as for Section 52790.